Indian National Mathematics Olympiad Conducted by HBCSE with Solved Papers of RMO INMO 2016-2019 Part 1 upto Page 335 Inequalities Problems

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Indian National MATHEMATICS OLYMPIAD

Conducted by **Homi Bhabha Centre for Science Education**

With

SOLVED PAPERS RMO & INMO 2016-2019

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In India, National Board of Higher Mathematics (NBHM) started National Mathematics Olympiad in 1986. It worked with Homi Bhabha Centre for Science Education, Mumbai. One aim of this activity is to support the mathematical talent among the senior secondary students in the country. The problems of the Olympiads, chosen from various areas of secondary school mathematics, require exceptional mathematical ability and mathematical knowledge on the part of the candidates.

This book is a collection of lessons, and includes challenging and stimulating problems from various National and Regional Mathematics Olympiad. The problems are provided with complete solutions, occasionally accompanied by remarks, alternative ways of doing the same problem or the generalization of the problem.

This book is an ideal source of study for all those aiming to participate in National and Regional Mathematics Olympiads as well as ones, who are preparing for entrances such as JEE Main & Advanced, where complex and tricky problems are routine...

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PREFACE

Since the year 1989 when India started participating in the Mathematical Olympiads, the interest of the school students in the country has tremendously increased in India National Mathematics Olympiad (INMO). Now, we find a number of really talented students in almost all the prestigious schools in India who are excited about this mega event and sincerely desire to participate in it, win a handful of medals and make the country proud. We just need to educate them about the competition, and provide them the relevant study material.

The Mathematical Olympiad tests the participant's level of mastery of the methods of Mathematics and the strategizing and tactical skills in plenty. The Olympiad is an open challenge to all those who love the problem solving.

This book has been written, keeping in mind the orientation required on the parts of students to face the Olympiads at national or regional level. This book has designed to give the student an insight and proficiency into almost all the areas of Mathematics. Exhaustive theory has been provided of selected and relevant chapters to clarify the basic concepts. Problems from recently held Olympiads have been given to increase awareness of what to expect in the event.

Revised Edition of This Book Has

1. Complete theory with support of good number of solved examples and exactly on the pattern and level of Indian National Mathematics Olympiads.
2. Each chapter has two level exercises divided according to RMO Regional Mathematics Olympiad and INMO (Indian National Mathematical Olympiad)
3. Solutions have been provided for selected questions.

First of all, I would like to thank Mr Deepesh Jain, Director Arihant Group the man with a distinct vision, for the idea to write this book, and then bringing it to reality. I am also thankful to my colleagues and students for the moral support they provided. I take this an opportunity to thank Sunil Chugh, Director, HMA, for the inspiration to write the book of this nature, and Sumit Malviya for the assistance he provided in the preparation of the manuscript.

It is hoped, this book will charge you up for the Olympiad juggernaut. I have tried my best to keep this book error-free. However, if any error or whatsoever is left I request the readers to bring forward to my notice. Suggestions for the further improvement of the book are welcome.

With best wishes

Rajeev Manocha

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Dedicated to My Parents
I.P.F. Manocha, Bimla Manocha
& My sons Robit and Sarthak

INTRODUCTION

MATHEMATICS OLYMPIADS

ABOUT THE EXAM & HOW TO SUCCEED IN IT ?

International Mathematical Olympiad

International Mathematical Olympiads was created by Dr. George Lenchner, a prominent Maths educator, in 1977 with the aim of stimulating enthusiasm and love for Mathematics. About 1,50,000 students from 50 states of USA and 25 other countries participate in this competitive examination every year. Through this examination, effort has been made to introduce important mathematical concepts and teach major strategies for problem solving. The examination also seeks to foster Mathematical creativity and ingenuity and provide satisfaction, joy and thrill through meeting challenges.

Every participant team comprises 35 students. Some schools send more than one team in the contest. However, the rule indicates that only schools or home-school associations may participate individuals are barred from entering into the contest.

Every team enters into competition in just one division. Those teams which have members from more than one school are called 'district teams' or 'institute teams' and they are not eligible for team awards. The team score is the total of ten highest individual scores taken after the fifth contest in a series of contests which involve selection.

The International Mathematics Olympiad invites students from 2nd to 12th class to participate and excel at international level.

Indian National Mathematical Olympiads

In India, National Board for Higher Mathematics (NBHM) started National Mathematics Olympiads in 1986. It worked with Homi Bhabha Centre for Science Education, Mumbai, for this purpose. One aim of this activity is to support the mathematical talent among the high school students in the country. NBHM take the responsibility of selecting, training and grooming of Indian team for participation in the International Mathematical Olympiad every year. There are certain regional bodies which provide voluntary service and play an important role at various stages. The country has been divided into 23 regions for conducting Mathematical Olympiads.

Stages

Stage I Regional Mathematical Olympiad (RMO)

A regional coordinator conducts tests in each region. Regional Mathematical Olympiads (RMO) is held in each region between September and first Sunday of December every year. The regional coordinator makes sure that at least one centre in each district of the region is provided for the test. All the students of high school upto Class XII can appear at Regional Mathematical Olympiads. The qualifying test is of three hours and consists of six to seven problems.

The regional coordinator has the liberty of making his own paper or obtaining paper from NBHM. The regions which choose to go for paper by NBHM conduct this contest on the first Sunday of December. The performance in RMO is judged and a certain number of students from every region are chosen to appear for the next round. There is a nominal fee to meet the expenses of the organising the contest.

Stage II Indian National Mathematical Olympiad

The second stage of selection involves holding of the contest at the national level. Indian National Mathematical Olympiads is (INMO) organized on the first Sunday of every February at various centres in different regions. Only those students who have been selected in Regional Mathematical Olympiads can appear at this contest. At this level, there is a 4-hour test which is common to all regions. Those who rank among the top 35, receive a certificate of merit.

Stage III International Mathematical Olympiad Training Camp (IMOTC)

The INMO awardees are invited to a month long training camp held in April-May each year at the Homi Bhabha Centre for Science Education (HBCSE), Mumbai. INMO awardees of the previous year who have satisfactorily gone through postal tuition throughout the year are invited again to a second round of training (Senior Batch). The senior batch participants who successfully complete the camp receive a prize of 5,000/- in the form of books and cash. On the basis of a number of selection tests through the camp, a team of the best six students is selected from the combined pool of junior and senior batch participants.

Stage IV Pre-departure Training Camp for IMO

The selected team of six students goes through another round of training and orientation for about 10 days prior to the departure for IMO.

Stage V International Mathematical Olympiad (IMO)

The six-members team selected at the end of the camp accompanied by a leader and a deputy leader represent the country at the IMO, held in July each year in a different member country of IMO. IMO consists of two 4 and 1/2-hour written tests held on two days. Travel to IMO venue and return takes about 2 weeks. India has been participating in IMO since 1989. Students of the Indian team who receive gold, silver and bronze medals at the IMO receive a cash prize of 5000/-, 4000/- and 3000/- respectively during the following year at a formal ceremony at the end of the training camp.

Ministry of Human Resource Development (MHRD) finances international travel of the 8-members Indian delegation, while NBHM (DAE) finances the entire in-country programme and other expenditure connected with international participation.

Awards and Recognition

- Participation Certificate to every student | Merit Certificate for students appearing in second level School Topper Medal to the class topper in each participating school (with more than 50 students).
- Top three students from each class are awarded Gold, Silver and Bronze medals.
- The top 500 winners (classwise) are endowed with cash prizes and scholarships, courtesy of the official sponsor.
- Every participant who scores 50% or more is awarded a Certificate of Participation. Toppers are awarded merit certificate.
- School/College of each medal winning participant is awarded the 'Intellect Trophy' for having groomed and nurtured the talent.

Career Prospects

Mathematical Olympiads were created to hunt out the talented students and to give them an opportunity to express their talent in creative and intuitive way. It reads well on a student's curriculum vitae to have participated in Mathematical Olympiads. Those who have participated in the Olympiads have a good chance of doing well in Engineering.

The top few students from Mathematical Olympiads are selected for admission to premier institutions. The career of the students takes up an upwards mobility curve after participating in this contest. The students become confident of their performance and rich with the feeling of self-worth after selection to represent at Mathematical Olympiads.

Number of Students Appearing

- Approximately 10,000 students appear for this examination at regional level, though the number is only eight when they represent the country at international level.
- Students from rural and urban areas alike participate in the regional competition. This competition is open for all the students from Class IX onwards.

Month and Frequency of Examination

International Mathematical Olympiads is held once every year in July. The qualifying rounds of Indian National Mathematics Olympiads is held in different months of the years starting from first Sunday of December.

Craze of the Examination

Mathematical Olympiads, whether they are held at national level or international level, attract interest not only from the mathematicians but also from the student community, the parents and society at large. Even those students who may not be participating in the contest are interested in it.

The students who participate in the contest start preparing for it at least three months prior to the date of examination. The craze for the examination is not limited only to a certain region, the verve engulfs the entire country. Those who are selected are viewed as the promising stars of future who will contribute to the progress of science and technology in years to come.

General Eligibility

The students must be from Class IX and must be at least 14 years of age to participate in this contest both at national and international levels. They must pass the different rounds of selection in order to be eligible.

At the first round, all students who are in Class IX are eligible to appear for the qualifying test. In the subsequent rounds, however, only those whom make it in the test would be eligible to carry forth.

Skill Sets Required

In Mathematics, the skill most required is ability to think fast, be spatially gifted and compute complex data without committing error. The students must develop the habit of calculating data mentally and simplify the methods in order to arrive at the correct answer with minimum effort. Another very important part is concentration; lack of concentration can cause the candidates to commit errors, so sufficient effort must be put into develop the power of concentration.

Difficulty Level of Problems

The problems of Olympiads, chosen from various areas of secondary school Mathematics, require exceptional mathematical ability and mathematical knowledge on the part of the contestants. Generally, there are six to seven questions asked and they are of extremely high level. To solve these questions, the students need to be thorough with their basics and have good knowledge of all the principles of Mathematics. Since two or three principles from different mathematical streams must be applied to solve the questions, the problems assume a sophistication which requires an agile and quick thinking mind with plenty of common sense to answer them.

How to Prepare for Different Subjects in Stipulated Time

The best way of dealing with this contest is to cultivate scientific thinking and to develop power of concentration. The students must try to deal with two or three mathematical principles simultaneously so that they become adept at solving complex problems which require application of higher mental faculty. There are many books in the market but Arihant books are the best as they help in development of scientific temper along with providing reading material and solution.

General Mental Set up Required for the Examination

The candidates must have a positive frame of mind and they must be able to deal with stress that comes as a package with these examinations. They must have the desire to succeed and ability to work for long lasting success. The candidates must be well versed in all the fundamentals of the principles of Mathematics. They must be original thinker and must have a scientific temperament. The power of concentration of the candidates must be of exceptionally high order. The calculation skill of the candidates must be extremely good since all Maths sums are based on reasoning and calculation.

Do's and Don't on the Day of Examination

On the day of the examination, the candidates must take care about a few things which are listed below:

- The candidates must reach the venue of the examination at least half an hour before time.
- The candidates must carry their admit cards to the examination hall.
- The candidates must carry their own pens, pencils, erasers, sharpeners and must refrain from borrowing these articles from the other candidates.
- The candidates must abstain from talking to other candidates in the examination hall while the examination is being conducted.
- The candidates must hand over their answer sheets to the invigilator as soon as the stipulated time is over.

How this Book is Useful for You

This book by Arihant is the best in the market for purpose of preparation. It has many complex problems and tricky questions requiring sophisticated methods of calculation.

This book coaches the students to think in multidimensional way and apply the principles of Mathematics to solve the most complex problems in the simplest way. It also gives tremendous exposure to different types of problems which exist in the realm of Mathematics.

This book has matter for study by the students and solved problems which help in understanding of the subject itself. It also has previous years papers from Regional and International Olympiads along with their solutions. It is an excellent book to have which will prove to be the best friend to the contestants.

Unit 1 Theory of Numbers

Unit 1

Theory of Numbers

Natural Numbers

The numbers 1, 2, 3, ..., which are used in counting are called *natural numbers* or *positive integers*.

Basic Properties of Natural Numbers

In the system of natural numbers, we have two 'operations' addition and multiplication with the following properties.

Let x, y, z denote arbitrary natural numbers, then

1. $x + y$ is a natural number *i.e.*, the sum of two natural number is again a natural number.
2. **Commutative law of addition** $x + y = y + x$
3. **Associative law of addition** $x + (y + z) = (x + y) + z$
4. $x \cdot y$ is a natural number *i.e.*, product of two natural numbers is a natural number.
5. **Commutative law of multiplication** $x \cdot y = y \cdot x$
6. **Associative law of multiplication** $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
7. **Existence of multiplicative identity** $a \cdot 1 = a$
8. **Distributive law** $x(y + z) = xy + xz$

Divisibility of Integers

An integer $x \neq 0$ divides y, if there exists an integer a such that $y = ax$ and thus we write as $x \mid y$ (x divides y).
If x does not divides y, we write as $x \nmid y$ (x does not divides y)
[This can also be stated as y is divisible by x or x is a divisor of y or y is a multiple of x].

Properties of Divisibility

1. $x \mid y$ and $y \mid z \Rightarrow x \mid z$
2. $x \mid y$ and $x \mid z \Rightarrow x \mid (ky \pm lz)$ for all $k, l \in z$. z = set of all integers.
3. $x \mid y$ and $y \mid x \Rightarrow x = \pm y$
4. $x \mid y$, where $x > 0, y > 0 \Rightarrow x \leq y$
5. $x \mid y \Rightarrow x \mid yz$ for any integer z.
6. $x \mid y$ iff $nx \mid ny$, where $n \neq 0$.

Test of Divisibility

Divisibility by certain special numbers can be determined without actually carrying out the process of division. The following theorem summarizes the result :

A positive integer N is divisible by

- 2 if and only if the last digit (unit's digit) is even.
- 4 if and only if the number formed by last two digits is divisible by 4.
- 8 if and only if the number formed by the last three digits is divisible by 8.
- 3 if and only if the sum of all the digits is divisible by 3.
- 9 if and only if the sum of all the digits is divisible by 9.
- 5 if and only if the last digit is either 0 or 5.
- 25 if and only if the number formed by the last two digits is divisible by 25.
- 125 if and only if the number formed by the last three digits is divisible by 125.
- 11 if and only if the difference between the sum of digits in the odd places (starting from right) and sum of the digits in the even places (starting from the right) is a multiple of 11.

Division Algorithm

For any two natural numbers a and b, there exists unique numbers q and r called respectively quotient and remainder, $a = bq + r$, where $0 \le r < b$.

Common Divisor

If a number 'c' divides any two numbers a and b *i.e.*, if $c \mid a$ and $c \mid b$, then c is known as a common divisor of a and b.

Greatest Common Divisor

If a number d divides a and b and is divisible by all the common divisors of a and b, then d is known as the *greatest common divisor* (GCD) of a and b or HCF of a and b.

The GCD of numbers a and b is the unique positive integer d with the following two properties.

(i) $d \mid a$ and $d \mid b$

(ii) If $c \mid a$ and $c \mid b$, then $c \mid d$

We write it as $(a, b) = d$

For example, $(12, 15) = 3$; $(7, 8) = 1$

Note 1. $(a, b) = (b, a)$ 2. If $a|b$; then $(a, b) = a$

Properties of GCD

1. If $(b, c) = g$ and d is a common divisor of b and c, then d is a divisor of g.
2. For any $m > 0$, $(mb, mc) = m\,(b, c)$
3. If $d|b$ and $d|c$ and $d > 0$, then $\left(\frac{b}{d}, \frac{c}{d}\right) = \left(\frac{1}{d}\right)(b, c)$
4. If $(b, c) = g$, then $\left(\frac{b}{g}, \frac{c}{g}\right) = 1$
5. If $(b, c) = g$, then there exists two integers x and y such that $g = xb + yc$
6. If $(a, b) = 1$ and $(a, c) = 1$, then $(a, bc) = 1$
7. If $a \mid bc = 1$ and $(a, b) = 1$, then $a \mid c = 1$

For example $\qquad a = 6, b = 21, c = 10$

$$6 \mid 21 \times 10 \text{ but } (6, 21) = 3$$

and $(6, 10) = 2$ and 6 divides neither 21 nor 10

LCM of two integers a, b is the smallest positive integer divisible by both a and b and it is denoted by $[a, b]$.

The **Euclidean algorithm** can be used to find the GCD of two integers as well as representing GCD as in 5th property. Consider 2 numbers 18, 28

$$28 = 1.8 + 10 ; 18 = 1.10 + 8$$
$$10 = 1 \cdot 8 + 2 ; 8 = 4 \cdot 2 + 0$$
$$(18, 28) = 2$$
$$(18, 28) = 2 = 10 - 1.8 = 10 - (18 - 1.10)$$
$$= 2.10 - 1.18 = 2(28 - 1.18) - 1.18$$
$$= 2.28 - 3.18 = 2.28 + (-3)18$$

Note The representation in 5th property is not unique. In fact we can represent (a, b) as $xa + yb$ in infinite number of ways where $x, y \in Z$

Z = set of all integers.

In above example, 252 is LCM of 18 and 28.

$$252 = 9.28$$
$$252 = 14.18$$
$$(18, 28) = 2.28 + (-3) \cdot 18$$
$$= 2.28 + 252K + (-3)18 - 252K$$
$$= (2 + 9K)28 + (-3 - 14K)18$$

where K is any integer.

Unit

1 is called unit in the set of positive integers.

Prime

A positive integer P is said to be prime, if

(i) $P > 1$

(ii) P has no divisors except 1 and P *i.e.*, A number which has exactly two different factors, itself and one, is called a *prime number*.

Thus, 2, 3, 5, 7, 11, ... are primes. 2 is the only even number which is prime. All other primes being odd.

But the converse is not true *i.e.*, every odd number need not be prime.

Composite

Every number (greater than one) which is not prime is called *composite number*. *i.e.*, a number which has more than two different factors is called *composite*. For example 18 is a composite number because 2, 3, 6, 9 are divisors of 18 other than 1 and 18.

We can also define a composite number as : A natural number n is said to be *composite*, if there exists integers l and m such that $n = lm$, where $1 < l < n$ and $l < m < n$.

Remark

- A prime number P can be written as a product only in one way namely $P.1$.
- A composite number n can also be written as $n.1$. But composite number can be written in one more way also as mentioned above.
- A composite number has at least three factors.

Note 1 is neither prime nor composite.

Twin Primes

A pair of numbers is said to be *twin primes*, if they differ by 2.

e.g., 3, 5 are twin primes.

Perfect Number

A number n is said to be perfect if the sum of all divisors of n (including n) is equal to $2n$.

For example 28 is a perfect number because divisors of 28 are 1, 2, 4, 7, 14, 28.

Sum of divisors of $n (= 28) = 1 + 2 + 4 + 7 + 14 + 28 = 56 = 2n$

Coprime Integers

Two numbers a and b are said to be coprime, if 1 is only common divisors of a and b.

i.e., if GCD of a and $b = 1$ *i.e.,* if $(a, b) = 1$

e.g., $(4, 5) = 1$, $(8, 9) = 1$.

Theorem 1 *If $a = qb + r$, then $(a, b) = (b, r)$.*

Proof Let $(a, b) = d$ and $(b, r) = e$

$\because \quad (a, b) = d \quad \therefore \quad d \mid a$ and $d \mid b$

$\therefore \quad d \mid a$ and $d \mid qb \quad \therefore \quad d \mid (a - qb)$

i.e., $\quad d \mid r \quad [\because a - qb = r]$

$\therefore$ d is common divisor of b and r.

$\therefore \quad d \mid e \qquad$ [$\because$ e is the GCD of b and r] ...(i)

Again $\because \quad (b, r) = e$

$\therefore \quad e \mid b$ and $e \mid r \quad \therefore \quad \frac{e}{bq}$ and $\frac{e}{r}$

$\therefore \quad e \mid bq + r$ *i.e.,* $e \mid a \quad [\because a = bq + r]$

$\therefore$ e is a common divisor of a and b.

$\therefore \quad e \mid d \qquad$ [$\because$ d is the GCD of a and b] ...(ii)

From Eqs. (i) and (ii), we have $d = e$

i.e., $\quad (a, b) = (b, r)$

Remark

The above result can also be stated as :

GCD of a and b is same as GCD of b and r, where r is remainder obtained on dividing a by b.

Corollary If $(a, b) = 1$, then $(b, r) = (a, b) = 1$

i.e., if a is coprime to b, then r is coprime to b. r is remainder obtained on dividing a by b.

Theorem 2 *If d is the greatest common divisor of a and b, then there exists integers x and y such that $d = xa + yb$ and d is the least positive value of $xa + yb$.*

Proof

Case I By successive application of division algorithm to numbers a and b.

Let $r_1, r_2, \ldots, r_n$ be successive remainders.

Therefore, $a = bq_1 + r_1, 0 < r_1 < b$ (Dividing a by b)

$b = r_1q_2 + r_2, 0 < r_2 < r_1$ (Dividing b by remainder r_1)

$r_1 = r_2q_3 + r_3, 0 < r_3 < r_2$ (Dividing r_1 by remainder r_2)

$\vdots$

$r_{n-2} = r_{n-1}q_n + r_n, \quad 0 < r_n < r_{n-1}$

$r_{n-1} = r_nq_{n+1} + r_{n+1}, \quad 0 < r_{n+1} < r_n$

Since $r_1 > r_2 > \ldots$ is a set of decreasing integers, this process must terminate after a finite number of step. *i.e.*, remainder must be zero after some stage.

So let $r_{n+1} = 0 \quad \therefore \quad r_{n-1} = r_nq_{n+1} + 0 = r_nq_{n+1}$

$r_n \mid r_{n-1} \quad \therefore \quad (r_{n-1}, r_n) = r_n$ [If $a \mid b$, then $(a, b) = a$]

Now, $(a, b) = (b, r_1) = (r_1, r_2) = (r_2, r_3) = \ldots = (r_{n-1}, r_n) = r_n$ ...(i)

i.e., GCD of a and b is r_n.

From first of above equations $r_1 = a - bq_1 = ax_1 + by_1$, where $x_1 = 1, y_1 = -q$,

Putting the value of $r_1 = a - bq_1$ in $r_2 = b - r_1q_2$

$$r_2 = b - r_1q_2 = b - (a - bq_1)q_2 = b - aq_2 + bq_1q_2$$

$$= -aq_2 + b(1 + q_1q_2) = ax_2 + by_2, \text{ where } x_2 = -q_2$$

$$y_2 = 1 + q_1q_2$$

Similarly, $r_3 = ax_3 + by_3$ and so on $r_n = ax_n + by_n$

or $r_n = ax + by$

where $x_n = x$ and $y_n = y$

i.e., GCD of a and $b = r_n$ can be expressed as $(a, b) = d = ax + by$ [By Eq. (i)]

Case II $(a, b) = d$ ($\because$ $d \mid a$ and $d \mid b$)

$\therefore$ $d \mid (ax + yb)$ for all values of x and y.

$\therefore$ $\exists$ an integer k such that $xa + yb = kd$ ...(ii)

But least value of k is 1.

Putting $k = 1$ in Eq. (ii), least value of $xa + yb$ is d.

Corollary If a and b are coprime integers *i.e.*, if $(a, b) = 1$, then there exists integers x and y such that

$$ax + by = 1$$

Example 1 *If $(a, b) = d$, then $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.*

Solution $\because$ $(a, b) = d$

$\therefore$ $d \mid a$ and $d \mid b$ [By definition of GCD].

$\therefore$ There exists integers a_1 and b_1 such that $a = da_1$ ...(i)

$b = db_1$ ...(ii)

Again $\because$ $(a, b) = d$

There exists integers x and y such that

$$ax + by = d \qquad \text{[By Theorem 2]}$$

Putting values of a and b from Eqs. (i) and (ii)

$$da_1x + db_1y = d.$$

or $$a_1x + b_1y = 1$$

$$(a_1, b_1) = 1 \qquad \text{[By corollary theorem 1]}$$

or $$\left(\frac{a}{d}, \frac{b}{d}\right) = 1 \qquad \left[\because \text{From Eq. (i)}, a_1 = \frac{a}{d}, \text{from Eq. (ii)}, b_1 = \frac{b}{d}\right]$$

Remark

$(a, b) = d$ and $a = a_1d, b = b_1d$, then a_1 and b_1 are coprime *i.e.*, $(a_1, b_1) = 1$

Example 2 *If $a|bc$ and $(a, b) = 1$, then $a|c$.*

Solution $\because$ $a|bc$

$\therefore$ There exists an integer d such that $bc = ad$ ...(i)

$\because$ $(a, b) = 1$

$\therefore$ There exist integers m and n such that $am + bn = 1$ ...(ii)

Multiplying both sides of Eq.(ii) by c, $acm + bcn = c$ ...(iii)

Putting $bc = ad$ from Eq. (i) in Eq. (iii)

$$acm + adn = c$$

$$a(cm + dn) = c$$

$\therefore$ $a|c$

Note If $a|bc$ and $(a, b) = 1$, then $a|c$. This result is also known as **Gauss Theorem**.

Example 3 *Prove that every two consecutive integers are coprime.*

Solution Let n and $n + 1$ be two consecutive integers.

Let $(n, n + 1) = d$

$\therefore$ $d|n$ and $d|n + 1$

$d|(n + 1) - n$ or $d|1$

$\therefore$ $d = 1$

$\therefore$ $(n, n + 1) = 1$

i.e., n and $(n + 1)$ are relatively prime.

Example 4 *Show that* GCD *of* $a + b$ *and* $a - b$ *is either* 1 *or* 2, *if* $(a, b) = 1$.

Solution Let $(a + b, a - b) = d$

$\therefore$ $d|(a + b)$ and $d|(a - b)$

$\therefore$ $d|(a + b + a - b)$ and $d|(a + b) - (a - b)$

or $d|2a$ and $d|2b$.

i.e., d is a common divisor of $2a$ and $2b$.

$\therefore$ $d|(2a, 2b)$ [By definition of GCD]

i.e., $d \mid 2(a, b)$ $[\because (ma, mb) = m(a, b)]$

But $(a, b) = 1 \;\therefore\; d \mid 2$

$\therefore$ $d = 1$ or $d = 2$.

Example 5 *Find* GCD *of* 858 *and* 325 *and express it in the form* $m\,858 + n325$.

Solution $858 = 325.2 + 208$ ...(i)

Dividing 858 by 325

$$325 = 208.1 + 117 \quad ...(ii)$$

Dividing 325 by 208

$$208 = 117.1 + 91 \quad ...(iii)$$

Dividing 208 by 117

$$117 = 91.1 + 26 \quad ...(iv)$$

Dividing 117 by 91

$$91 = 26.3 + 13 \quad ...(v)$$

Dividing 91 by 26

$$26 = 13.2$$

$\therefore$ GCD of 858 and 325 is $d = 13$

From Eq. (v), $d = 13 = 91 - 26.3$

Substituting the value of 26 from Eq. (iv)

$\Rightarrow$ $91 - 3(117 - 91.1) = 91 - 3.117 + 3.91 = 4.91 - 3.117$

Substituting the value of 91 from Eq. (iii)

$$= 4(208 - 117) - 3.117 = 4.208 - 7.117$$

Substituting the value of 117 from Eq. (ii)

$$= 4.208 - 7(325 - 2081) = 4.208 - 7.325 + 7.208$$

$$= 11.208 - 7.325$$

$$= 11(858 - 325.2) - 7.325 \quad \text{[Putting the value of 208 from Eq. (i)]}$$

$$= 11.858 - 22.325 - 7.325 = 11.858 - 29.325$$

$$= m858 + n.325 \text{ where } m = 11, n = -29$$

Example 6 *If a and b are relatively prime, then any common divisor of ac and b is a divisor of c.*

Solution a and b are relatively prime

$\therefore \exists$ integers x and y such that

$$ax + by = 1 \quad ...(i)$$

Let d be any common divisor of ac and b.

$\because d \mid ac$, $\therefore \exists$ an integer m such that

$$ac = dm \quad ...(ii)$$

$\because d \mid b$, $\therefore \exists$ an integer n such that

$$b = dn \quad ...(iii)$$

Multiplying both sides of Eq. (i) by c

$$acx + bcy = c \quad ...(iv)$$

Putting the values of ac and b from Eqs. (ii) and (iii) in Eq. (iv).

$$dmx + dncy = c$$

or $$d(mx + ncy) = c$$

$\therefore$ $$d \mid c$$

Example 7 *If a and b are any two odd primes, show that* $(a^2 - b^2)$ *is composite.*

Solution $a^2 - b^2 = (a - b)(a + b)$

$\because$ $a.0$ and b are odd primes.

So, let $a = 2k + 1$

$b = 2k' + 1$

$\therefore$ $a - b = 2k + 1 - 2k' - 1 = 2k - 2k' = 2(k - k')$ is even

$a + b = 2k + 1 + 2k' + 1 = 2k + 2k' + 2 = 2(k + k' + 1)$ is even

$\therefore$ Neither $(a - b)$ nor $(a + b)$ is equal to 1.

$\therefore$ Neither of the two divisors $(a - b)$ and $(a + b)$ of $(a^2 - b^2)$ is equal to 1.

$\therefore$ $(a^2 - b^2)$ is composite.

[$\because$ Out of the two divisors of a prime number p, one must be equal to 1]

Example 8 *If* $a|c$, $b|c$ *and* $(a, b) = 1$, *then* $ab|c$.

Solution $\because$ $a|c$

$\therefore$ There exists an integers d such that

$$c = ad \quad \text{...(i)}$$

$\because$ $$b|c$$

$\therefore$ There exists an integer e such that

$$c = be$$

$\because$ $(a, b) = 1$, therefore there exist integers m, n such that

$$am + bn = 1 \quad \text{...(ii)}$$

Multiplying both sides by c

$$acm + bcn = c \quad \text{...(iii)}$$

Putting $c = be$ from Eq. (ii) in acm and $c = ad$ from Eq. (i) in bcn, Eq. (iii) becomes

$$abem + band = c$$

$$ab(em + dn) = c$$

$\therefore$ $ab|c$

Example 9 *If* $a^2 - b^2$ *is a prime number, show that* $a^2 - b^2 = a + b$, *where* a, b *are natural numbers.*

Solution $a^2 - b^2 = (a - b)(a + b)$ ...(i)

$\because$ $(a^2 - b^2)$ is a prime number

$\therefore$ One of the two factors $= 1$

$\because$ $a - b = 1$ [$\because a - b < a + b$]

$\because$ The only divisor of a prime number are 1 and itself.

Eq. (i) becomes $a^2 - b^2 = 1(a+b)$

or $\qquad a^2 - b^2 = a + b$

e.g., $\qquad 3^2 - 2^2 = 5$ (which is prime)

$\Rightarrow \qquad 3^2 - 2^2 = 3 + 2,\ 3, 2 \in N.$

Example 10 *Prove that an integer is divisible by 9 if and only if the sum of its digits is divisible by 9.*

Solution Let $a = a_n \ldots a_3a_2a_1$ be an integer

[Note a is not the product of $a_1, a_2, a_3, \ldots, a_n$ but $a_1, a_2, a_3, \ldots, a_n$ are digits in the value of a. For example 368 is not the product of 3, 6 and 8 rather 3, 6, 8 are digits in value of 368

$$= 8 + 6 \times 10 + 3 \times (10)^2]$$

$$\begin{aligned} a &= a_n \ldots a_3a_2a_1 \\ &= a_1 + (10)^1a_2 + (10)^2a_3 + (10)^3a_4 + \ldots + (10)^{n-1}a_n \\ &= a_1 + 10a_2 + 100a_3 + 1000a_4 + \ldots \\ &= a_1 + (a_2 + 9a_2) + (a_3 + 99a_3) + (a_4 + 999a_4) + \ldots \\ &= (a_1 + a_2 + a_3 + a_4 + \ldots) + (9a_2 + 99a_3 + 999a_4 + \ldots) \qquad \ldots\text{(i)} \end{aligned}$$

or $\qquad a = S + 9(a_2 + 11a_3 + 111a_4 + \ldots)$

where $\qquad S = a_1 + a_2 + a_3 + a_4 + \ldots$

is the sum of digits in the value of a

$\therefore \qquad a - S = 9(a_2 + 11a_3 + 111a_4 + \ldots)$

$\therefore \qquad 9 | (a - S) \qquad \ldots$(ii)

Case I a is divisible by 9

i.e., $\qquad 9 | a \qquad \ldots$(iii)

$\therefore \qquad 9 | [a - (a - S)]$ [From Eqs. (ii) and (iii)]

i.e., $\qquad 9 | S$ *i.e.,* sum of digits is divisible by 9.

Case II S (sum of digits) is divisible by 9

i.e., $\qquad 9 | S \qquad \ldots$(iv)

From Eqs. (ii) and (iv), $9 | [(a - S) + S]$

i.e., $\qquad 9 | a$

i.e., the integer a is divisible by 9.

Theorem 3 *Prove that the product of any r consecutive numbers is divisible by r!.*

Proof Let

$$P_n = n(n+1)(n+2)\ldots(n+r-1) \qquad \ldots\text{(i)}$$

be the product of r consecutives integers beginning with n.

We shall prove the theorem by Induction method.

For $r = 1$, $P_n = n$ is divisible by $1!$ for all n.

$\therefore$ The theorem is true for $r = 1$ *i.e.,* the product of 1 (consecutive) integer is divisible by $1!$.

Let us assume the theorem to be true for the product of $(r - 1)$ consecutive integers.

i.e., every product of $(r-1)$ consecutive integers is divisible by $(r-1)!$.

Changing n to $n+1$ in Eq. (i)

$$\therefore \quad P_{n+1} = (n+1)(n+2)\ldots(n+r)$$

Multiplying both sides by n

$$nP_{n+1} = n(n+1)(n+2)\ldots(n+r)$$
$$= n(n+1)(n+2)\ldots(n+r-1)(n+r)$$
$$nP_{n+1} = (n+r)P_n$$
$$= nP_n + rP_n$$

or $$n(P_{n+1} - P_n) = r \cdot P_n$$

or $$P_{n+1} - P_n = r \cdot \frac{P_n}{n}$$

$$= r \cdot \frac{n(n+1)(n+2)\ldots(n+r-1)}{n}$$ [Using value of P_n]

or $$P_{n+1} - P_n = r(n+1)(n+2)\ldots(n+r-1)$$

or $$P_{n+1} - P_n = r$$

Product of $(r-1)$ consecutive integers.

$$= rP \quad \ldots(i)$$

Where P denotes the product of $(r-1)$ consecutive integers.

But the product P of $(r-1)$ consecutive integers is divisible by $(r-1)!$. [By assumption]

$$\therefore \quad P = k(r-1)!$$

$\therefore$ Eq. (i) becomes $P_{n+1} - P_n = rk(r-1)! = kr(r-1)! = k(r)!$

i.e., $$r! \mid (P_{n+1} - P_n), \forall n$$

Put $$n = 1$$

$$\therefore \quad r! \| (P_2 - P_1)$$

But $P_1 = 1 \cdot 2 \cdot 3 \ldots r = r!$ is divisible by $r!$

i.e., $$r_1! \| P_1$$

$$\therefore \quad r! \| (P_2 - P_1) + P_1 \text{ i.e., } r! \| P_2$$

Put $n = 2$,

$$\therefore \quad r! \| (P_3 - P_2)$$

But $r! \| P_2$

$$\therefore \quad r! \| (P_3 - P_2) + P_2$$

i.e., $r! \| P_3$ and so on.

Generalising we can say that $r! \| P_n$ for all n.

Corollary *${}^nC_r$ is an integer*

$${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{r! \qquad (n-r)!}$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$
$$= \frac{\text{the product of } r \text{ consecutive integers}}{r!} = \text{An integer}$$

($\because$ The product of r consecutive integer is divisible by $r!$).

Note Therefore the product of two consecutive integers is divisible by $2! = 2$; the product of any three consecutive integers is divisible by $3 = 6!$ and so on.

Example 1 *Prove that product of two odd numbers of the form $4n + 1$ is of the form $(4n + 1)$.*

Solution Let $a = 4k + 1$, $b = 4k' + 1$

be two numbers of the form $(4n + 1)$

$$\therefore \quad ab = (4k + 1)(4k' + 1)$$
$$= 16kk' + 4k + 4k' + 1$$
$$= 4(4kk' + k + k') + 1 = 4l + 1$$

(where $l = 4kk' + k + k'$)

Which is in form $(4n + 1)$.

Example 2 *Prove that square of each odd number is of the form $8j + 1$.*

Solution Let $n = 2m + 1$ be an odd number.

$$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1$$
$$= 4m(m + 1) + 1$$

Now, $m(m + 1)$ being product of two consecutive integers, is divisible by $2! = 2$

$$\therefore \quad m(m + 1) = 2j$$
$$\Rightarrow \quad n^2 = 4(2j) + 1 = 8j + 1$$

Example 3(a) *Show that sum of an integer and its square is even.*

Solution Let n be any integer.

So, we have to prove that $n^2 + n$ is even.

$\Rightarrow n^2 + n = n(n + 1)$ which is product of two consecutive numbers n and $n + 1$ and hence divisible by $2! = 2$

Hence, $n^2 + n$ is an even number.

Example 3(b) *If n is an integer. Prove that product $n(n^2 - 1)$ is multiple of 6.*

Solution $n(n^2 - 1) = n(n - 1)(n + 1) = (n - 1)n(n + 1)$

Which being the product of three consecutive integers is divisible by $3! = 6$

$\therefore$ $n(n^2 - 1)$ is divisible by 6.

i.e., $n(n^2 - 1)$ is a multiple of 6.

Note If n is a multiple of p, we shall write $n = M(p)$.

Example 4 *If r is an integer, show that $r(r^2 - 1)(3r + 2)$ is divisible by 24.*

Solution $r(r^2 - 1)(3r + 2) = r(r - 1)(r + 1)(3r + 2)$

$$= (r - 1)r(r + 1)\{3(r + 2) - 4\}$$
$$= 3(r - 1)r(r + 1)(r + 2) - 4(r - 1)r(r + 1)$$

$(r - 1)r(r + 1)(r + 2)$ being the product of four consecutive integers is divisible by $4! = 24$

$\therefore$ $3(r - 1)r(r + 1)(r + 2)$ is also divisible by 24.

Again $(r-1)r(r+1)$ is divisible by $3! = 6$

$\therefore$ $4(r-1)r(r+1)$ is also divisible by $4 \cdot 6 = 24$

$\therefore$ $r(r^2-1)(3r+2) = 3(r-1)r(r+1)(r+2) - 4(r-1)r(r+1)$

is also divisible by 24.

Example 5 *If m, n are positive integers, show that* $(m+n)!$ *is divisible by* $m!\,n!$.

Solution

$$\frac{(m+n)!}{m!\,n!} = \frac{1 \cdot 2 \cdot 3 \ldots m(m+1)(m+2)\ldots(m+n)}{m!\,n!}$$

$$= \frac{m!(m+1)(m+2)\ldots(m+n)}{m!\,n!}$$

$$= \frac{(m+1)(m+2)\ldots(m+n)}{n!}$$

$$= \frac{\text{The product of } n \text{ consecutive integers}}{n!}$$

$$= \text{An integer}$$

$\therefore$ $(m+n)!$ is divisible by $m!\,n!$.

Example 6 *If* $(4x-y)$ *is a multiple of* 3, *show that* $4x^2+7xy-2y^2$ *is divisible by* 9.

Solution $\because$ $4x-y$ is a multiple of 3.

$\therefore$ $$4x - y = 3m$$

$\therefore$ $$y = 4x - 3m$$

On putting value of y in $4x^2+7xy-2y^2$

$$= 4x^2 + 7x(4x-3m) - 2(4x-3m)^2$$
$$= 4x^2 + 28x^2 - 21xm - 2(16x^2 + 9m^2 - 24xm)$$
$$= 4x^2 + 28x^2 - 21xm - 32x^2 - 18m^2 + 48xm$$
$$= 27mx - 18m^2 = 9m(3x-2m)$$

$\therefore$ $4x^2 - 7x - 2y^2$ is divisible by 9.

Example 7 *If* n *is an integer, prove that* $n(n+1)(2n+1)$ *is divisible by* 6.

Solution

$$n(n+1)(2n+1) = n(n+1)[(n+2)+(n-1)]$$
$$= n(n+1)(n+2) + n(n+1)(n-1)$$
$$= n(n+1)(n+2) + (n-1)n(n+1)$$

Each of the two $(n-1)n(n+1)$ and $(n+2)(n+1)n$ being the product of three consecutive integers is divisible by $3! = 6$

$\therefore$ $n(n+1)(2n+1)$ is also divisible by 6.

Example 8 *Prove that* 4 *does not divide* (m^2+2) *for any integer* m.

Solution $\because$ m is an integer.

$\therefore$ Either m is even or m is odd.

Case I m is even

So let $$m = 2k$$

$\therefore \quad m^2 + 2 = (2k)^2 + 2 = 4k^2 + 2 = 2(2k^2 + 1)$

$= (2 \times \text{an odd integer})$

Which is not divisible by 4.

Case II m is odd

Let $m = 2k + 1$

$\therefore \quad m^2 + 2 = (2k+1)^2 + 2 = 4k^2 + 1 + 4k + 2$

$= (2 + 4k + 4k^2) + 1.$

Which being an odd integer is not divisible by 4.

Note Two important formulae.

1. If n is either even or odd,

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + y^{n-1})$$

2. If n is odd,

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \dots + y^{n-1})$$

Example 9 *Prove that $8^n - 3^n$ is divisible by 5.*

Solution $8^n - 3^n = (8-3)(8^{n-1} + 8^{n-2}.3 + \dots + 3^{n-1})$

or $8^n - 3^n = 5(8^{n-1} + 8^{n-2}.3 \dots + 3^{n-1})$

$\therefore$ $8^n - 3^n$ is divisible by 5.

Example 10 *If $p > 1$ and $2^p - 1$ is prime, then prove that p is prime.*

Solution If possible, let p be not prime

$\therefore \quad p$ is composite. $(\because p > 1)$

$\therefore \quad p = mn$, where $m > 1$ and $n > 1$

$\therefore \quad 2^p - 1 = 2^{mn} - 1 = (2^m)^n - 1^n$

Putting $2^m = a = a^n - 1^n$ where $a = 2^m > 2$

$= (a-1)(a^{n-1} + a^{n-2} + \dots + 1^{n-1})$

Now, each of the two factors on RHS is greater than 1.

$\therefore \quad 2^p - 1$ is composite.

But it is contrary to given

$\therefore \quad p$ must be prime.

Remark

- But converse is not true *i.e.*, when p is prime, $2^p - 1$ need not be prime.
- For example, $p = 11$ is prime but $2^{11} - 1$ is divisible by 23 and hence is not prime.

Theorem 4 *The number of primes is infinite.*

Proof If possible suppose that the numbers of prime is finite.

$\therefore$ $\exists$ the greatest prime say q.

Let b denote the product of these primes 2, 3, 5, ..., q.

i.e., let $\quad b = 2 \cdot 3 \cdot 5 \dots q \quad$ ...(i)

let $\quad a = b + 1 \quad$ ...(ii)

Surely, $\quad a \neq 1 \quad (\because a = b + 1 > 1)$

$\therefore$ The number a must have a prime say factor p *i.e.*, $p \mid a$.

Now, p is one of the primes 2, 3, 5, 7 ... q (Because according to our assumption 2, 3, 5, 7, ... q are the only primes).

$\therefore$ $\quad p \mid b \qquad (\because b = 2.3.5 \ldots q)$

Now $\quad p \mid a$ and $p \mid b$

$\therefore$ $\quad p \mid a - b$ or $p \mid 1 \qquad [\because$ from Eq. (ii), $a - b = 1]$

$p = 1$ (which is impossible) $\qquad (\because 1$ is not prime)

So our supposition is false.

$\therefore$ The number of primes is infinite.

Theorem 5 ***Fundamental Theorem of Arithmetic*** *each natural number greater than 1 can be expressed as a product of primes in one and only one way (except for the order of the factors).*

Example 1 *Every natural number other than 1 admits of a prime factor.*

Solution Suppose that $n \neq 1$ is a natural number.

If n itself is a prime number, the example is proved in as much as the prime number n is a factor of itself.

If n is composite, then n must have factors other than 1 and n.

Let l be the least of these factors of n other than 1 and n.

i.e., $\quad 1 < l < n$ and $l \mid n$

Now, we have to prove, l is prime.

If possible let l be not prime.

But $\quad l > 1$

$\therefore$ l is composite

$\therefore$ $\quad \exists$ integers l_1 and l_2 such that

$l = l_1 l_2$ where $1 < l_1 < l$ and $1 < l_2 < l$

$\Rightarrow$ $\quad l_1 \mid l$ but $l \mid n$

$\therefore$ $\quad l_1 \mid n$ where $1 < l_1 < l < n$

i.e., $l_1 (< l)$ is a divisor of n other than 1 and n.

But this is a contradiction because $l_1 < l$ and l is the least divisor of n other than 1 and n.

$\therefore$ $\quad$ Our supposition is wrong.

$\therefore l$ is a prime factor of n.

Example 2 *Show that every odd prime can be put either in the form $4k + 1$ or $4k + 3$ (i.e., $4k - 1$), where k is a positive integer.*

Solution Let n be any odd prime. If we divide any n by 4, we get

$$n = 4k + r$$

where $\quad 0 \le r < 4$ *i.e.*, $r = 0, 1, 2, 3$

$\therefore$ Either $\quad n = 4k$ or $n = 4k + 1$

or $\quad n = 4k + 2$ or $n = 4k + 3$

Clearly, $4n$ is never prime and $4n + 2 = 2(2n + 1)$

cannot be prime unless $n = 0 \qquad (\because$ 4 and 2 can not be factors of an odd prime)

$\therefore$ An odd prime n is either of the form

$$4k+1 \text{ or } 4k+3.$$

But $\quad 4k+3 = 4(k+1) - 4 + 3 = 4k' - 1 \qquad$ (where $k' = k+1$)

$\therefore$ An odd prime n is either of the form $4k+1$ or $(4k+3)$ *i.e.*, $4k'-1$.

Note 1. Every number of the form $4k+3$ is of the form $4k-1$ and conversely.
2. Every number of the form $4k+1$ or $4k-1$ is not necessarily prime.
3. The above result should be committed to memory.

Example 3 *Show that there are infinitely many primes of the form* $4n+3$.

Solution If possible, let number of primes of form $(4n+3)$ be finite.

These primes are 3, 7, 11,..., q (put $n = 0, 1, 2, \ldots$)

Let q be the greatest of these primes of the form $(4n+3)$.

Let $a = 3, 7, 11, \ldots, q$ be the product of all primes of the form $(4n+3)$.

Let $\quad b = 4a - 1 \qquad$ ...(i)

$\Rightarrow \quad b > 1 \qquad [\because a \geq 3, \therefore b = 4a - 1, b \geq 11]$

$\therefore$ By fundamental theorem b can be expressed as a product of primes say $p_1 . p_2 . p_3 \ldots p_r$.

i.e., $\quad b = p_1 . p_2 \ldots p_r \qquad$ ...(ii)

Now, $b = 4a - 1$ is odd and hence 2 can't be a factor of b.

$\therefore$ None of the prime factors in RHS of Eq. (ii) is 2.

i.e., Every prime factor in RHS of Eq. (ii) is odd.

$\therefore$ Each of $p_1, p_2, \ldots, p_r$ is of the form $(4n+1)$ or $(4n+3)$.

Again, all $p_1, p_2, \ldots, p_r$ can't be of the form $(4n+1)$.

[$\because$ If it were so, then b (their product) will also be of the form $(4n+1)$].

But this is contrary to Eq. (i) as $b = 4a - 1$ is of the form $(4n+3)$.

$\therefore$ At least one of $p_1, p_2 \ldots p_r$ (say p) is (a prime factor of b) of the form $(4n+3)$ *i.e.*, $p|b$.

Also $p|a$ [$\because$ p is one prime of the form $(4n+3)$ and a is product of all such primes].

$\therefore \quad p|4a$

$\therefore \quad p|(4a-b)$

$\therefore \quad p|1 \qquad [\because$ from Eq. (i), $4a - b = 1]$

Which is impossible $\qquad [\because p$ being prime $> 1]$

$\therefore$ Our supposition is wrong.

$\therefore$ Number of primes of the form $(4n+3)$ is infinite.

Theorem 6 ***The number of divisors of a composite number n*** *: If n is a composite number of the order* $n = p_1^{\alpha_1} . p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, *then the number of divisors denoted by* $d(n)$ *is*

$$(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1)$$

Proof Let n be any composite number, let $d(n)$ denote the number of divisors of composite number n by Fundamental Theorem of Arithmetic. n can be expressed as the product of the powers of primes. $n = p_1^{\alpha_1} . p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2 \ldots, \alpha_k$ are non negative integers.

$\because p_1$ is a prime number, therefore, the only divisors of $p_1^{\alpha_1}$ are $1, p_1, p_1^2, p_1^3, \dots, p_1^{\alpha_1}$

The number of these divisors of $p_1^{\alpha_1} = \alpha_1 + 1$

Similarly, the number of divisors of $p_2^{\alpha_2} = \alpha_2 + 1$

The number of divisors of $p_k^{\alpha_k} = \alpha_k + 1$

Therefore, the total number of divisors of $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_k^{\alpha_k} = (\alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_k + 1)$

[$\because$ Every divisor of $p_i^{\alpha_i}$ $(1 \le i \le k)$ is a divisor of n]

i.e., $d(n) = (\alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_k + 1)$

Note Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3}$ [p_1, p_2, p_3 are distinct prime numbers]

Let $d(n)$ denotes number of divisor .

1. If n is a perfect square then $d(n)$ is odd

($\because$ all the d_i are even)

2. If n is not a perfect square then, $d(n)$ is even.

[The number of ways of writing n are the product of two factors.]

If n is a perfect square, then number of ways are equal to $\dfrac{d(n)+1}{2}$.

If n is not a perfect square, then number of ways are equal to $\dfrac{d(n)}{2}$.

Theorem 7 *The sum of the divisors of any composite number n is denoted by $\sigma(n)$ which is equal to*

$$\left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right)\dots\left(\frac{p_k^{\alpha_k+1}-1}{p_k-1}\right).$$

Proof Let n be any composite number and let $\sigma(n)$ is sum of positive divisors of n. By Fundamental Theorem of Arithmetic n can be expressed as the product of the powers of primes.

$$\therefore \qquad n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

[where $p_1, p_2, \dots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers]

$\because$ p_1 is prime number

$\therefore$ divisors of $p_1^{\alpha_1}$ are only $1, p_1, p_1^2, \dots, p_1^{\alpha_1}$

Sum of these divisors of $p_1^{\alpha_1} = 1 + p_1 + p_1^2 + \dots + p_1^{\alpha_1} = \dfrac{1[p_1^{\alpha+1} - 1]}{p-1}$

$\left[\because \text{ RHS is a Geometric Progression with } a = 1,\ r = p_1, n = \alpha_1 + 1 \text{ and } S_n = \dfrac{a(r^n - 1)}{r-1}\right]$

Similarly sum of divisors of $p_2^{\alpha_2} = \dfrac{(p_2^{\alpha_2+1} - 1)}{p_2 - 1}$

Sum of divisors of $p_k^{\alpha_k} = \dfrac{(p_k^{\alpha_k+1} - 1)}{p_k - 1}$

$$\therefore \qquad \sigma(n) = \text{Sum of divisors of } n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

$$= \frac{(p_1^{\alpha_1+1} - 1)}{(p_1 - 1)} \cdot \frac{(p_2^{\alpha_2+1} - 1)}{(p_2 - 1)} \dots \frac{(p_k^{\alpha_k+1} - 1)}{(p_k - 1)}.$$

[$\because$ Every divisor of $p_i^{\alpha_i}$ $(1 \le i \le k)$ is divisor of n]

Note If $d_k(n)$ denotes the sum of kth power of divisor of n, then

$$d_k(n) = \frac{(p_1^{k(\alpha_1+1)} - 1)}{(p_1^k - 1)} \cdot \frac{(p_2^{k(\alpha_2+1)} - 1)}{(p_2^k - 1)} \cdots \frac{(p_m^{k(\alpha_m+1)} - 1)}{(p_m^k - 1)}$$

Example 1 *Find the sum of the cubes of divisor of 12.*

Solution $12 = 2^2 \times 3$

$$d_k(12) = \frac{2^{3(2+1)} - 1}{2^3 - 1} \times \frac{3^{3(1+1)} - 1}{3^3 - 1} = 2044$$

Example 2 *Find the number of divisor of 600.*

Solution $600 = 2^3 \times 3^1 \times 5^2$

$$\alpha_1 = 3, \alpha_2 = 1, \alpha_3 = 2$$

$$\text{Number of divisors} = (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)$$

$$= (3+1)(1+1)(2+1) = 4 \times 2 \times 3 = 24$$

Greatest Integer Function

Greatest integer function is also known as *Bracket Function.*

If x is any real number, then the largest integer which does not exceed x is called the ***integral part of*** x and will be denoted by $[x]$.

The function which associates with each real number x, the integer $[x]$ is often called the bracket function.

For example, $[3] = 3, [-4] = -4, [3.7] = 3$

$$[-4.2] = -5, \left[\frac{5}{3}\right] = 1, [-\pi] = -4$$

Note 1. $[x]$ is the largest integer $\le x$.

2. If a and b are positive integer, such that

$$a = qb + r, 0 \le r < b$$

Then, $\frac{a}{b} = q + \frac{r}{b}$, where $0 < \frac{r}{b} < 1$

$$\therefore \qquad \left[\frac{a}{b}\right] = q$$

i.e., $\left[\frac{a}{b}\right]$ is the quotient in the division of a by b.

Properties of Greatest Integer Function

(1) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x, 0 \le x, 0 \le x - [x] < 1$

(2) If $x \ge 0, [x] = \sum_{1 \le i \le x} 1$

(3) $[x + m] = [x] + m$, If m is an integer.

(4) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(5) $[x] + [-x] = 0$, If x is an integer $= -1$ otherwise.

(6) $\left[\frac{[x]}{m}\right] = \left[\frac{x}{m}\right]$, if m is a positive integer.

(7) $-[-x]$ is the least integer greater than or equal to x.

This is denoted as (x) (read as ceiling x)

For example, $(2.5) = 3$, $(-2.5) = -2$

(8) $[x + 0.5]$ is the nearest integer to x.

If x is midway between two integers, $[x + 0.5]$ represents the larger of the two integers.

(9) The number of positive integers less than or equal to n and divisible by m is given by $\left[\frac{n}{m}\right]$.

(10) If p is a prime number and e is the largest exponent of p such that

$$p^e | n!, \text{ then } e = \sum_{i=1}^{\infty} \left[\frac{n}{p_i}\right]$$

Theorem 1 *If a is real number, c is natural number, then* $\left[\frac{[a]}{c}\right] = \left[\frac{a}{c}\right]$

Proof Let $[a] = n$ i.e., n is largest integer $\leq a$

$\therefore \quad a = n + r, \; 0 \leq r < 1$ ...(i)

Let $\quad \left[\frac{[a]}{c}\right] = \left[\frac{n}{c}\right] = m$

$\therefore \quad \frac{n}{c} = m + s$, where $0 \leq s < 1$

$\therefore \quad n = mc + cs$, where $0 \leq cs < c$ ...(ii)

Now, $\quad \text{LHS} = -\left[\frac{[a]}{c}\right] = \left[\frac{n}{c}\right] = m$

$\text{RHS} = \left[\frac{a}{c}\right] = \left[\frac{n+r}{c}\right]$ [Putting value of a from Eq. (i)]

Putting the value of n from Eq. (ii).

$$= \left[\frac{mc + cs + r}{c}\right]$$

$$= \left[m + \frac{cs + r}{c}\right]$$

From Eq. (ii), $cs \leq (c - 1)$ and $r < 1$

Adding $cs + r < c - 1 + 1$

$\Rightarrow \quad cs + r < c$

$\therefore \quad \frac{cs + r}{c} < 1$

$\therefore \quad \text{RHS} = \left[m + \frac{cs + r}{c}\right] = m$

$\therefore \quad \text{LHS} = \text{RHS}$

Theorem 2 *For every positive real number*

$$\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]=[x]$$

Proof First suppose that $x = 2m + y$, where m is an integer and $0 \le y < 1$.

$$[x]=2m,\left[\frac{x}{2}\right]=m$$

$$\left[\frac{x+1}{2}\right]=\left[\frac{2m+1+y}{2}\right]=m$$

Since $$\frac{1}{2}\le\frac{1+y}{2}<1$$

$\therefore$ $$\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]=[x]$$

Next, let $x = (2m + 1) + y$, where m is an integer and $0 \le y < 1$.

Then, $$\left[\frac{x}{2}\right]=m,\left[\frac{x+1}{2}\right]=\left[\frac{(2m+2)+y}{2}\right]=m+1$$

$$[x]=2m+1$$

$\therefore$ $$\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]=[x]$$

The desired equality holds for all possible values of x.

Example 1 *Find the highest power of* 3 *contained in* 1000!.

Solution $p = 3, n = 1000$

$$\left[\frac{n}{p}\right]=\left[\frac{1000}{3}\right]=\left[333\frac{1}{3}\right]=333$$

$$\left[\frac{n}{p^2}\right]=\left[\frac{333}{3}\right]=[111]=111$$

$$\left[\frac{n}{p^3}\right]=\left[\frac{111}{3}\right]=[37]=37$$

$$\left[\frac{n}{p^4}\right]=\left[\frac{37}{3}\right]=\left[12\frac{1}{3}\right]=12$$

$$\left[\frac{n}{p^5}\right]=\left[\frac{12}{3}\right]=[4]=4$$

$$\left[\frac{n}{p^6}\right]=\left[\frac{4}{3}\right]=\left[1\frac{1}{3}\right]=1$$

$$\left[\frac{n}{p^7}\right]=\left[\frac{1}{3}\right]=0$$

$\therefore$ Highest power of 3 contained in 1000!

$$=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\left[\frac{n}{p^4}\right]+\left[\frac{n}{p^5}\right]+\left[\frac{n}{p^6}\right]+\left[\frac{n}{p^7}\right]$$

$$= 333 + 111 + 37 + 12 + 4 + 1 + 0 = 498$$

Theorem 3 *If n and k are positive integers and k is greater than 1, then* $\left[\frac{n}{k}\right]+\left[\frac{n+1}{k}\right]\le\left[\frac{2n}{k}\right]$

Proof Let $n = qk + r$, q and r are integers and

$$0 \le r \le k-1$$

Then,
$$\frac{n}{k}=q+\frac{r}{k},\ \frac{n+1}{k}=q+\frac{r+1}{k},$$
$$\frac{2n}{k}=2q+\frac{2r}{k}$$

(i) $r < k-1$, then $\left[\frac{n}{k}\right]=q,\ \left[\frac{n+1}{k}\right]=q,\ \left[\frac{2n}{k}\right]\ge 2q$

The desired result is immediate.

(ii) $r = k-1$, then

$$\left[\frac{n}{k}\right]=q,\ \left[\frac{n+1}{k}\right]=q+1,$$
$$\left[\frac{2n}{k}\right]=2q+\left[\frac{2(k-1)}{k}\right]=2q+1$$

$\therefore$
$$\left[\frac{n}{k}\right]+\left[\frac{n+1}{k}\right]=\left[\frac{2n}{k}\right]$$

From which the desired result is immediate.

Theorem 4 *If n be any positive integer, then show that*

$$\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\left[\frac{n+8}{16}\right]+\ldots=n$$

Proof We know that,

$$[x]=\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]$$

Applying above formula to $n, \frac{n}{2}, \frac{n}{4}, \frac{n}{8}, \frac{n}{16}, \ldots$

$$[n]=\left[\frac{n}{2}\right]+\left[\frac{n+1}{2}\right]$$
$$\left[\frac{n}{2}\right]=\left[\frac{n}{4}\right]+\left[\frac{(n/2)+1}{2}\right]$$
$$\left[\frac{n}{4}\right]=\left[\frac{n}{8}\right]+\left[\frac{(n/4)+1}{2}\right]$$
$$\left[\frac{n}{8}\right]=\left[\frac{n}{16}\right]+\left[\frac{(n/8)+1}{2}\right]$$

Adding corresponding sides and cancelling out the terms $\left[\frac{n}{2}\right],\left[\frac{n}{4}\right],\left[\frac{n}{8}\right],\ldots$ from both sides, we have

$$n=\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\ldots$$

$\because [n]=n$

Theorem 5 *For every real number x*

$$[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx]$$

Proof Let $x=[x]+y$, where $0\le y<1$

Let p be an integer such that

$$p-1\le ny<p$$

(This is always possible because given a real number, we can always find two consecutive integers between which the number lies).

Now, $$x+\frac{k}{n}=[x]+y+\frac{k}{n}$$

Also, $y+\frac{k}{n}$ lies between $\frac{p-1+k}{n}$ and $\frac{p+k}{n}$

So long as $$\frac{p-1+k}{n}<1,$$

i.e., $$k<n-(p-1)$$

$y+\frac{k}{n}$ is less than 1 and consequently

$$\left[x+\frac{k}{n}\right]=[x]$$

i.e., $$\left[x+\frac{k}{n}\right]=[x]\quad\text{for } k=0,1,\ldots,n-p$$

But $\left[x+\frac{k}{n}\right]=[x]+1$, for $k=n-p+1,\ldots,n-1$

$\therefore$ $$[x]+\left[x+\frac{1}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]$$

$$=[x]+\ldots+[x](n-p+1\text{ times})+([x]+1)+([x]+1)+\ldots(p-1)\text{ times}$$

$$=n[x]+(p-1)\qquad\ldots\text{(i)}$$

Also, $$[nx]=[n[x]+ny]=n[x]+(p-1)$$

Since $$p-1\le ny<p\qquad\ldots\text{(ii)}$$

From Eqs. (i) and (ii)

$$[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx]$$

Theorem 6 *The highest power of a prime number p contained in n! is given by*

$$k(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots$$

Proof Let $k(n!)$ denote the highest power of p contained in $n!$ $n!$ is the product of the factors $1, 2, 3, \ldots, n$. The factors in $n!$ which will be divisible by p are

$$p, 2p, 3p, \ldots \left[\frac{n}{p}\right]p$$

$\therefore$ $$k(n!)=\left[\frac{n}{p}\right]+k\left(\left[\frac{n}{p}\right]!\right)\qquad\ldots\text{(i)}$$

Changing n to $\frac{n}{p}$ in Eq. (i)

$$k\left(\left[\frac{n}{p}!\right]\right)=\left[\frac{n}{p^2}\right]+k\left(\left[\frac{n}{p^2}\right]!\right) \quad \ldots \text{(ii)}$$

Putting the value from Eq. (ii) in Eq. (i)

$$k(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+k\left(\left[\frac{n}{p^2}\right]!\right)$$

Continuing this process, we get $\quad k(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots$

This process must end after a finite number of steps.

Congruences

If a and b are two integers and m is a positive integer, then a is said to be congruent to b modulo m, if m divides $a-b$ denoted by $m \mid (a-b)$.

In notation form we express it as $a \equiv b \bmod m$ or $a - b \equiv 0 \bmod m$.

Note 1. $a \equiv b \bmod m$, then $m \mid (a-b)$ or $(a-b)$ is a multiple of m.
2. If $m \nmid (a-b)$ [m does not divides $(a-b)$], then a is said to be incongruent to b mod m and this fact is expressed as a is not congruent to b mod m.
3. If $m \mid a$, then $a \equiv 0 \bmod m$

For example :
(i) $13 \equiv 1 \bmod 4$ $\quad (\because 4 \mid (13-1)=12)$
(ii) $4 \equiv -1 \bmod 5$ $\quad (\because 5 \mid (4-(-1))=5)$
(iii) $12 \equiv 0 \bmod 4$ $\quad (\because 4 \mid 12)$
(iv) 17 is not congruent to 3 mod 5 $\quad (\because 5 \nmid (17-3))$

Theorem 7 *If $a \equiv b$ mod m, then*

(i) $a + c = b + c$ mod m
(ii) $ac \equiv bc$ mod m, where c is any integer.

Proof (i) $\because a \equiv b \bmod m$

$$m \mid (a-b)$$

$\Rightarrow$
$$m \mid \{(a+c)-(b+c)\} \quad [\because a+c-(b+c)=a-b]$$
$$a+c \equiv b+c \bmod m$$

(ii) $\because a \equiv b \bmod m$

$$m \mid (a-b)$$
$$m \mid c(a-b)$$
$$m \mid (ac-bc)$$

$\therefore$
$$ac \equiv bc \bmod m$$

Note The converse of theorem 15 (ii) is not true.
Theorem states that if $a \equiv b \bmod m$, then $ac \equiv bc \bmod m$.
i.e., a congruence can always be multiplied by an integer
But the converse is not true *i.e.*,. It is not always possible to cancel a common factor from a congruence.
For example : $\quad 16 \equiv 8 \bmod 4 \quad [\because 4 \mid 16-8]$
But if we cancel the common factor 8 from numbers 16 and 8, we get $2 \equiv 1 \bmod 4$ which is a false result because $4 \mid (2-1)$

Theorem 8 *If $a \equiv b \bmod m$ and $c \equiv d \bmod m$, then*
(i) $a + c \equiv b + d \bmod m$ (ii) $a - c \equiv b - d \bmod m$ (iii) $ac \equiv bd \bmod m$

Proof (i) $\because a \equiv b \bmod m$ and $c \equiv d \bmod m$

$\therefore$ $m \mid (a - b)$ and $m \mid (c - d)$

$m \mid ((a - b) + (c - d))$

or $m \mid ((a + c) - (b + d))$

$\therefore$ $a + c \equiv b + d \bmod m$

(ii) $a \equiv b \bmod m$ and $c \equiv d \bmod m$

$\therefore$ $m \mid (a - b)$ and $m \mid (c - d)$

$\therefore$ $m \mid (a - b) - (c - d)$ or $m \mid (a - c) - (b - d)$

$a - c \equiv b - d \bmod m$

(iii) $\because a \equiv b \bmod m$ and $c \equiv d \bmod m$

$m \mid (a - b)$ and $m \mid (c - d)$

$\therefore$ There exists integer h and k such that

$a - b = mh$ and $c - d = mk$

$a = b + mh$ and $c = d + mk$

Multiplying the two equations, we get

$$ac = (b + mh)(d + mk) = bd + mbk + mhd + m^2hk$$

$$ac - bd = m(bk + hd + mhk)$$

$\therefore$ By definition of divisibility $m \mid (ac - bd)$

or $ac \equiv bd \bmod m$

Corollary *If $a \equiv b \bmod m$, then $a^2 \equiv b^2 \bmod m$*

$\because$ $a \equiv b \bmod m$ and again

$a \equiv b \bmod m$

Multiplying the two congruence $a^2 \equiv b^2 \bmod m$

Theorem 9 *(i) Prove that $a \equiv a \bmod m$ i.e., every integer is congruent to itself.*
(ii) If $a \equiv b \bmod m$, then prove that $b \equiv a \bmod m$
(iii) If $a \equiv b \bmod m$, $b \equiv c \bmod m$ prove that $a \equiv c \bmod m$

Proof (i) We know that $m/0$ $(m \neq 0)$

$m \mid (a - a)$

$a \equiv a \bmod m$ [by definition of congruence]

(ii) Let $a \equiv b \bmod m$

$m \mid (a - b)$

$\therefore$ $m \mid -(a - b)$ or $m \mid (b - a)$

$\therefore$ $b \equiv a \bmod m$

(iii) Let $a \equiv b \bmod m$ and $b \equiv c \bmod m$

$\therefore$ $m \mid (a - b)$ and $m \mid (b - c)$

$\therefore$ $m \mid (a - b) + (b - c)$ or $m \mid (a - c)$

$\therefore$ $a \equiv c \bmod m$

Theorem 10 *If $a \equiv b \bmod m$, then $a^k \equiv b^k \bmod m$ for every positive integer k.*

Proof We know that,

$$a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + a^{k-3}b^2 + \ldots + b^{k-1}) \qquad \ldots\text{(i)}$$

But $a \equiv b \bmod m \Rightarrow m \mid (a-b)$

$\therefore$ There exists an integer t such that

$$a - b = mt \qquad \ldots\text{(ii)}$$

Putting this value of $(a-b)$ from Eq. (ii) in Eq. (i)

$$a^k - b^k = mt(a^{k-1} + a^{k-2}b + \ldots b^{k-1})$$

$$\therefore \quad m \mid (a^k - b^k)$$

$$\therefore \quad a^k \equiv b^k \bmod m$$

Theorem 11 *If $a \equiv b \bmod m$ and $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$ is an integral rational function of an indeterminate x with integral coefficients, then $f(a) \equiv f(b) \bmod m$*

Proof $\because f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$

Putting $x = a$

$$\therefore \quad f(a) = p_0a^n + p_1a^{n-1} + p_2a^{n-2} + \ldots + p_{n-1}a + p_n \qquad \ldots\text{(i)}$$

Putting $x = b$

$$\therefore \quad f(b) = p_0b^n + p_1b^{n-1} + p_2b^{n-2} + \ldots + p_{n-1}b + p_n \qquad \ldots\text{(ii)}$$

Subtract Eq. (i) from Eq. (ii), we get

$$f(a) - f(b) = p_0(a^n - b^n) + p_1(a^{n-1} - b^{n-1}) + \ldots + p_{n-1}(a-b)$$

$$= p_0(a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1}) +$$

$$p_1(a-b)(a^{n-2} + a^{n-3}b + \ldots + b^{n-2}) + \ldots + p_{n-1}(a-b)$$

or

$$f(a) - f(b) = (a-b)[p_0(a^{n-1} + a^{n-2}b + \ldots + b^{n-2}) + p_1(a^{n-2} + a^{n-3}b + \ldots + b^{n-2}) + \ldots + p_{n-1}]$$

$$= (a-b)t \text{ (say)} \qquad \ldots\text{(iii)}$$

But $a \equiv b \bmod m$ (given)

$$m \mid (a-b)$$

$\therefore$ $\exists$ an integer k such that $a - b = mk$

Putting this value of $a - b = mk$ in Eq. (iii)

$$f(a) - f(b) = mkt$$

$$\therefore \quad m \mid f(a) - f(b)$$

$$\Rightarrow \quad f(a) \equiv f(b) \bmod m$$

Theorem 12 ***Fermat Theorem***

If p is prime, then

$$(a+b)^p = (a^p + b^p) \bmod p.$$

Proof Expanding by binomial theorem

$$(a+b)^p = a^p + {}^pC_1a^{p-1}b + {}^pC_2a^{p-2}b^2 + \ldots + {}^pC_{p-1}ab^{p-1} + b^p$$

or

$$(a+b)^p = (a^p + b^p) + \sum_{r=1}^{p-1} {}^pC_r a^{p-r}b^r \qquad \ldots\text{(i)}$$

Now, $${}^pC_r = \frac{p!}{r!(p-r)!}; \quad 1 \le r \le (p-1)$$

But $p! = 1.2.3 \ldots p$ is divisible by p

p is coprime to $r!$

$\because$ p is coprime to 1, 2, 3, ... r $\quad (\because r < p,\ p$ is prime$)$

$\therefore$ p is co prime to their product $= r!$

Also for the same reason p is coprime to $(p-r)!$

$$\therefore \quad {}^pC_r = \frac{p!}{r!(p-r)!} \text{ is divisible by } p.$$

$\therefore$ $\exists$ an integer k_r such that

$${}^pC_r = pk_r$$

Putting this value of ${}^pC_r$ in Eq. (i)

$$(a+b)^p - (a^p + b^p) = p \sum_{r=1}^{p-1} k_r a^{p-r} b^r$$

which is divisible by p.

$$\therefore \quad (a+b)^p \equiv (a^p + b^p) \bmod p$$

Generalization

If p is a prime number, prove that

$$(a_1 + a_2 + a_3 + \ldots + a_n)^p$$
$$\equiv (a_1^p + a_2^p + a_3^p + \ldots + a_n^p) \bmod p$$
$$(a_1 + a_2 + a_3 + \ldots + a_n)^p = (a_1 + b_1)^p$$

where $$b_1 = a_2 + a_3 + \ldots + a_n$$
$$\equiv (a_1^p + b_1^p) \bmod p.$$
$$\equiv [a_1^p + (a_2 + a_3 + \ldots + a_n)^p] \bmod p$$
$$\equiv [a_1^p + (a_2 + c_2)^p] \bmod p$$

where $$c_2 = a_3 + a_4 + \ldots + a_n \equiv (a_1^p + a_2^p + c_2^p) \bmod p$$

continuing like this, we get

$$(a_1 + a_2 + a_3 + \ldots + a_n)^p \equiv (a_1^p + a_2^p + \ldots + a_n^p) \bmod p$$

Theorem 13 *If p prime number, then*

$$a^p = a \bmod p$$

Proof We know that,

$$(a_1 + a_2 + a_3 + \ldots + a_n)^p \equiv (a_1^p + a_2^p + a_3^p \ldots + a_n^p) \bmod p \quad \ldots(i)$$

Putting $$a_1 = a_2 = a_3 = \ldots = a_n = 1 \text{ in Eq. (i)}$$
$$(1 + 1 + 1 + \ldots + 1)^p$$
$$\equiv (1^p + 1^p + 1^p + \ldots + 1^p) \bmod p$$

or $$n^p \equiv (1 + 1 + 1 + \ldots + 1) \bmod p$$

or $n^p \equiv n \bmod p$ for every natural number n.

Replacing n by a.

$$a^p = a \bmod p$$

Theorem 14 *Fermat Little Theorem*

If p is a prime number and $(a, p) \equiv 1$, prove that $a^{p-1} = 1 \bmod p$.

Proof As p is prime.

$$\therefore \quad a^p = a \bmod p$$

Cancelling a from both sides. [$\because a$ is coprime to p]

We have $a^{p-1} \equiv 1 \bmod p$.

Theorem 15 $n! = n^n - {}^nC_1(n-1)^n + {}^nC_2(n-2)^n - \ldots + (-1)^{n-2}\,{}^nC_{n-2}2^n + (-1)^{n-1}\,{}^nC_{n-1}$

Proof Expanding by binomial theorem

$$(e^x - 1)^n = e^{nx} - {}^nC_1e^{(n-1)x} + {}^nC_2e^{(n-2)x} - \ldots + {}^nC_{n-2}(-1)^{n-2}e^{2x} + {}^nC_{n-1}(-1)^{n-1}e^x + (-1)^n \quad \ldots\text{(i)}$$

We know that
$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots$$

Using this expansion of e^θ, Eq. (i) becomes

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots - 1\right)^n$$
$$= \left(1 + \frac{nx}{1!} + \frac{(nx)^2}{2!} + \ldots + \frac{(nx)^n}{n!} + \ldots\right) - {}^nC_1\left[1 + \frac{(n-1)x}{1!} + \frac{[(n-1)x]^2}{2!} + \ldots + \frac{[(n-1)x]^n}{n!} + \ldots\right]$$
$$+ {}^nC_2\left[1 + \frac{(n-2)x}{1!} + \frac{[(n-2)x]^2}{2!} + \ldots + \frac{[(n-2)x]^n}{n!} + \ldots\right]$$
$$+ \ldots + {}^nC_{n-2}(-1)^{n-2}\left[1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \ldots + \frac{(2x)^n}{n} + \ldots\right]$$
$$+ (-1)^{n-1}\,{}^nC_{n-1}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\right) + (-1)^n$$

Comparing coefficient of x^n on both sides

$$1 = \frac{n^n}{n!} - {}^nC_1\frac{(n-1)^n}{n!} + {}^nC_2\frac{(n-2)^n}{n!} - \ldots +$$
$$(-1)^{n-2}\,{}^nC_{n-2}\frac{2^n}{n!} + \frac{(-1)^{n-1}\,{}^nC_{n-1}}{n!}$$

Multiplying both sides by $n!$

$$n! = n^n - {}^nC_1(n-1)^n + {}^nC_2(n-2)^n - \ldots + (-1)^{n-2}\,{}^nC_{n-2}2^n + (-1)^{n-1}\,{}^nC_{n-1}$$

Theorem 16 *Wilson Theorem*

If p is prime, then $(p-1)! + 1 \equiv 0 \bmod p$

Proof

Case I when $p = 2$

$$(2-1)! + 1 \equiv 0 \bmod 2 \qquad [\text{Putting } p = 2 \text{ in } (p-1)! + 1 \equiv 0 \bmod p\,]$$
$$\Rightarrow \quad 1! + 1 \equiv 0 \bmod 2$$
$$\Rightarrow \quad 2 \equiv 0 \bmod 2$$

which is true

$\therefore$ Wilson theorem is true for $p = 2$.

Case II If p is an odd prime.

$$n! = n^n - {}^nC_1(n-1)^n + {}^nC_2(n-2)^n - \dots + (-1)^{n-2}\,{}^nC_{n-2}2^n + (-1)^{n-1}\,{}^nC_{n-1}$$

Put $n = p - 1$ on both sides

$\therefore$
$$(p-1)! = (p-1)^{p-1} - {}^{p-1}C_1(p-2)^{p-1} + {}^{p-1}C_2(p-3)^{p-1} + \dots + (-1)^{p-3}\,{}^{p-1}C_{p-3}2^{p-1} + (-1)^{p-2}\,{}^{p-1}C_{p-2} \quad \dots\text{(i)}$$

$\because$ p is prime.

$\therefore$ p is coprime to all numbers $< p$.

i.e., p is coprime to $p-1, p-2, p-3, \dots 2, 1$.

$\therefore$ Putting $a = p-1, p-2, p-3, \dots 2$ in Fermat Theorem

$$a^{p-1} \equiv \text{mod } p$$

$$(p-1)^{p-1} \equiv 1 \text{ mod } p$$

or
$$(p-1)^{p-1} - 1 = M(p)$$

$$(p-1)^{p-1} = M(p) + 1$$

Similarly,
$$(p-2)^{p-1} = M(p) + 1$$

$$(p-3)^{p-1} = M(p) + 1$$

....

....

....

$$2^{p-1} = M(p) + 1$$

Putting these values of $(p-1)^{p-1}, (p-2)^{p-1}, \dots, 2^{p-1}$ in Eq. (i).

$\therefore$
$$(p-1)! = [M(p)+1] - {}^{p-1}C_1[M(p)+1] + {}^{p-1}C_2[M(p)+1] + \dots + (-1)^{p-3}\,{}^{p-1}C_{p-3}[M(p)+1] + (-1)^{p-2}\,{}^{p-1}C_{p-2}$$

or
$$(p-1)! = M(p) + 1 - {}^{p-1}C_1 + {}^{p-1}C_2 - \dots + (-1)^{p-3}\,{}^{p-1}C_{p-3} + (-1)^{p-2}\,{}^{p-1}C_{p-2}$$

Adding and Subtracting $(-1)^{p-1}$ in RHS.

$$(p-1)! = M(p) + [(1)^{p-1} - {}^{p-1}C_1 + {}^{p-1}C_2 - {}^{p-1}C_3 + \dots + (-1)^{p-3}\,{}^{p-3}C_{p-3} + (-1)^{p-2}\,{}^{p-1}C_{p-2} + (-1)^{p-1}] - (-1)^{p-1}$$

$$(p-1)! = M(p) + (1-1)^{p-1} - (-1)^{p-1}$$

$\therefore$
$$(p-1)! = M(p) + 0 - (-1)^{p-1} = M(p) - 1$$

$\because$ p is odd. $\therefore p-1$ is even.

$\therefore$
$$(-1)^{p-1} = 1$$

or
$$(p-1)! + 1 = M(p)$$

$(p-1)! + 1$ is divisible by p.

$\therefore$
$$(p-1)! + 1 \equiv 0 \text{ mod } p.$$

Theorem 17 ***Converse of Wilson Theorem***

If $p > 1$ and $(p-1)! + 1 \equiv 0$ mod p, then p is a prime number.

Proof If possible let p be not prime.

$\therefore$ p is composite ($\because p > 1$)

So let $p = p_1p_2$, where $(1 < p_1 < p, 1 < p_2 < p)$ or $1 < p_1 \le p-1, 1 < p_2 \le p-1$

Now, $1 < p_1 \le p - 1$

$\therefore p_1$ is one of the factors in the value of $(p-1)!$ and therefore p_1 divides $(p-1)!$.

Also $$p = p_1 p_2 \qquad \text{...(i)}$$

$\Rightarrow$ $$p_1 \mid p \qquad \text{...(ii)}$$

But $(p-1)! + 1 \equiv 0 \bmod p$

$\therefore$ $$p \mid (p-1)! + 1 \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii)

$$p_1 \mid (p-1)! + 1 \qquad \text{...(iv)}$$

From Eqs. (iv) and (i)

$$p_1 \mid (p-1)! + 1 - (p-1)!$$

i.e., $$p_1 \mid 1$$

But this is impossible. ($\because p_1 > 1$)

$\therefore$ p is a prime number.

Euler's Function

Definition : The number of integers $\le n$ and coprime to n is called *Euler's function* for n and is denoted by $\phi(n)$.

Examples

$$\phi(1) = 1$$

[$\because$ 1 is the only integer ≤ 1 and coprime to 1].

$$\phi(2) = 1$$

[$\because$ 1 is the only integer < 2 and coprime to 2].

$$\phi(8) = 4$$

[$\because$ 1, 3, 5, 7 are the only four integers < 8 and coprime to 8].

Remark

- If p is a prime number, then $1, 2, 3, \ldots (p-1)$ are all less than p and coprime to p and are $(p-1)$ in total.

$\therefore$ $$\phi(p) = p - 1$$

Theorem 18 *Prove that* $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right)$ *where* $p_1, p_2, \ldots, p_r$ *are distinct prime factors of n.*

Proof $\because$ $p_1, p_2, \ldots, p_r$ are distinct prime factors of n.

$\therefore$ $$n = p_1^{k_1} . p_2^{k_2} \ldots p_r^{k_r}$$

$\therefore$ $$\phi(n) = \phi(p_1^{k_1} . p_2^{k_2} \ldots p_r^{k_r}) = \phi(p_1^{k_1})\phi(p_2^{k_2})\ldots\phi(p_r^{k_r})$$

[$\because$ $p_1, p_2, \ldots, p_r$ are distinct primes and hence are coprime to each other and $\phi(ab) = \phi(a)\phi(b)$, if a and b are coprime to each other.]

$$= p_1^{k_1}\left(1 - \frac{1}{p_1}\right)p_2^{k_2}\left(1 - \frac{1}{p_2}\right)\ldots p_r^{k_r}\left(1 - \frac{1}{p_r}\right)$$

$$= p_1^{k_1} . p_2^{k_2} \ldots p_r^{k_r}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right)$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right) \qquad [\because n = p_1^{k_1} . p_2^{k_2} \ldots p_r^{k_r}]$$

Theorem 19 *Prove that* $\phi(p^k) = p^k\left(1 - \frac{1}{p}\right)$, where p is prime.

Proof Number of integers from 1 to p^k which are not coprime to p^k are $p.1, p.2, p.3, \ldots p.p^{k-1}$.
Total number of such integers, which are not coprime to $p^k = p^{k-1}$.

$$\therefore \quad \phi(p^k) = \text{Number of integers coprime to } p^k \text{ and} < p^k.$$
$$= p^k - p^{k-1} = p^k(1 - 1/p)$$

Remark

If a and b are coprime to each other, then $\phi(ab) = \phi(a)\phi(b)$.

Example 1 *Find the number of positive integers* ≤ 3600 *that coprime to* 3600.

Solution $n = 3600 = 2^4 \times 3^2 \times 5^2$

$$\phi(n) = \phi(3600) = \phi(2^4 \times 3^2 \times 5^2)$$
$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) = 3600\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

[Here $p_1 = 2, p_2 = 3, p_3 = 5$]

$$= 3600 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5}$$
$$\therefore \quad \phi(3600) = 960$$

Example 2 *If* $m > 2$, *show that* $\phi(m)$ *is even.*

Solution If $(a, m) = 1$, then $(m - a, m) = 1$

$\therefore$ Integers coprime to m occur in pairs of type a and $m - a$.

$\therefore$ $\phi(m)$ is even.

Example 3 *For what values of* m *is* $\phi(m)$ *odd.*

Solution If $m > 2$, $\phi(m)$ is even.

$$\phi(1) = 1$$
$$\phi(2) = 1$$

Only for $m = 1, m = 2$

$\phi(m)$ is odd.

Concept Let $a = \frac{10^n - 1}{10 - 1} = \frac{10^n - 1}{9}$

We can express any a of the form $\frac{10^n - 1}{9}$ in terms of perfect square.

$$a = \frac{10^n - 1}{9} \Rightarrow 9a = 10^n - 1$$
$$9a + 1 = 10^n$$

Let
$$b = 9a + 1$$
$$c = 8a + 1$$

Now, consider $4ab + c = 4a(9a + 1) + 8a + 1 = 36a^2 + 12a + 1 = (6a + 1)^2$

Verification $(6 \times 1 + 1)^2 = 7^2 = 49 = (6 \times 11 + 1)^2 = 67^2$

$$(6 \times 111 + 1)^2 = 667^2$$

Now, consider

$$(a-1)b + c = (a-1)(9a+1) + 8a + 1$$
$$= 9a^2 + a - 9a - 1 + 8a + 1 = 9a^2 = (3a)^2$$

Verification $a = 1 \Rightarrow 3^2$

$$a = 11 \Rightarrow (33)^2$$

Now, consider $(16ab + c)$

$$16a(9a+1) + 8a + 1$$
$$(12a+1)^2$$

This is also a perfect square.

Concept Prove that every number of the sequence 49, 4489, 44489, 4448889 is a perfect square.
If there are n fours and $(n-1)$ eight and one 9.
Let us denote 444889 as $4_3 8_2 9$.
Consider 667 written as $6_2 7$
We know $444889 = (667)^2$.
$\therefore$ We develop $(6n_{m-1}7)^2 = 4_n 8_{n-1} 9$
If this is true then

$$(6_{n-1}7)^2 = \left[\frac{6(10^n - 1)}{9} + 1\right]^2 = \left[\frac{6 \times 10^n + 3}{9}\right]^2 = \left[\frac{2.10^n + 1}{9}\right]^2$$
$$= \frac{4.10^{2n} + 4.10^n + 1}{9} = \frac{40_{n-1}40_{n-1} + 1}{9} = 4_n 8_{n-1} 9$$

Example 1 *Let n be the natural number. If $2n+1$ and $3n+1$ are perfect square. Then prove that n is divided by 40.*

Solution $40 = 2^3 \times 5$. It is sufficient to prove that n is divisible by 8 and 5.

Let $\quad 2n + 1 = x^2 \quad$ ...(i)

and $\quad 3n + 1 = y^2 \quad$ ...(ii)

$\Rightarrow x^2$ is odd.

$\Rightarrow x$ is odd.

Let $\quad x = 2a + 1$

$$(2n+1) = (2a+1)^2$$
$$2n + 1 = 4a^2 + 4a + 1$$
$$n = 2a^2 + a$$

$\Rightarrow n$ is even.

If n is even $\Rightarrow 3n+1$ is odd

$\Rightarrow \quad y^2$ is odd $\Rightarrow y$ is odd

Let $\quad y = 2b + 1$

Subtract Eq. (i) from Eq. (ii), we get

$$n = y^2 - x^2 \qquad \text{...(iii)}$$

$\Rightarrow \quad n = (2b+1)^2 - (2a+1)^2$

We know square difference of odd number is always divisible by 8.

$\therefore \quad n$ is divisible by 8. ...(iv)

If we eliminate n between 1 and 2

$$3x^2 - 2y^2 = 1$$

Since square of odd number ends with 1, 5 or 9·

$\therefore \quad 3x^2$ ends with 3, 5 or 4, 7

$\Rightarrow \quad 2y^2$ ends with 2, 0, 8

$\Rightarrow \quad x^2$ ends with 1 and y^2 ends with 1

$\Rightarrow \quad n = y^2 - x^2$ [from Eq. (iii)]

$\quad = 0$

$\therefore$ It is divisible by 5.

Example 2 *Prove that there are infinitely many squares in the sequence* 1, 3, 6, 10, 15, 21, 28,...

Solution Suppose T_n is a square

Let T_n of the above sequence be $\frac{n(n+1)}{2}$

$$\Rightarrow \quad T_n = \frac{n(n+1)}{2}$$

If it is a square then $T_n = (m)^2$

$$\Rightarrow \quad \frac{n(n+1)}{2} = (m)^2$$

$$\Rightarrow \quad n(n+1) = 2(m)^2$$

Also, $T_{4n(n+1)}$ is also a square.

$$\therefore \quad T_{4n(n+1)} = \frac{4n(n+1)[4(n)(n+1)+1]}{2}$$

$$= \frac{4(2m^2)[4n^2+4n+1]}{2} = 4m^2(2n+1)^2$$

$T_{4n(n+1)}$ is also a perfect square.

$\therefore$ perfect squares are $T_1 = 1$

$T_8 = 36$ is a perfect square.

T_{288} is also a perfect square.

Example 3 *If $N = 12^3 \times 3^4 \times 5^2$, find the total number of even factor of N.*

Solution If $N = 12^3 \times 3^4 \times 5^2$.

Then, $\quad N = 2^6 \times 3^7 \times 5^2$

$\therefore$ Total Number of factors are $= (6+1)(7+1)(2+1) = 7 \times 8 \times 3 = 168$

In above factors, some of these are odd multiple and some are even.

The odd multiples are formed only with combination of 35 and 55.

So total number of odd multiples is

$$(7+1)(2+1) = 24$$

$\therefore$ Even multiples $= 168 - 24 = 144$

Example 4 *Show that $n^2 - 3n - 19$ is not a multiple of 289 for any integer n.*

Solution Suppose $17^2 | n^2 - 3n - 19$

Since $n^2 - 3n - 19 = (n+7)(n-10) + 51$

$17 | (n+7)(n-10);$

$\therefore$ $17^2 | (n+7)(n-10)$ $(\because n+7 \equiv n-10 \pmod{17})$

$\Rightarrow$ $17^2 | (n^2 - 3n - 19) - (n+7)(n+10)$

i.e., $17^2 | 51$ which is a contradiction

Consequently $n^2 - 3n - 19$ is not a multiple of 289.

Example 5 *Determine all integers n such that $n^4 - n^2 + 64$ is the square of an integer.*

Solution Since $n^4 - n^2 + 64 > n^4 - 2n^2 + 1 = (n^2 - 1)^2$ for some non negative integer k,

$(n^4 - n^2 + 64) = (n^2 + k)^2 = n^4 + 2n^2k + k^2$

i.e., $n^2 = \dfrac{64 - k^2}{2k + 1}$ from which we find that the possible values 64, 1, 0 for n^2 are obtained when $k = 0, 7, 8$ respectively.

Hence, $n \in (0, \pm 1, \pm 8)$

Example 6 *Let a, b, c, d, e be consecutive positive integers such that $b + c + d$ is a perfect square and $a + b + c + d + e$ is a perfect cube. Find smallest possible value of c.*

Solution a, b, c, d, e are consecutive positive integer $b + c + d = 3c$ and $a + b + c + d + e = 5c$

Now, $3 | 3c \Rightarrow 3^2 | 3c$ $(\because 3c$ is a square)

$\Rightarrow$ $3 | c \Rightarrow 3 | 5c \Rightarrow 3^3 | 5c$ $(\because 5c$ is a cube)

$\Rightarrow$ Also $5 | 5c \Rightarrow 5^3 | 5c \Rightarrow 5^2 | c$

$\therefore$ $3^3 5^2 | c$ *i.e.*, $675 | c$

$\therefore$ 675 being a possible value of c is the smallest of such numbers.

Example 7 *If $11 + 11\sqrt{11a^2 + 1}$ is an odd integer where a is a rational number. Prove that a is perfect square.*

Solution Let $\lambda = 11 + 11\sqrt{11a^2 + 1}$

Then, $(\lambda - 11)^2 = 11^2(11a^2 + 1)$

Simplifying, we get $\lambda(\lambda - 22) = 11^3 a^2$

Putting $|a| = \dfrac{r}{s}, r, s \in N$ such that $(r, s) = 1$ gives $\lambda(\lambda - 22)s^2 = 11^3 r^2$.

Since $11^2 | s$ because otherwise 11 would divide r, $11 | \lambda$. Writing $11\mu = \lambda$, we get $\mu(\mu - 2)s^2 = 11r^2$

Since $11 | s$ for otherwise we would have $11 | r$. It follows that $s = 1$. Thus we have $\mu(\mu - 2) = 11r^2$. Since $\mu - 2$ and μ are consecutive odd integers they are relatively prime.

If $11 | \mu - 2$, then μ is a square of form $11n + 2$ which is not possible.

$\therefore$ $11 | \mu$ and hence $\mu = 11n^2$ for some $n \in N$.

Thus, we have $\lambda = 11\mu = 11^2 n^2$

Example 8 *Determine all pairs of positive integers (m, n) for which $2^m + 3^n$ is a perfect square.*

Solution Let $\qquad 2^m + 3^n = k^2$

Since $\qquad (-1)^m \equiv 2^m \equiv k^2 \equiv 1 \pmod 3 \qquad (\because 3 \mid k)$

m is even, say $2p$.

Now, $\qquad (k - 2^p)(k + 2^p) = 3^n$

$\Rightarrow k - 2^p = 1$ and $k + 2^p = 3^n \Rightarrow 2^{p+1} + 1 = 3^n$

Since $(-1)^n \equiv 3^n \pmod 4) = 2^{p+1} + 1 \equiv 1$, n is even, say $2q$.

Now $\quad (3^q - 1)(3^q + 1) = 2^{p+1} \Rightarrow 3^q - 1 = 2$

$\Rightarrow 3^q = 3 \Rightarrow q = 1$ and hence $p = 2$

So, we have only one solution $(4, 2)$.

Example 9 *Determine the set of integers n for which $n^2 + 19n + 92$ is a square.*

Solution Let $n^2 + 19n + 92 = m^2$, m is a non negative integer. Then, $n^2 + 19n + 92 - m^2 = 0$

Solving for n, we get $n = \frac{1}{2}(-19 \pm \sqrt{4m^2 - 7})$

$\therefore$ $4m^2 - 7$ is a square *i.e.,* $4m^2 - 7 = p^2$

Where $p \in N$

$\therefore \qquad (2m - p)(2m + p) = 7$

$\therefore$ $2m + p$ being positive therefore $(2m+p)$ is 7 and $2m - p = 1$

Hence, $\qquad 4m = 8 \Rightarrow m = 2$

Thus, we have $n^2 + 19n + 92 = 4$

$\Rightarrow \qquad n^2 + 19n + 88 = 0$

$\Rightarrow \qquad (n + 8)(n + 11) = 0$

$\Rightarrow \qquad n = -8$ or -11

Example 10 *Find n, if $2^{200} - 2^{192} \cdot 31 + 2^n$ is a perfect square.*

Solution $2^{200} - 2^{192}\, 31 + 2^n = 2^{192}(2^8 - 31) + 2^n = 2^{192}(256 - 31) + 2^n = 2^{192} \cdot 225 + 2^n$

$\therefore$ For some $m \in N$

$$2^n = m^2 - 2^{192} \cdot 225 = m^2 - (2^{96} \cdot 15)^2 = (m - 2^{96} \cdot 15)(m + 2^{96} \cdot 15)$$

So, $m = 2^{96} \cdot 15 = 2^{\alpha}$ and $m + 2^{96} \cdot 15 = 2^{\alpha + \beta}$ for some non negative integers α, β.

Hence, $2^{97} \cdot 15 = 2^{\alpha + \beta} - 2^{\alpha} = 2^{\alpha}(2^{\beta} - 1)$

$\Rightarrow \qquad 2^{\alpha} = 2^{97}$ and $2^{\beta} - 1 = 15.$

i.e., $\qquad \alpha = 97$ and $\beta = 4$

$\therefore \qquad n = 2\alpha + \beta = 198$

Example 11 *Find the number of values of n for which $2^{11} + 2^8 + 2^n$ is a perfect square.*

Solution We can write $2^{11} + 2^8 + 2^n$ as

$$2^8(2^3 + 1) + 2^n$$

$\Rightarrow \qquad 2^8 \cdot 9 + 2^n$

$\Rightarrow \qquad 2^n(2^{8-n} \cdot 9 + 1)$

Note that for any $k < 8$, $2^k(2^{8-k}9 + 1)$ is not a square, when k is odd, 2^k is not a square and in the other case, the second factor is not a square. Hence $n \geq 8$. Now write $2^{11} + 2^8 + 2^n$ as $2^8(9 + 2^{n-8})$. Then the problem is to find the number of non negative integers k such that $9 + 2^k$ is a square.

$$9 + 2^k = t^2 \Rightarrow 2^k = (t-3)(t+3)$$

$\Rightarrow t - 3 = 2^p$ and $t + 3 = 2^{p+q}$ for some non negative integers p and q.

$\therefore 2^p(2^q - 1) = 6$ implying $p = 1$ from which it follows that $t = 5$.

Hence, there is a unique solution.

Example 12 *Find all positive integers n for which $n^2 + 96$ is a perfect square.*

Solution Suppose m is a positive integer, such that $n^2 + 96 = m^2$

Then, $m^2 - n^2 = 96 \Rightarrow (m-n)(m+n) = 96$

since $m - n < m + n$ and $m - n, m + n$ must be both even [as $m + n = (m - n) + 2n$. Therefore $m - n, m + n$ must be both odd or both even; also if both of them are odd, then the product cannot be even.

$\therefore$ Only possibilities are

$$m - n = 2, m + n = 48 \Rightarrow m = 25, n = 23$$
$$m - n = 4, m + n = 24 \Rightarrow m = 14, n = 10$$
$$m - n = 6, m + n = 16 \Rightarrow m = 11, n = 5$$
$$m - n = 8, m + n = 12 \Rightarrow m = 10, n = 2$$

Example 13 *Give with justification, a natural number n for which $3^9 + 3^{12} + 3^{15} + 3^n$ is a perfect cube .*

Solution

$$\begin{aligned}3^9 + 3^{12} + 3^{15} + 3^n &= 3^9(1 + 3^4 + 3^6 + 3^{n-9})\\ &= (3^3)^3\{1 + 3\cdot 3^2 + 3(3^2)^2 + (3^2)^3 + 3^{n-9} - 3(3^2)^2\}\\ &= (3^3)^3(1 + 3^2)^3 \text{ provided } 3^{n-9} - 3^5 = 0\\ &= (270)^3 \text{ provided } 3^{n-9} = 3^5\end{aligned}$$

i.e., provided $n = 14$

So, given number is a perfect cube when $n = 14$

Example 14 *Prove that $2^p + 3^p$ is not a perfect power if p is a prime number.*

Solution If $p = 2$, $2^p + 3^p = 2^2 + 3^2 = 13$ (not a perfect power)

Let now p be a prime > 2..

$x + a$ divides $x^p + a^p$, whenever p is odd [factor theorem]

$\therefore$ $2^p + 3^p$ is divisible by $2 + 3 = 5$. We shall show that $2^p + 3^p$ is not divisible by 5^2.

$$x^p + 3^p = (x+3)(x^{p-1} - 3x^{p-2} + 3^2x^{p-3} + \ldots + (-3)^{p-1})$$

When $x = -3$, then

$$\begin{aligned}&x^{p-1} - 3x^{p-2} + \ldots + (-3)^{p-1}\\ &= (-3)^{p-1} - 3(-3)^{p-2} + \ldots + (-3)^{p-1}\\ &= p3^{p-1}\end{aligned}$$

Showing that $x + 3$ does not divide

$$x^{p-1} - 3x^{p-2} + 3^2x^{p-3} + \ldots + (-3)^{p-1}$$

Consequently $(x + 3)^2$ does not divide $x^p + 3p$. So, $(2 + 3)^2$ does not divide $2^p + 3^p$. Since $2^p + 3^p$ is a multiple of 5 but is not a multiple of 5^2.

$\therefore$ it cannot be a perfect power.

Example 15 *A 4 digit number has the following properties (I) It is a perfect square (II) its first 2 digit are equal to each other (III) its last two digit are equal to each other. Find all such four digit number.*

Solution We want to find positive integers x and y such that $1 \le x \le 9, 0 \le y \le 9$ and $xxyy$ is a perfect square. Since, $10^2 = 100, 100^2 = 10000$. It follows that $xxyy$ must be the square of a 2 digit number. Suppose that $(ab)^2 = xxyy$.

The number $xxyy$ is clearly a multiple of 11.

Since, it is a perfect square it must be a multiple of 11^2 *i.e.*, 121.

$\therefore$ It must be of the form

$$121 \times 1, 121 \times 4, 121 \times 9, 121 \times 16, 121 \times 25,$$
$$121 \times 36, 121 \times 49, 121 \times 64, 121 \times 81$$

Out of these 121×64 *i.e.*, 7744 is of the form $xxyy$, we conclude that 7744 is the desired number.

Example 16 *Show that for any integer n, the number $n^4 - 20n^2 + 4$ is not a prime number.*

Solution

$$\begin{aligned} n^4 - 20n^2 + 4 &= (n^4 - 4n^2 + 4) - 16n^2 \\ &= (n^2 - 2)^2 - 16n^2 \\ &= (n^2 - 4n - 2)(n^2 + 4n - 2) \quad \ldots(i) \end{aligned}$$

Note It can be easily seen that none of the factors $n^2 - 4n - 2, n^2 + 4n - 2$ can have the value ± 1, whatever integral value n may have. Here four cases arises.

(i) $n^2 - 4n - 2 = 1 \Rightarrow n = \dfrac{4 \pm \sqrt{28}}{2}$

(ii) $n^2 - 4n - 2 = -1 \Rightarrow n = \dfrac{4 \pm \sqrt{20}}{2}$

(iii) $n^2 + 4n - 2 = 1 \Rightarrow n = \dfrac{-4 \pm \sqrt{28}}{2}$

(iv) $n^2 + 4n - 2 = -1 \Rightarrow n = \dfrac{-4 \pm \sqrt{20}}{2}$

From the above four cases, we find that whatever integral value n may have, $n^4 - 20n^2 + 4$ is the product of the integers $n^2 - 4n - 2$ and $n^2 + 4n - 2$ neither of which equals ± 1..

Example 17 *Prove that the product of four consecutive natural numbers cannot be a perfect cube.*

Solution Consider the product

$P = n(n + 1)(n + 2)(n + 3)$, where n is a natural number.

If possible, that P is a perfect cube $= k^3$ Two cases arises.

Case I If n is odd. $n, (n + 1), (n + 3)$ are all prime to $n + 2$

Now, we know that every common divisor of $n + p$ and $n + q$ must divide $q - p$.

$\therefore$ $n + 2$ and $n(n+1)(n+3)$ are relatively prime.

Since, their product is a perfect cube, each of them must be a perfect cube.

Since, $n^3 < n(n+1)(n+3) < (n+3)^3$

$$\therefore \qquad n(n+1)(n+3) = (n+1)^3 \text{ or } (n+2)^3$$

As $n(n+1)(n+3)$ and $(n+2)^3$ are relatively prime, so second possibility ruled out.

Also $n(n+1)(n+3) = (n+1)^3 \Rightarrow n = 1$. Since $P = 24$, when $n = 1$ which is not a perfect cube. So the possibility $n = 1$ is also ruled out. So n cannot odd.

Case II *If* n is even.

Then $n + 1$ is prime to $n, n + 2$ and $n + 3$. Consequently $n + 1$ is relatively prime to $n(n+2)(n+3)$. Since the product of relatively prime numbers $n + 1$ and $n(n+2)(n+3)$ is a perfect cube, each of them must be a perfect cube.

$$n^3 < n(n+2)(n+3) < (n+3)^3$$

$$\therefore \qquad n(n+2)(n+3) = \text{either } (n+1)^3$$

or $(n+2)^3$since $n(n+2)(n+3)$ and $n + 1$ are relatively prime

$\therefore$ First possibility ruled out.

Also $n(n+2)(n+3) = (n+2)^3 \Rightarrow n + 4 = 0$ which is out of question. Consequently n cannot be even.

Thus, we find that the product of 4 consecutive integers cannot be a perfect cube.

Example 18 *Find all primes p for which the quotient* $(2^{p-1} - 1) | p$ *is a square.*

Solution Suppose $m(m+1) = 7n^2$, m and n are integers since m and $m + 1$ are relatively prime.

$$\therefore \qquad m \text{ and } m+1 \text{ must be the numbers } 7p^2, q^2$$

(in some order) p and q are relatively prime and $pq = n$; Since the product of 2 consecutive integer is even.

$\therefore$ $m(m+1)$ is even, which means that one of the numbers $m, m + 1$ must be even.

Suppose $m = q^2$ (so that $m + 1 = 7p^2$). Since every square number is of one of the forms $4k$, $4k + 1$. Consequently $m + 1$ must be of one of the forms $4k + 1$, $4k + 2$. However this is not possible for if p is even, then $7p^2$ is of the form $4k$. If p is odd, then $7p^2$ is of the form $4k + 3$.

$$\therefore \qquad m + 1 \neq 7p^2. \text{ So } m = 7p^2 \text{ and } m + 1 = q^2$$

Concept of Finding Number of Positive Integral Solutions for the Equation of the Form $x^2 + y^2 = k$

We know that,

$$(2n)^2 \equiv 0 \bmod 4$$

and

$$(2n+1)^2 \equiv 1 \bmod 4$$

Now, if

(a) x and y are both even then,

$$x^2 + y^2 \equiv 0 \bmod 4$$

(b) x and y are both odd then,

$$x^2 + y^2 \equiv 2 \bmod 4$$

(c) one is even and other is odd then,

$$x^2 + y^2 \equiv 1 \bmod 4$$

$$\{\because x^2 + y^2 \equiv 0, 1, 2 \bmod 4 \text{ and } x^2 + y^2 \not\equiv 3 \text{ and } 4\}$$

the above discussion implies, if

$$x^2 + y^2 = k$$

and if k is of the form of $(4m + 3)$, then $x^2 + y^2 = k$ does not have any integral solution.

e.g., suppose, we are asked to find integral solution for equation

$x^2 + y^2 = 19$, then it will not have any integral solution because 19 is of the form $(4m + 3)$.

Now, if we have $x^4 + y^4 = k$, then we know that

$$(2n)^4 \equiv 0 \bmod 16 \quad \text{and} \quad (2n + 1)^4 \equiv 1 \bmod 16$$

Again, if (a) x and y are both even then,

$$x^4 + y^4 \equiv 0 \bmod 16$$

(b) x and y are both odd then,

$$x^4 + y^4 \equiv 2 \bmod 16$$

(c) one is even and other is odd, then

$$x^4 + y^4 \equiv 1 \bmod 16$$

$\therefore$ $\quad x^4 + y^4 \equiv 0, 1, 2 \bmod 16$

and $\quad x^4 + y^4 \not\equiv i \bmod 16,$

where $\quad i = (3, 4, 5, \ldots, 15)$

So, the above discussion implies, if

$$x^4 + y^4 = k$$

and k is of the form $(16m + i)$, where $i = 3, 4, 5, \ldots, 15$, then $x^4 + y^4 = k$ will not have any integral solution.

e.g., suppose we are asked to find integral solutions for equation.

$x^4 + y^4 = 16003$, then it will not give any integral solution because 16003 is of the form $(16m + 3)$.

The concept can be extended to more than two variables expression, suppose the equation is

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 + \ldots + x_{14}^4 = 1599$$

now, we know that

$$(2n)^4 \equiv 0 \bmod 16$$

and $\quad (2n + 1)^4 \equiv 1 \bmod 16$

$\therefore$ $\quad \sum_{i=1}^{14} x_i^4 \equiv 0, 1, 2 \ldots 14 \bmod 16$

but our RHS is $1599 \equiv 15 \bmod 16$.

$\therefore$ No integral solution can be obtained for the above equation.

Reason

[If all the variables are considered to be odd, then maximum remainder which can come out is "14" and if any of the variable is an even number then remainder will be less than 14.]

Now, let us consider another discussion.

If we have $\quad a^2 + b^2 + c^2 = a^2 b^2$

we know, $\quad (2n)^2 \equiv 0 \bmod 4 \quad \text{and} \quad (2n + 1)^2 \equiv 1 \bmod 4$

Case I If a, b and c all are odd then,

$$a^2 + b^2 + c^2 \equiv 3 \mod 4$$

whereas $a^2b^2 \equiv 1 \mod 4$.

It will never give an equality, so the given equation has no integral solution.

Case II If two numbers are odd and one is even, then

$$a^2 + b^2 + c^2 \equiv 2 \mod 4$$

whereas $a^2b^2 \equiv 0, 1 \mod 4$

Again we get no integral solution.

Case III If two even and one odd.

$$a^2 + b^2 + c^2 \equiv 1 \mod 4$$

whereas $a^2b^2 \equiv 0 \mod 4$

Again, no integral solution

Case IV If all are even then

$$a^2 + b^2 + c^2 \equiv 0 \mod 4$$

whereas $a^2b^2 \equiv 0 \mod 4$

Now, there is a possibility to have a solution and the only possible solution is (0, 0, 0) which is the only trivial solution.

Now, let us come again to the discussion of

$$x^2 + y^2 = k$$

we have seen, if $k = 4m + 3$ there is no integral solution now, if k is even then it will be either of the form

$$k = 4m$$

or $k = 4m + 2$

considering $k = 4m$

If m can be expressed as $i^2 + j^2$, where i and j are non-negative integers such that

(i) $i \neq j$, then there will be four integral solutions and 8 ordered pairs.

(ii) if $i = j$ or m can be written as $i^2 + 0^2$, then there will be two integral solutions and 4 ordered pairs.

Let us consider some examples.

e.g., $x^2 + y^2 = 20$

here, 20 is of the form 4(5) and 5 can be expressed as $5 = 1^2 + 2^2$ which gives $i = 1$ and $j = 2$ so, it implies there are 4 integral solutions, which will be of the form $\pm 2i$ and $\pm 2j$ also we have 8 ordered pairs. Therefore in this case we have (2, 4) as one of the solutions and other solutions are

(2, −4), (−2, −4), and (−2, 4) also (4, 2), (4, −2), (−4, −2) and (−4, 2) **keep this thing in mind there are only 4 integers used.**

e.g., $x^2 + y^2 = 8$

Here, $8 = 4(2)$

and $2 = 1^2 + 1^2$

which gives $i = j$.

So, it implies there are 2 integral solutions which will be of the form $2i$ and $2j$ also we have 4 ordered pairs therefore in this case we have our solutions are

(2, 2), (−2, −2), (2, −2) and (−2, 2)

keep this thing in mind there are only 2 integers used.

e.g., $$x^2 + y^2 = 4$$

Here, $$4 = 4(1)$$

and $$1 = 1^2 + 0^2$$

which gives $i = 1$ and $j = 0$

So, it implies there are 2 integral solutions, which are of the form $2i$ and $2j$ also we have 4 ordered pairs. Therefore in this case the solutions are

$$(2, 0), (-2, 0), (0, 2) \text{ and } (0, -2).$$

e.g., $$x^2 + y^2 = 24$$

Here, $$24 = 4 \times 6$$

and 6 can't be represented as $i^2 + j^2$ so it will not have any integral solution.

e.g., $$x^2 + y^2 = 12$$

Here, $$12 = 4 \times 3$$

and 3 again can't be expressed as $i^2 + j^2$ so it will also not have any integral solution.

Now, considering $k = 4m + 2$ and if m can be written as $(i^2 + j^2 + i + j)$ and if $i \neq j$, then these will be 4 integers and 8 ordered pairs of solutions.

And if $i = j$, then there will be 2 integers and 4 ordered pairs of solutions.

e.g., $$x^2 + y^2 = 10$$

Here, $$10 = 4(2) + 2$$

and $$2 = (1)^2 + (0)^2 + (1) + (0)$$

Therefore there will be four integral solutions which will be given as $\pm (2i + 1)$ and $\pm (2j + 1)$ and in this case the solutions are ± 3 and ± 1 which will give eight ordered pairs as (3, 1), (3, -1), (-3, 1) and (-3, -1) also (1, 3), (1, -3), (-1, 3) and (-1, -3).

e.g., $$x^2 + y^2 = 2$$

Here, $$2 = 4(0) + 2$$

and $$0 = (0)^2 + (0)^2 + (0) + (0)$$

Therefore there are only two integral solutions which will again be given as $\pm (2i + 1)$ and $\pm (2j + 1)$ and in this case the solutions are ± 1 and ± 1, which will give four ordered pairs as (1, 1), (1, -1), (-1, 1) and (-1, -1).

e.g., $$x^2 + y^2 = 18$$

Here, $$18 = 4(4) + 2$$

and $$4 = (1)^2 + (1)^2 + (1) + (1)$$

So $$i = 1 \text{ and } j = 1$$

$\Rightarrow$ $$i = j$$

Therefore it will have 2 integral solutions which will be given as $\pm (2i + 1)$ and $\pm (2j + 1)$ and in this case the solutions are ± 3 and ± 3 which gives four ordered pairs as

$$(3, 3), (3, -3), (-3, 3) \text{ and } (-3, -3)$$

e.g., $$x^2 + y^2 = 14$$

Here, $$14 = 4(3) + 2$$

and 3 can't be expressed as $(i^2 + j^2 + i + j)$ as it is always an even number and an even number can't be equal to an odd number. So it implies if right hand side is $(4m + 2)$ and m is an odd number. So the equation will never produce any integral solution.

Now, we will extend this concept for an odd number in right hand side of the equation.

$$x^2 + y^2 = k$$

i.e., $k = 4m + 1$

or $k = 4m + 3$

As it has been already discussed $(k = 4m + 3)$ will not produce any integral solutions.

So, considering $k = 4m + 1$, only.

If $m = i^2 + j^2 + j$,

then there will be an odd integer and an even integer, if $i \neq 0$ and $j \neq 0$ or $i \neq 0$ and $j = 0$, then there are four integers and 8 ordered pairs which will satisfy the equation.

So, one of the integral solution is $\pm 2i$ and other is $\pm (2j + 1)$.

Now, if $i = 0$, $j \neq 0$, then there are two integers and four ordered pairs which will satisfy the equations.

So, one of the integral solution is 0 and other is $\pm (2j + 1)$.

e.g., $x^2 + y^2 = 21$

Here, $21 = 4 \times 5 + 1$

But 5 cannot be written as $i^2 + j^2 + j$, so it will not give any integral solution.

Exceptional case :

If $x^2 + y^2 = k$

and k is an odd and a perfect square, then perform the following test always.

Take square root of k, which will come out to be as $\sqrt{k}$, now subtract "1" from it we get $(\sqrt{k} - 1)$ always double it, so it becomes $2(\sqrt{k} - 1)$, now add "1" to it which becomes $2(\sqrt{k} - 1)$. If this value is a perfect square say, it is a^2, then the equation will always have 6 integers and 12 ordered pairs as its solutions and the integers will be $\pm a$, $\pm (\sqrt{k} - 1)$, $\pm \sqrt{k}$ and 0. Always keep this thing in mind $\sqrt{k}$ is an integer.

And if the test fails, then equation will be solved by the method discussed earlier.

e.g., $x^2 + y^2 = 169$

here $169 = 4(42) + 1$, which is of the form $(4m + 1)$ and also it is an odd perfect square so we will have to perform the mentioned test.

e.g.,
$$\sqrt{169} = 13$$
$$13 - 1 = 12$$
$$12 \times 2 = 24$$
$$24 + 1 = 25$$

and $25 = 5^2$

$\therefore$ we will have 4 integers in which ± 5, ± 12, will form 8 ordered pairs (5, 12), (5, −12), (−5, 12), (−5, −12), (12, 5), (12, −5), (−12, 5), (−12, −5) also there will be three ± 13, 0 which will form four pairs (13, 0), (−13, 0), (0, 13), (0, −13)

e.g., $x^2 + y^2 = 49$

here $49 = 4 \times 12 + 1$, which is of the form $(4m + 1)$ and also an odd perfect, so we will again perform the mentioned test.

$$\sqrt{49} = 7$$
$$7 - 1 = 6$$
$$6 \times 2 = 12$$
$$12 + 1 = 13$$

but 13 is not a perfect square therefore the solution will be checked by the earlier method.

$$12 = (0)^2 + (3)^2 + (3)$$

Here, $i = 0$ and $j = 3$.

so the solutions will be 0 and ± 7. Also the ordered pairs will be (0, 7), (0, −7), (7, 0) and (−7, 0)

Concept Solving of the equation of the form $xy = n$.

If we are asked to find the number of positive integral solution for $xy = n$, we first write n is the form $p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}$. The number of positive integral solution is same as the number of divisors of n which is equal to $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)$...

Let us consider example $xy = 8 = 2^3$

Number of divisors of 8 are $3 + 1 = 4$. So there are 4 integral solution and 4 ordered pair namely (1, 8) (8, 1) (2, 4) (4, 2)

Now let us consider another example $xy = 72(x + y)$

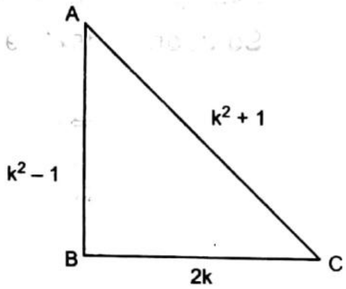
we can write it as $(x - 72)(y - 72) = 72^2$.

Let $$x - 72 = X, y - 72 = 72^2$$

$$\therefore \quad XY = 72^2 = 2^6 3^4$$

Number of solutions are 35.

Concept The area of a Δ formed by pythogorean triplet with integer sides is always divisible by 3.



$$\text{Area of } \Delta\, ABC = \frac{1}{2}\, BC \times AB$$
$$= \frac{1}{2}\, 2x\, (x^2 - 1)$$
$$= (x^3 - x)$$

Let $$p(x) = x^3 - x$$

For $x > 1$ we use induction

$p(2) = 8 - 2 = 6$, $p(2)$ is true

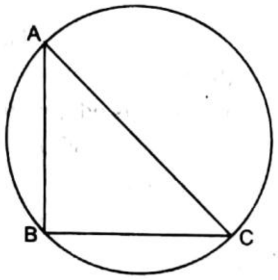
Let $p\,(x)$ be true for $n = m$

$$p(m) = m^3 - m = 3c$$

$$\therefore \quad p(m + 1) = (m + 1)^3 - (m + 1)$$
$$= (m + 1)^3 - (m + 1) = m^3 + 3m^2 + 2m$$
$$= 3c + m + 3m^2 + 2m = 3(c + m + m^2)$$

$\therefore$ $P\,(m + 1)$ is true.

Concept The radius of the circumcircle of a Δ formed by pythogorean triplet cannot be integer.



The hypotenuse of the Δ ABC is the diameter of the circle.

Let us consider the pythogorean triplet $x^2 - 1$, $2x$, $x^2 + 1$

Here $x^2 + 1$ is hypotenuse. Since x is an even number its square is also even, therefore an even number plus one is an odd number.

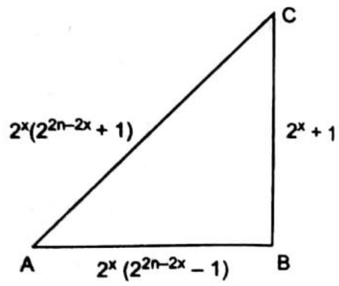
$\therefore$ $x^2 + 1$ is an odd number

$\therefore$ Radius of circumcircle is $\dfrac{x^2 + 1}{2}$

Concept For any natural number x for

$$x = 0, 1, 2, \ldots, n$$
$$2^n + 1,\ 2^x\,(2^{2x - 2x} - 1),\ 2^x\,(2^{2n - 2x} + 1)$$
$$AC^2 = [2^x\,(2^{2n - 2x} + 1)]^2$$

$$= 2^{2x} (2^{4n-4x} + 2^{2n-2x+1} + 1)$$
$$= (2^{4n-4x+2x} + 2^{2n-2x+1+2x} + 2^{2x})$$
$$AC^2 = (2^{4n-2x} + 2^{2n+1} + 2^{2x})$$
$$AB^2 + BC^2 = [2^{2x} (2^{2n-2x} - 1)^2] + (2^{n+1})^2$$
$$= 2^{2x} (2^{4n-4x} - 2 \cdot 2^{2n-2x} + 1) + 2^{2n+2}$$
$$= 2^{4n-2x} + 2^{2x} + 2^{2n} \cdot 2^2 - 2^{2n} \cdot 2$$
$$= 2^{4n-2x} + 2^{2x} + 4 \cdot 2^{2n} - 2 2^{2n}$$
$$= 2^{4n-2x} + 2^{2x} + 2 2^{2n}$$
$$= 2^{4n-2x} + 2^{2x} + 2^{2n+1}$$



Example 1 *Prove that there are no natural numbers, which are solutions of* $15x^2 - 7y^2 = 9$.

Solution $15x^2 - 9 = 7y^2$

$$3(5x^2 - 3) = 7y^2$$

$\Rightarrow 7y^2$ is a multiple of 3.

$\Rightarrow$ y is a multiple of 3.

Let $y = 3z$

$$3(5x^2 - 3) = 7 \times 9z^2$$
$$5x^2 - 3 = 21z^2$$
$$5x^2 = 21z^2 + 3$$

$5x^2$ is a multiple of 3.

$\Rightarrow$ x is a multiple of 3

Let $x = 3u$

$$15u^2 = 1 + 7z^2$$
$$15u^2 - 6z^2 = 1 + z^2$$

$1 + z^2$ is a multiple of 3.

But for any z between 0 to 9, $1 + z^2$ is not a multiple of 3.

For any z, the given equation has no integral solution.

Aliter

Since RHS is odd, x and y must be opposite

i.e., one even and one odd.

As $3|15$ and $3|9$

$\therefore$ 3 must divide $7y^2$.

Let $y = 3y_1$ so $5x^2 - 21y_1^2 = 3$

Again, since 3 divide 21 so 3 must divide $5x^2$.

Let $x = 3x_1$, we get

$$15x_1^2 - 7y_1^2 = 1$$

$$\Rightarrow \quad 15x_1^2 = 7y_1^2 + 1$$

Last digit of perfect square y_1^2 may be one of these values 0, 1, 4, 9, 6, 5.

Hence, last digit of $7y_1^2 + 1$ will be 1, 8, 9, 4, 3, 6 respectively.

But $15x_1^2$ ends in 0 or 5.

$\therefore$ $15x_1^2 = 7y_1^2 + 1$ has no solutions.

Example 2 *Show that $x^2 + 1 = 3y$ has no solutions in integers.*

Solution Since LHS cannot be a multiple of 3 for any x between 0 to 9.

RHS is always a multiple of 3.

$\therefore$ $x^2 + 1 = 3y$ has no integral solutions.

Example 3 *Show that $21x^2 - 10y^2 = 9$ has no solution.*

Solution $21x^2 - 9 = 10y^2$

$\Rightarrow$ $3(7x^2 - 3) = 10y^2$

$\Rightarrow$ $10y^2$ is a multiple of 3.

$\Rightarrow$ y is a multiple of 3.

Let $y = 3y_1$

$$3(7x^2 - 3) = 10 \times 9y_1^2$$

$$7x^2 - 3 = 30y_1^2$$

$$7x^2 = 3 + 30y_1^2 \Rightarrow 7x^2 = 3(1 + 10y_1^2)$$

$\Rightarrow$ $7x^2$ is a multiple of 3

So, x is multiple of 3.

Let $x = 3x_1$

$$7 \times 9x_1^2 = 3(1 + 10y_1^2)$$

$$21x_1^2 = 1 + 10y_1^2$$

$$21x_1^2 - 9y_1^2 = 1 + y_1^2$$

$$3(7x_1^2 - 3y_1^2) = 1 + y_1^2$$

$\Rightarrow$ $1 + y_1^2$ is a multiple of 3.

But $1 + y_1^2$ is not a multiple of 3.

$\therefore$ The given equation has no integral solution.

Note Every integer m can be written in the form $x^2 + y^2 - 5z^2$.

If $m = 2n$, then

$$= 2n = (n-2)^2 + (2n-1)^2 - 5(n-1)^2$$

If $m = 2n + 1 = (n+1)^2 + (2n)^2 - 5n^2$

Verification,

$$7 = 2(3) + 1 = (3+1)^2 + (2 \times 3)^2 - 5(3)^2$$

$$= 16 + 36 - 45$$

$$= 52 - 45 = 7$$

Similarly, every integer can be written in the form of

$$x^2 + y^2 + z^2 - 5u^2$$

Example 4 *Prove $\frac{1}{n+1}\,{}^{2n}C_n$ is an integer.*

Solution If a and b are integers and $a - b = c$, then c is also an integer.

Let $$a = {}^{2n}C_n;\ b = {}^{2n}C_{n-1}$$

$$ {}^{2n}C_n - {}^{2n}C_{n-1} = \frac{2n!}{n!n!} - \frac{2n!}{(n+1)!(n-1)!} = \frac{2n!}{n!\,n!}\left[1 - \frac{n}{n+1}\right]$$

$$= \frac{2n!}{n!n!}\left[\frac{1}{n+1}\right] \Rightarrow \text{ it is an integer.}$$

$$ {}^{kn}C_n - {}^{kn}C_{n-1} = \frac{(kn)!}{(kn-n)!n!} - \frac{kn!}{(kn-n+1)!(n-1)!}$$

$$= \frac{(kn)!}{(kn-n)!n!}\left[1 - \frac{(kn-n)!n!}{(kn-n+1)!(n-1)!}\right]$$

$$= \frac{(kn)!}{(kn-n)!n!}\left[1 - \frac{n}{kn-n+1}\right]$$

$$= {}^{kn}C_n\left[\frac{kn-n+1-n}{kn-n+1}\right] = {}^{kn}C_n\left[\frac{kn-2n+1}{kn-n+1}\right].$$

It is an integer.

Example 5 *If $xy = 2^2 \cdot 3^4 \cdot 5^7(x+y)$, find the number of integral solution.*

Solution Let $$N = 2^2 \cdot 3^4 \cdot 5^7$$

$$xy = N(x+y)$$
$$xy = Nx + Ny$$
$$xy - Nx - Ny = 0$$
$$(x-N)(y-N) = N^2$$
$$(x-N)(y-N) = 2^4.3^8.5^{14}$$

Number of integral solution

$$= (4+1)(8+1)(14+1) = 5 \times 9 \times 15 = 675$$

Example 6 *Find all positive integers x, y satisfying*

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}$$

Solution Suppose x, y are two positive integers such that

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}} \qquad \text{...(i)}$$

then $$\frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}} - \frac{1}{\sqrt{x}} = \frac{\sqrt{x} - \sqrt{20}}{\sqrt{20}\cdot\sqrt{x}}$$

$\therefore$ $$\frac{1}{y} = \frac{x + 20 - 4\sqrt{5x}}{20x}$$

Implying that $\sqrt{5x}$ is rational.

Now $x \in N \Rightarrow \sqrt{5x} \in N$. Hence $5x$ is the square of an integer which is divisible by 5.

$\therefore$ $5x = (5a)^2$ for some $a \in N$ *i.e.*, $x = 5a^2$ similarly $y = 5b^2$ for some $b \in N$.

Now, Eq. (i) becomes

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} \quad \Rightarrow \quad 2(a+b) = ab$$

$$(a-2)(b-2) = 4$$

$\Rightarrow$ $\quad (a, b) \in \{(3, 6), (4, 4), (6, 3)\}$.

$\therefore$ Solution set is $\{(45, 180), (80, 80), (180, 45)\}$.

Example 7 *Find the number of solutions in positive integers of the equation* $3x + 5y = 1008$.

Solution Let $x, y \in N$ such that $3x + 5y = 1008$ then $3 \mid 5y \Rightarrow 3 \mid y \Rightarrow y = 3k$ for some $k \in N$

Now, $\quad 3x + 15k = 1008$

$\Rightarrow \quad x + 5k = 336$

$\Rightarrow \quad 5k \le 335$

$\Rightarrow \quad k \le 67$

Thus, any solution pair is given by $(x, y) = (336 - 5k, 3k)$ where $1 \le k \le 67$.

$\therefore$ Number of solutions is 67.

Example 8 *Prove that there do not exist positive integrs x, y, z satisfying*

$$2xz = y^2 \text{ and } x + z = 997.$$

Solution $2 \mid y^2 \Rightarrow 4 \mid 2xz \Rightarrow 2 \mid x$ or $z \Rightarrow 2 \mid x$ and z $\qquad [\because 2 \mid (x+z)]$

Let $\quad x = 2x_1, y = 2y_1$ and $z = 2z_1$

Then $\quad x_1 z_1 = y_1^2$ and $x_1 + z_1 = 997$

Again y_1 and one of x_1, z_1, say x_1, are even writing $x_1 = 2x_2$ and $y_1 = 2y_2$

We have $x_2 z_1 = y_2^2$ and $2x_2 + z_1 = 997$

Since, 997 is a prime, x_2 and z_1 are relatively prime.

$\therefore$ Each is a square since their product is square.

Since any square is of the form $8n$, $8n + 1$ or $8n + 4$, $2x_2 \equiv 0$ or $2 \pmod 8$ and $z_1 \equiv 1 \pmod 8$ ($\because z_1$ is odd).

Hence $\quad 2x_2 + z_1 \equiv 1$ or $3 \pmod 8$.

A contradiction as $997 \equiv 5 \pmod 8$.

Example 9 *Show that the equation* $3x^{10} - y^{10} = 1991$ *has no integral solution.*

Solution Suppose the existence of $x, y \in Z$ such that $3x^{10} - y^{10} = 1991$. Note that $11 \mid 1991$.

$\therefore$ Neither x nor y is divisible by 11 for otherwise 11 would divide both .

$\Rightarrow \quad 11^{10} \mid 3x^{10} - y^{10} = 11^{10} \mid 1991$

an impossibility.

Hence x and y are prime to 11.

$\therefore \quad x^{10} \equiv y^{10} \equiv 1 \pmod{11}$

$\Rightarrow \quad 1991 = 3x^{10} - y^{10} \equiv 3 - 1 = 2$ a contradiction.

Example 10 *Find all integral solutions of*

$$x^4 + y^4 + z^4 - t^4 = 1991$$

Solution Let n be any integer ; when it is odd $n^4 - 1 = (n-1)(n+1)(n^2+1)$ is divisible by 16 as $n-1, n+1, n^2+1$ are all even and $n-1, n+1$ being consecutive even integers one of them is divisible by 4 . When n is even $n^4 \equiv 0$ (mod-16).

Thus, $n^4 \equiv 0$ or 1, Now for any $x, y, z, t, \in Z$

$$x^4 + y^4 + z^4 - t^4 \equiv \alpha$$

where $\alpha \in \{-1, 0, 1, 2, 3\}$, since $1991 \equiv 7$

$$x^4 + y^4 + z^4 - t^4 \neq 1991$$

Example 11 *For $n \in N$, let $s(n)$ denote the number of ordered pairs (x, y) of positive integers for which $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$. Determine the set of positive integers n for which $s(n) = 5$.*

Solution $\frac{1}{x} + \frac{1}{y} = \frac{1}{n} \Rightarrow x, y > n$

$\therefore$ $x = n + a$ and $n + b$, $a, b \in N$.

Now, $$\frac{1}{n+a} + \frac{1}{n+b} = \frac{1}{n}$$

$$\Rightarrow \quad (n+b+n+a)n = (n+a)(n+b)$$

$$\Rightarrow \quad n^2 = ab$$

$\therefore$ $s(n)$ is the number of divisors of n^2

Let $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ be prime factorization of n where $\alpha_1 \geq \dots \geq \alpha_m$.

Then $s(n) = (1 + 2\alpha_1) \dots (1 + 2\alpha_m)$

$$\therefore \quad s(n) = 5$$

$$\Rightarrow \quad 1 + 2\alpha_1 = 5$$

$$\text{and} \quad m = 1$$

$$\therefore \quad n = p_1^2$$

Required set is $\{p^2 : p \text{ is prime}\}$.

Example 12 *If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$; a, b, c are positive integers with no common factors. Prove that $(a + b)$ is a square.*

Solution By the hypothesis $\frac{a+b}{ab} = \frac{1}{c}$

i.e., $$(a+b)c = ab$$

Let p be any prime which divides $(a + b)$; then p divides one of a, b and therefore both.

Since $gcd(a, b, c) = 1$; p does not divide c.

$$\therefore \quad \text{for any } k \in N, \; p^k | a \Leftrightarrow p^k | b$$

Hence the maximum power of p which divides $a + b$ is the square of the maximum power of p which divides a.

$\therefore$ $a + b$ is a square.

Example 13 *Find all integers n such that* $\sqrt{\dfrac{3n-5}{n+1}}$ *is also an integer.*

Solution Since, $\sqrt{\dfrac{3n-5}{n+1}} = \sqrt{3 - \dfrac{8}{n+1}}$ is an integer.

$$\frac{8}{n+1} \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$$

and $\qquad 3 - \dfrac{8}{n+1}$ is a square.

Consequently $\dfrac{8}{n+1} = -1$ or 2;

i.e., $\qquad n = -9$ or 3

Example 14 *Find the number of pairs of integers* (a, b) *such that*

$$a^3 + a^2b + ab^2 + b^3 + 1 = 2002.$$

Solution $a^3 + a^2b + ab^2 + b^3$

$$= (a + b)(a^2 + b^2) = 2001$$
$$= 3 \times 23 \times 29$$

$a + b$ is therefore one of the three numbers 3, 23, or 29.

If $a + b = 3$, then $a = 1, b = 2$

(or $a = 2, b = 1$) so that $a^2 + b^2 = 5$.

But in this case $a^2 + b^2 = 23 \times 29$

$\therefore a + b$ is not 3.

If $a + b = 23$, then $a^2 + b^2 = 87$

so that both a and b will be less than 10 and $a + b < 20$, a contradiction.

If $a + b = 29$, then $a^2 + b^2 = 69$ so that both a and b will be less than 10 and $a + b < 20$, a contradication.

Thus, the number of pairs (a, b) satisfying the given condition is zero.

Additional Solved Examples

Additional Solved Examples

Example 1. *How many zeros does* 100, *ends with ?*

Solution If e is the maximum power of 5 in 100!, then

$$e = \sum_{i=1}^{\infty}\left[\frac{100!}{5^i}\right] = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \left[\frac{100}{5^3}\right] = 20 + 4 + 0 = 24$$

Hence, 100! ends with 24 zeros.

Example 2. *If n! has exactly* 20 *zeros at the end, find n. How many such n are there ?*

Solution Let e be the maximum power of 5 in $n!$ then

$$e = \sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right] < \sum_{i=1}^{\infty}\left(\frac{n}{5^i}\right) = \frac{n}{5}\sum_{i=1}^{\infty}\frac{1}{5^{i-1}}$$

$$\Rightarrow \qquad e = \frac{\frac{n}{5}}{1-\frac{1}{5}} = \frac{n}{4} \qquad \left(\because S_\infty = \frac{a}{1-r}\right)$$

$$\therefore \qquad n > 4e$$

Here e is given to be 20.

$$\therefore \qquad n \geq 80 \text{ for } 80,\ e = 19$$

$\therefore$ 85 is the required answer.

86, 87, 88, 89 are also valid values of n. If solution exist for this type of problem there will be 5 solutions.

Example 3. *Find number of zeros at the end of* $(5^n - 1)!$.

Note First student should remember $[x + k] = x$

$[x - k] = x - 1, 0 \leq k < 1$.

Solution We will find highest power of 5 in $(5^n - 1)!$

$$e = \sum_{i=1}^{\infty}\left[\frac{5^n - 1}{5^i}\right]$$

$$= \left[\frac{5^n - 1}{5}\right] + \left[\frac{5^n - 1}{5^2}\right] + \left[\frac{5^n - 1}{5^3}\right] + \ldots + \left[\frac{5^n - 1}{5^{n-1}}\right]$$

$$= \left[5^{n-1} - \frac{1}{5}\right] + \left[5^{n-2} - \frac{1}{5^2}\right] + \left[5^{n-3} - \frac{1}{5^3}\right] + \ldots + \left[5 - \frac{1}{5^{n-1}}\right]$$

$$= (5^{n-1} - 1) + (5^{n-2} - 1) + (5^{n-3} - 1) + \ldots + (5 - 1) \qquad [\because [x - h] = x - 1]$$

$$= (5^{n-1} + 5^{n-2} + \ldots + 5) - (n - 1)$$

$$= \frac{5(5^{n-1} - 1)}{5 - 1} - (n - 1)$$

$$= \frac{(5^n - 4^n - 1)}{4}$$

$$\therefore \qquad \text{Total number of zeros} = \frac{5^n - 4^n - 1}{4}$$

Example 4. *Find highest power of* 15 *in* 100!.

Solution Here, $15 = 3 \times 5$ we first find highest power of 3.

$$E_3(100!) = 48 \qquad (\because E_3 \text{ means exponent of 3.})$$

and $$E_5(100!) = 24$$

$\therefore$ $$E_{15}(100!) = \min(24, 48) = 24$$

Example 5. *Find the exponent of* 6 *in* 33!.

Solution Here, $$6 = 2 \times 3$$

$$E_2(33!) = 31$$

$$E_3(33!) = 15$$

$$E_6(33!) = \min(31, 15) = 15$$

Example 6. *Prove that* 33! *is divisible by* 2^{15}.

Solution We first find highest power of 2 in 33!

$$= \left[\frac{33}{2}\right] + \left[\frac{33}{2^2}\right] + \ldots + \left[\frac{33}{2^5}\right] = 16 + 8 + 4 + 2 + 1 = 31$$

Hence, exponent of 2 in 33! is 31.

i.e., 33! is divisible by 2^{31}.

But 2^{31} is also divisible by 2^{15}.

$\therefore$ 33! is divisible by 2^{15}.

Exmaple 7. *Let* $N = 2^{n-1}(2^n - 1)$ *and* $(2^n - 1)$ *is a prime number.*

$1 < d_1 < d_2 < \ldots < d_k = N$ *are divisor of N. Show that* $1 + \frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_k} = 2$

Solution Let $2^n - 1 = q$

$\therefore$ Divisor of N are

$$1, 2, 2^2, \ldots, 2^{n-1}, q, 2q, 2^2q, \ldots, 2^{n-1}q$$

Let $$S = 1 + \frac{1}{d_1} + \frac{1}{d_2} \ldots + \frac{1}{d_k}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}\right) + \frac{1}{q}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}\right)$$

$\Rightarrow$ $$S = 1\frac{\left[1 - \frac{1}{2^n}\right]}{1 - \frac{1}{2}} + \frac{1}{q}\frac{\left(1 - \frac{1}{2^n}\right)}{\left(1 - \frac{1}{2}\right)}$$

$\Rightarrow$ $$S = \frac{2^n - 1}{2^{n-1}} + \frac{1}{q}\frac{2^n - 1}{2^{n-1}}$$

$$= \frac{q(2^n - 1) + (2^n - 1)}{q2^{n-1}} = \frac{(2^n - 1)(q + 1)}{q \cdot 2^{n-1}}$$

$$= \frac{(2^n - 1)(2^n)}{(2^n - 1)2^{n-1}} = \frac{2^n}{2^{n-1}} = 2$$

Example 8. *Show that* $n = 2^{m-1}(2^m - 1)$ *is a perfect number, if* $(2^m - 1)$ *is a prime number.*

Solution As we know by the definition of perfect numbers that if the sum of the divisors of a number n, other than itself, is equal to n, then n is called a perfect number.

Let $n = 2^{m-1} \times p$ where $p = 2^m - 1$ is prime number. Divisors of $2^{m-1} \times p$ are $1, 2, 2^2, 2^3, \ldots, 2^{m-1}, p, 2p, 2^2p, \ldots, 2^{m-2}p, 2^{m-1}p$.

Now, we should sum all these divisors excepting the last one, *i.e.*, $2^{m-1}p$

$$S = (1 + 2 + 2^2 + \ldots + 2^{m-1}) + p(1 + 2 + 2^2 + \ldots + 2^{m-2})$$
$$= \frac{1(2^m - 1)}{2-1} + \frac{p[1(2^{m-1} - 1)]}{2-1} = 2^m - 1 + p(2^{m-1} - 1)$$
$$= 2^m + p2^{m-1} - p - 1 = 2^{m-1}(2 + p) - (p + 1)$$
$$= 2^{m-1}(1 + 2^m) - 2^m \qquad [\because p = 2^m - 1]$$
$$= 2^{m-1}(2^m - 1) = n$$

Example 9. *Prove that product of four consecutive positive integers increased by* 1 *is a perfect square.*

Solution Let the consecutive positive integers be

$$n, n+1, n+2, \text{and } n+3.$$

Consider the expression, $N = n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n)(n^2 + 3n + 2) + 1$

$$= (n^2 + 3n)^2 + 2(n^2 + 3n) + 1 = [(n^2 + 3n) + 1]^2 = (n^2 + 3n + 1)^2$$

Example 10. *Three consecutive positive integers raised to the first, second and third powers respectively, when added make a perfect square, the square root of which is equal to the sum of three consecutive integers, find these integers.*

Solution Let $(n-1), n, n+1$ be the three consecutive integers.

Then, $(n-1)^1 + n^2 + (n+1)^3 = (3n)^2 = 9n^2$

$\Rightarrow$ $n - 1 + n^2 + n^3 + 3n^2 + 3n + 1 = 9n^2$

$\Rightarrow$ $n^3 - 5n^2 + 4n = 0 \Rightarrow n(n-1)(n-4) = 0$

$\Rightarrow$ $n = 0$ or $n = 1$ or $n = 4$

But $n = 0$ and $n = 1$ will make the consecutive integers $-1, 0, 1$ and $0, 1, 2$ which contradicts the hypothesis that the consecutive integers are all greater than zero.

Hence $n = 4$, corresponding to which the consecutive integers are 3, 4 and 5.

Example 11. *Determine the sum of all the divisors* d *of* $19^{88} - 1$ *which are of the form* $d = 2^a \cdot 3^b$ *with* $a, b > 0$.

Solution $19^{88} - 1 = (1 + 18)^{88} - 1 = (1 - 20)^{88} - 1$

In the binomial expansion of $(1 + 18)^{88}$, first term is 1 and second term is 18 all other terms are divisible by 18^2.

$\therefore$ Maximum power of 3 divides $(1 + 18)^{88} - 1$ is 3^2. In the expansion of $(1 - 20)^{88}$, first term is 1 and second term is $-20.88 = -2^5.55$ and third term is $20^2 \times \frac{88.87}{2} = 2^6.5^2.11.87$ and therefore $(1.20)^{88} - 1$ divisible by 2^6. Hence maximum power of 2 which divides $(1 - 20)^{88} - 1$ is 2^5.

Hence the sum of factors of $19^{88} - 1$ which are of the form $a_2{}^a \cdot 3^b$ where $a, b > 0$ is

$$(2 + 2^2 + 2^3 + 2^4 + 2^5)(3 + 3^2) = 744$$

Example 12. *Determine with proof all the arithmetic progression with integer terms, with the property that for each positive integer n, the sum of first n terms is a perfect square.*

Solution When $n = 1$, the first term itself is a perfect square. Which is k^2(say)

The sum to n terms of the AP is $S_n = \frac{n}{2}[2a + (n-1)d]$, where $a = k$

Since, S_n is a perfect square for every n the nth term $2a + (n-1)d > 0$ for every n and hence $d > 0$.

If n is an odd prime, say p, then $S_p = \frac{p}{2}[2a + (p-1)d]$.

Since, S_p is a perfect square $p|[2a + (p-1)d]$ *i.e.*, $p|[(2a - d) + pd]$

But $p|pd$, so $p|(2a - d)$. This is possible for all prime p, if and only if, $2a - d = 0$ or $2a = d$ *i.e.*, $d = 2k^2$.

So, the required AP is

$$k^2, 3k^2, 5k^2, \ldots, (2n-1)k^2$$

k is any natural number.

Example 13. *Show that* 13 *divides* $2^{70} + 3^{70}$.

Solution We can write $2^{70} + 3^{70}$ as $4^{35} + 9^{35}$ and 35 is odd. Now $a^n + b^n$ is divisible by $a + b$ when n is odd.

$\therefore$ It follows $4^{35} + 9^{35}$ is divisible by 13.

Example 14. *Determine all positive integer n for which* $2^n + 1$ *is divisible by* 3.

Solution $2^n + 1$ is divisible by 3.

$\Rightarrow$ $(3-1)^n + 1$ is divisible by 3.

$\Rightarrow$ $(-1)^n + 1$ is divisible by 3.

$\Rightarrow n$ is odd.

$\therefore$ Set of all odd natural number in the set of all those positive integer for which $2^n + 1$ is divisible by 3.

Example 15. *Prove that for every positive integer* $1^n + 8^n - 3^n - 6^n$ *is divisible by* 10.

Solution Since, 10 is the product of two prime 2 and 5. It is sufficient to show that given expression is divisible by both 2 and 5.

So we use the fact $a^n - b^n$ is always divisible by $a - b$, when n is an integer.

Let $A = 1^n + 8^n - 3^n - 6^n$ or $A = (8^n - 3^n) - (6^n - 1^n)$

We find that $8^n - 3^n$ and $6^n - 1^n$ is divisible by $(8-3) = (6-1) = 5$

Again writing $A = (8^n - 6^n) - (3^n - 1^n)$ is divisible by 2

$\therefore$ A is divisible by 10.

Example 16. *Show that* $1^{1997} + 2^{1997} + \ldots + 1996^{1997}$ *is divisible by* 1997.

Solution We shall make groups of the terms

$$(1^{1997} + 1996^{1997}) + (2^{1997} + 1995^{1997}) + \ldots + (998^{1997} + 999^{1997})$$

Here each bracket is of the form $a_i^{2n+1} + b_i^{2n+1}$. It is divisible by $(a_i + b_i)$

But $a_i + b_i = 1997$ for all i

$\therefore$ Each bracket and hence their sum is divisible by 1997.

Example 17. *Find the remainder of* 2^{100} *when divided by* 3.

Solution We use the concept of congruent modulo.

$$2 \equiv 2 \bmod 3.$$

$\therefore$ $$a \equiv a \bmod m$$

Raising the power to 5

$$2^5 \equiv 32 \bmod 3$$

or $32 \equiv 2 \bmod 3$ $\therefore$ 2 is the remainder when 32 is divided by 3.

$\Rightarrow$ $$2^5 \equiv 2 \bmod 3$$

Raising the power to 4.

$$2^{20} \equiv 16 \bmod 3$$

or $16 \equiv 1$ and 3 $\therefore$ 1 is the remainder when 16 is divided by 3.

$\Rightarrow$ $$2^{20} \equiv 1 \bmod 23$$

Raising the power to 5.

$$2^{100} \equiv 1 \bmod 3$$

$\therefore$ 1 is the remainder when 2^{100} is divided by 3.

Example 18. *Find the remainder when* $2^{100} + 3^{100} + 4^{100} + 5^{100}$ *is divided by* 7.

Solution

$$2^{100} \equiv 2 \bmod 7$$

$$3^{100} \equiv 4 \bmod 7, \bmod m$$

$$4^{100} \equiv 4 \bmod 7,$$

$$5^{100} \equiv 2 \bmod 7,$$

$\therefore$ $$2^{100} + 3^{100} + 4^{100} + 5^{100} \equiv 12 \bmod 7$$

$$\left\{\begin{array}{l} \therefore \quad a \equiv b \bmod m \\ \quad c \equiv d \bmod m \\ a + c \equiv b + d \bmod m \end{array}\right\}$$

But $12 \equiv 5 \bmod 7$

$\therefore$ $$2^{100} + 3^{100} + 4^{100} + 5^{100} \equiv 5 \bmod 7$$

$\therefore$ Remainder is 5.

Example 19. *Find remainder when* $1! + 2! + 3! + \ldots + 100!$ *is divided by* 24.

Solution

$$4! \equiv 0 \bmod 24 \qquad [\because 4! = 4 \times 3 \times 2 \times 1]$$

$$5! \equiv 0 \bmod 24 = 24$$

$$100! \equiv 0 \bmod 24$$

$\therefore$ $$4! + 5! + \ldots + 100! \equiv 0 \bmod 24$$

Now, $$1! + 2! + 3! \equiv 9 \bmod 24$$

$\therefore$ $$1! + 2! + 3! + 4! + 5! + \ldots + 100! \equiv 9 \bmod 24$$

$\therefore$ 9 is the remainder when

$$1! + 2! + 3! + \ldots + 100! \text{ is divided by } 24.$$

Example 20. *Show that* $6^n \equiv 6 \ mod\ 10 \ \forall n \in N$

Solution Let $P(n): 6^n \equiv 6 \bmod 10$

$$P(1): 6 \equiv 6 \bmod 10$$

Let us assume $P(n)$ is true for $n = k$

$$6^k \equiv 6 \bmod 10$$

$\therefore$ $6^k - 6 \equiv 10\text{m}$ for some $m \in Z$

$\therefore$ $6^k \equiv 10\text{m} + 6$

Now, consider $6^{k+1} - 6 = 6^k \cdot 6 - 6 = 6(10m + 6) - 6 = 60m + 30$ or $10(6m + 3)$

Thus, 10 divides $6^{k+1} - 6$

$\therefore$ $6^{k+1} \equiv 6 \bmod 10$

Example 21. *What is the remainder when*

$1^5 + 2^5 + 3^5 + \ldots + 99^5 + 100^5$ is divided by 4 ?

Solution We observe $(2n)^5 \equiv 0 \bmod 4$

and $(2n-1)^5 \equiv (2n-1) \bmod 4$

Now $\sum\limits_{r=1}^{100} r^5 = \sum\limits_{r=1}^{50} (2r)^5 + \sum\limits_{r=1}^{50} (2r-1)^5$

$$\sum_{r=1}^{100} r^5 \equiv 0 + \sum_{r=1}^{50} (2r-1) (\text{mod } 4)$$

$$\equiv 50^2 \bmod 4 \equiv 0 \bmod 4 \qquad \left[\because \sum_{r=1}^{n} (2r-1) = n^2\right]$$

$\therefore$ Remainder is 0.

Example 22. *Find the remainder when* $(2222)^{5555}$ *is divided by* 7.

Solution $2222 \equiv 3 \bmod 7$

$$(2222)^3 \equiv 27 \bmod 7$$

and $27 \equiv -1 \bmod 7$

$\therefore$ $(2222)^3 \equiv -1 \bmod 7$

$$(2222)^{5553} \equiv (-1)^{1851} \bmod 7$$

$$(2222)^{5553} \equiv -1 \bmod 7$$

$$(2222)^2 \equiv 9 \bmod 7$$

$$(2222)^{5555} \equiv -9 \bmod 7 \quad -9 \equiv 5 \bmod 7$$

$\therefore$ $(2222)^{5555} = 5 \bmod 7$

Example 23. *Show that, if the sum the square of two whole numbers is divisible by* 3, *then each of them is divisible by* 3.

Solution Let x and y be any two integers.

$$x \equiv 0 \bmod 3$$
$$x \equiv 1 \bmod 3$$
$$x \equiv 2 \bmod 3$$

$$x^2 \equiv 0 \bmod 3$$
$$x^2 \equiv 1 \bmod 3$$
$$x \equiv 2 \bmod 3$$
$$x^2 \equiv 4 \bmod 3$$
$$x^2 \equiv 1 \bmod 3$$
$$x \equiv 1 \bmod 3$$
$$x^2 \equiv 1 \bmod 3$$
$$y^2 \equiv 0 \bmod 3$$
$$y^2 \equiv 1 \bmod 3$$
$$x^2 + y^2 \equiv 0 \bmod 3 \quad \text{...(i)}$$
$$x^2 + y^2 \equiv 1 + 1 = 2 \bmod 3 \quad \text{...(ii)}$$
$$x^2 + y^2 = 1 \bmod 3 \quad \text{...(iii)}$$

In Eqs. (ii) and (iii), $x^2 + y^2$ is not a multiple of 3. In Eq. (i) $x^2 + y^2$ is multiplying of 3. But Eq. (i) is the result of adding $x^2 \equiv 0 \pmod 3$ and $y^2 \equiv 0 \pmod 3$ implying both x^2 and y^2 and hence both are divisible by 3.

Example 24. *Find the last digit of* 43^{17}.

Solution For finding last digit always use congruence with 10.

$$43 \equiv 3 \bmod 10$$
$$(43)^{17} \equiv 3^{17} \bmod 10$$

i.e., last digit of $(43)^{17}$ is same as 3^{17}

Now,
$$3^4 = 81 \equiv 1 \bmod 10$$
$$3^{17} = 3^{4 \times 4 + 1} \equiv 1 \times 1 \times 1 \times 1 \times 3 \bmod 10$$
$$3^{17} \equiv 3 \bmod 10$$

$\therefore$ Number in unit place is 3.

Example 25. *Find the last two digit of* 3^{1997}.

Solution This is same as asking what is remainder when $3^{1997} \div 100$

$$3^4 \equiv 81 \bmod 100$$
$$3^8 \equiv 61 \bmod 100$$
$$3^{12} \equiv 41 \bmod 100$$
$$3^{16} \equiv 21 \bmod 100$$
$$3^{20} \equiv 1 \bmod 100$$

Now, $3^{40}, 3^{60}, 3^{80}, 3^{100}, \dots, 3^{1980}$ all are $\equiv 1 \bmod 100$

We know,
$$3^{16} \equiv 21 \bmod (100)$$
$$3^{17} \equiv 21 \times 3 \bmod 100$$
$$3^{17} \equiv 63 \bmod 100$$

$\therefore$
$$3^{1997} = 3^{1980} \times 3^{17}$$

$\because$
$$3^{1980} \equiv 1 \bmod 100$$

and $$3^{17} \equiv 63 \bmod 100$$

$\therefore$ $$3^{1997} \equiv 63 \bmod 100$$

$\therefore$ Last two digit's is 63.

Example 26. *Find the unit digit and ten's digit of* $1! + 2! + 3! + 4! + \ldots + 1997!$.

Solution Let $S = 1! + 2! + 3! + 4! + \ldots + 1997!$

From 5! all the numbers will have unit digit 0 and also $1! + 2! + 3! + 4! = 33$

$\therefore$ Unit digit of S is 3

Now, from 10! all the unit and ten's digit will be zero and also

$$1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! = 33 + 120 + 720 + 5040 + 40320 + 362880$$

So, to get the ten's digit of S add only the tens digit of $33 + 120 + \ldots + 362880$ which is

$$3 + 2 + 2 + 4 + 2 + 8 = 21$$

$\therefore$ Tens digit is 1.

Example 27. *Find last two digit of* $(1! + 2! + 3! + \ldots + 100!)^2$

Solution We know that,

$$1! + 2! + 3! + \ldots + 100! \equiv 13 \bmod 100$$

$\Rightarrow$ $$(1! + 2! + 3! + \ldots + 100!)^2 \equiv 169 \bmod 100$$

$\Rightarrow$ $$169 \equiv 69 \bmod 100$$

$\Rightarrow$ $$(1! + 2! + 3! + \ldots + 100!)^2 \equiv 69 \bmod 100$$

$\therefore$ Last two digit is 69.

Example 28. *Find the remainder when* $P = 1^1 + 2^2 + 3^3 + 4^4 + \ldots + 50^{50}$ *is divided by* 8.

Solution All odd number are going to be of the type $4n \pm 1$.

Now, $$(4n \pm 1)^2 = 16n^2 \pm 8n + 1$$

Thus, any even power of $4n \pm 1$ will leave remainder 1. Thus, odd power of $4n \pm 1$ will leave remainder $4n \pm 1$ *i.e.*, 1, 3, 5, 7 when we reduce the remainder less than 8.

Now in this problem all terms with odd base will leave remainder 1,3, 5, 7, ... and so on. There will be 6 sets of (1, 3, 5, 7) as remainder and one extra 1 as remainder from 49^{49}.

Since in the expression all the terms are added we will have the sets of $(1 + 3 + 5 + 7)$ leaving zero as the remainder. Thus, the remainder from all terms with odd base is 1.

Terms with even base 2^2 will leave remainder 4 and all terms from 4^4 will be fully divisible by 8 as they would be multiple of 2^3.

Thus, net remainder is, $1 + 4 = 5$.

Example 29. *Show that* $(1! + 2! + 3! + 4!)^5$ *is of the form* $5k + 3$.

Solution We know if p is prime, then

$$(a + b + c)^p = a^p + b^p + c^p + m(p) \qquad \text{[Fermat theorem]}$$

$\therefore$ $$(1! + 2! + 3! + 4!)^5 = (1!)^5 + (2!)^5 + (3!)^5 + (4!)^5 + m(5)$$

Now $$1^5 \equiv 1 \bmod 5$$

$$(2!)^5 \equiv 2 \bmod 5$$

$$(3!)^5 \equiv 1 \bmod 5$$

$$(4!)^5 \equiv 4 \bmod 5$$

$\therefore$ $$(1!)^5 + (2!)^5 + (3!)^5 + (4!)^5 \equiv 8 \bmod 5$$

or $$(1!)^5 + (2!)^5 + (3!)^5 + (4!)^5 \equiv 3 \bmod 5$$

$\therefore$ $$(1! + 2! + 3! + 4!)^5 = 3 \bmod 5$$

$\therefore$ $$(1! + 2! + 3! + 4!)^5 \text{ is of the form } 5k + 3$$

Example 30. *Prove that* $\sum_{n=14}^{100} n!$ *is divisible by* 1001 *but not by* 1000.

Solution $1001 = 7 \cdot 11 \cdot 13$

$\therefore$ Every term of $\sum_{n=14}^{100} n!$ is divisible by 1001

The highest power of 5 which divides the first term is 5^2 $\left(\because \left[\frac{14}{5}\right] + \left[\frac{14}{5^2}\right] + \ldots = 2\right)$

$\therefore$ 1000 does not divide 14 ! but every other term divisible by 1000.

$\therefore$ $$\sum_{n=14}^{100} n! \text{ is not divisible by 1000.}$$

Example 31. $(1 + 10!)(1 + (10!)^2)(1 + (10!)^3)\ldots(1 + (10!)^{100})]^{100}$ *is divided by* 10!. *what is the remainder?*

Solution

$$(1 + x)^n \equiv 1 \bmod x$$
$$(1 + x^2)^n \equiv 1 \bmod x^2$$
$$\vdots$$
$$(1 + x^k)^n \equiv 1 \bmod x^k$$

....

....

....

$\therefore$ $$[(1 + x)(1 + x^2)\ldots(1 + x^k)]^n \equiv 1 \bmod x$$

Where x is HCF of $x, x^2, \ldots, x^k$.

If we put $x = 10!$, then $[(1 + 10!)(1 + (10!)^2)(1 + (10!)^3)\ldots(1 + (10!)^{100})]^{100} \equiv 1 \bmod (10!)$

Therefore, 1 is the remainder.

Example 32. *Let* $n = 640640640643$, *without actually computing* n^2. *Prove that* n^2 *leave a remainder* 1 *when divided by* 8.

Solution Since, 640640640000 is a multiple of 8 $643 \equiv 3 \pmod 8$

$\therefore$ $n = 8k + 3$ for some positive integer k.

$\therefore$ $n^2 = 64k^2 + 48k + 9 = 1 + a$ multiple of 8.

Example 33. *Prove that* $n^{16} - 1$ *is divisible by* 17, *if* $(n, 17) = 1$.

Solution $(n, 17) = 1$ and 17 is a prime number.

$\therefore$ By Fermat theorem $n^{17-1} \equiv 1 \bmod 17$

[If p is prime and $(a, p) = 1$, then $a^{p-1} \equiv 1 \bmod p$] or $n^{16} \equiv 1 \bmod 17$.

or $17 \mid (n^{16} - 1)$ *i.e.,* $(n^{16} - 1)$ is divisible by 17.

Example 34. *If a and b are coprime to the prime number p, then show that $a^{p-1} - b^{p-1} = M(p)$.*

Solution $\because$ $(a, p) = 1$ and p is prime.

$\therefore$ By Fermat theorem, $a^{p-1} \equiv 1 \bmod p$ ...(i)

Similarly $b^{p-1} \equiv 1 \bmod p$ ...(ii)

Subtracting congruence (i) and (ii)

$$a^{p-1} - b^{p-1} \equiv 0 \bmod p$$

$\therefore$ p divides $a^{p-1} - b^{p-1}$

$\therefore$ $a^{p-1} - b^{p-1}$ is a multiple of p.

Example 35. *If p is a prime number, show that the difference of the pth powers of any two numbers exceeds the difference of the numbers by a multiple of p.*

Solution Let x, y be two numbers

Difference of the pth powers of these numbers

$$= x^p - y^p$$

Difference of these number $= x - y$

We have to show that $(x^p - y^p) - (x - y)$ is a multiple of p.

i.e., $x^p - y^p \equiv (x - y) \bmod p$.

$\because$ p is a prime.

$$x^p \equiv x \bmod p \quad \text{...(i)}$$

$$y^p \equiv y \bmod p \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$x^p - y^p \equiv (x - y) \bmod p$$

Example 36. *Show that $n^7 \equiv n \bmod 42$.*

Solution $a^p \equiv a \bmod p, n^7 \equiv n \bmod 7$ [$\because$ 7 is prime number]

i.e., $n^7 - n$ is divisible by 7 ...(i)

Again,

$$\begin{aligned} n^7 - n &= n(n^6 - 1) \\ &= n(n^3 - 1)(n^3 + 1) \\ &= n(n-1)(n^2 + n + 1)(n + 1)(n^2 - n + 1) \\ &= (n-1)n(n+1)(n^2 + n + 1)(n^2 - n + 1) \end{aligned}$$

Now, $(n-1)n(n+1)$ being the product of three consecutive integers is divisible by $3! = 6$ and hence

$$n^7 - n = (n-1)n(n+1)(n^2 + n + 1)(n^2 - n + 1) \text{ is divisible by } 6 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $n^7 - n$ is divisible by $42 (= 6 \times 7)$

and $(6, 7) = 1$

{We know that if $a|c$, $b|c$ and $(a, b) = 1$, then $ab|c$}

Example 37. *Show that 4th power of every number is of the form $5k$ or $5k+1$ where k is any positive integer.*

Solution Let a be any number.

$\because$ 5 is prime and a is any number

$\therefore$ either $(a, 5) = 1$ or $5|a$

Case I $(a, 5) = 1$

By Fermat theorem, $a^4 \equiv 1 \bmod 5$

$\therefore \quad 5 | a^4 - 1$

$\therefore \quad a^4 - 1 = 5k$ or $a^4 = 5k + 1$

Case II $5/a$

$\therefore \quad 5 | a^4$

$\therefore \quad \exists$ an integer k such that $a^4 = 5k$

Hence the result.

Example 38. *If m is a prime number and a, b are two numbers less than m, prove that*

$$a^{m-2} + a^{m-3}b + a^{m-4}b^2 + \ldots + b^{m-2}$$

is a multiple of m.

Solution $\because$ m is prime and a, b are two numbers both less than m.

$\therefore \quad (a, m) = 1$ and $(b, m) = 1$

[$\because$ A prime number is coprime to every number less than it]

$\therefore$ By Fermat theorem,

$$a^{m-1} \equiv 1 \bmod m \quad \ldots(i)$$

$$b^{m-1} \equiv 1 \bmod m \quad \ldots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$a^{m-1} - b^{m-1} \equiv 0 \bmod m$$

$\therefore \quad m | (a^{m-1} - b^{m-1})$

or $\quad m | (a-b)[a^{m-2} + a^{m-3}b + a^{m-4}b^2 + \ldots + b^{m-2}]$

But $\quad (m, a-b) = 1 \quad$ [$\because a < m, b < m \Rightarrow a - b < m$ and m is prime]

$\therefore$ By Gauss Theorem

$m | [a^{m-2} + a^{m-3}b + \ldots + b^{m-2}] \quad$ [$\because$ If $a | bc$ and $(a, b) = 1$; then $a | c$]

$\therefore \quad (a^{m-2} + a^{m-3}b + \ldots + b^{m-2})$

is a multiple of m.

Example 39. *If p is prime, prove that $a^p(p-1)! + a$ is divisible by p.*

Solution $\because$ p is prime

By Wilson Theorem, $\quad (p-1)! \equiv -1 \bmod p \quad \ldots(i)$

Again, $\because$ p is prime

Now $\quad a^p \equiv a \bmod p \quad \ldots(ii)$

Multiplying Eqs. (i) and (ii)

$$a^p(p-1)! \equiv -a \bmod p$$

$\Rightarrow \quad a^p(p-1)! + a$ is divisible by p.

Example 40. *If p is a prime number, show that*

$$1^{p-1} + 2^{p-1} + 3^{p-1} + \ldots + (p-1)^{p-1} + 1 \equiv 0 \text{ mod } p.$$

Solution $\because$ p is a prime number and 1, 2, 3, ..., are all less than p.

$$\therefore \quad (1, p) = 1\,; (2, p) = 1\,; (3, p) = 1\,; \ldots (p-1, p) = 1$$

[$\because$ A prime number is coprime to every number less than it]

$\therefore$ Putting $a = 1, 2, 3, \ldots (p-1)$ in Fermat theorem

$$a^{p-1} \equiv 1 \text{ mod } p$$
$$1^{p-1} \equiv 1 \text{ mod } p$$
$$2^{p-1} \equiv 1 \text{ mod } p$$
$$3^{p-1} \equiv 1 \text{ mod } p$$
$$\vdots \quad \vdots \quad \vdots$$
$$(p-1)^{p-1} \equiv 1 \text{ mod } p$$

Adding up all these $(p-1)$ congruence, we get

$$1^{p-1} + 2^{p-1} + (p-1)^{p-1} + \ldots + \equiv (p-1) \text{ mod } p$$

Also $0 \equiv p$ mod p

Subtracting these congruence

$$1^{p-1} + 2^{p-1} + 3^{p-1} + \ldots + (p-1)^{p-1} \equiv -1 \text{ mod } p$$

or
$$1^{p-1} + 2^{p-1} + 3^{p-1} + \ldots + (p-1)^{p-1} + 1 \equiv 0 \text{ mod } p$$

Example 41. *Show that no square number is of the form* $3n - 1$.

Solution Let us suppose if possible $N^2 = 3n - 1$, N^2 represents any square number.

Then, $\quad N^2 + 1 = 3n$

i.e., $N^2 + 1$ is divisible by 3.

But $\quad N^2 + 1 = (N^2 - 1) + 2$

Now $N^2 - 1$ is divisible by 3 when N is prime to 3 (Fermat theorem)

Thus, $N^2 + 1$ exceeds a multiple of 3 by 2 and as such N^2 must be of the form $3n + 1$ if N is not prime to 3

$\therefore$ It must be of the form $n^2 = 3n$

Thus, no square number is of the form $3n - 1$.

Example 42. *Prove that every even power of every odd number is of the form* $8r + 1$.

Solution Any odd number is of the form $2p + 1$ and every even number is of the form $2n$

$$\therefore \quad (2p+1)^{2n} = [(2p+1)^2]^n = (4p^2 + 4p + 1)^n$$
$$= [4p(p+1) + 1)]^n \quad \ldots \text{(i)}$$

Since $p(p+1)$ is always even.

$\therefore$ $p(p+1)$ is always even

Also, $p(p+1)$ is of the form $2k$.

$\therefore$ A becomes $(8k+1)^n$.

Now, $(8k+1)^n$ is always of the form $8k + 1$.

Example 43. *If m and n are positive integer. If $(mn-1)$ is divided by n, then remainder is always $n-1$.*

Solution We can write $m\,n - 1 = n(m-1) + n - 1$

$\therefore$ Remainder is always $n - 1$

Example 44. *Prove that $p!$ and $(p-1)! - 1$ are relatively prime if p is an odd prime.*

Solution Let $(p!, (p-1)! - 1) = d$

$\therefore$ $d \mid p!$ and $d \mid (p-1)! - 1$

$\therefore$ There exist integers k and k such that

$$p! = kd \text{ or } p(p-1)! = kd \quad \text{...(i)}$$

and $(p-1)! - 1 = k'd$

or $(p-1)! = k'd + 1 \quad$ ...(ii)

Putting the value of $(p-1)!$ from Eq. (ii) in Eq. (i)

$$p(k'd + 1) = kd \quad \text{or} \quad pk'd + p = kd$$

or $d(k - pk') = p$

$\therefore$ $d \mid p$

$\therefore$ $d = 1$ or $d = p$ $\quad$ [$\because$ p is a prime number.]

If possible, let $d = p$

$\therefore$ $d \mid (p-1)! - 1 \Rightarrow p \mid (p-1)! - 1 \quad$ ...(iii)

But by Wilson theorem $(p-1)! + 1 \equiv 0 \bmod p$

i.e., $p \mid (p-1)! + 1 \quad$ ...(iv)

From Eqs. (iv) and (iii)

$$p \mid (p-1)! + 1 - (p-1)! + 1$$

i.e., $p \mid 2$ which is impossible. ($\because$ p is an odd prime and hence ≥ 3)

$\therefore$ $d = 1$

i.e., $(p!, (p-1)! - 1) = 1$

i.e., $p!$ and $(p-1)! - 1$ are relatively prime.

Example 45. *Find the number of positive integers $n \leq 1991$ such that 6 is a factor of $n^2 + 3n + 2$.*

Solution $n^2 + 3n + 2 \equiv 0 \pmod 6$

$\Leftrightarrow$ $(n+1)(n+2) \equiv 0 \bmod 2$

and $(n+1)(n+2) \equiv 0 \bmod 3$

$\Leftrightarrow$ $(n+1)(n+2) \equiv 0 \bmod 3$

Note : $(n+1)(n+2)$ is even for all $n \Leftrightarrow 3 \mid n$

$\therefore$ Required number $= 1991 - \left[\dfrac{1991}{3}\right] = 1991 - 663 = 1328$

Example 46. *Let a, b be odd integers and n a natural number. Prove that $a - b$ is divisible by 2^n if and only if $a^3 - b^3$ is divisible by 2^n.*

Solution $a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)\{(a-b)^2 + 3ab\}$

Since a, b are odd integers therefore $3ab$ is an odd integer. Also $a - b$ is an even integer and consequently $(a-b)^2 + 3ab$ is an odd integer.

$\therefore$ $a^3 - b^3$ is divisible by 2^n if and only if $a - b$ is divisible by 2^n.

Example 47. *Let a, b be odd integers. If 4 does not divide a – b, then prove that 4 cannot divide $a^3 - b^3$.*

Solution Suppose that 4 does not divide $a - b$.

Now,
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
$$= (a - b)[(a - b)^2 + 3ab] \quad \ldots\text{(i)}$$

Since, a and b are both odd, $3ab$ is odd .

Also, $a - b$ being even, $(a - b)^2$ is a multiple of 4 $(a - b)^2 + 3ab$ is odd .

Since, $(a - b)$ is not a multiple of 4, it follows from Eq. (i) that $a^3 - b^3$ is not a multiple of 4.

Example 48. *Prove that the sum of first n natural numbers $(n \geq 3)$ is never a prime.*

Solution Sum of first n natural number is $\frac{1}{2}n(n + 1)$, which is obviously a composite number.

(If n is even, it can be factorized as $\left(\frac{1}{2}n\right)(n + 1)$; if n is odd, it can be factorized as $n\frac{1}{2}(n + 1)$, which cannot be zero).

Example 49. *Consider any 3 consecutive natural numbers the smallest of which is greater than 3. Then prove that the square of the largest cannot be the sum of the squares of the other two.*

Solution Let $n, n + 1, n + 2$ be three consecutive natural numbers with $n > 3$
$$(n + 2)^2 - [n^2 + (n + 1)^2] = 3 + 2n - n^2 = -(n + 1)(n - 3)$$

Which cannot be zero since $n > 3$.

Example 50. *Prove that every natural number $n \geq 12$ is a sum of two composite numbers.*

Solution If n is odd, write $n = (n - 9) + 9$

Since n is odd, $\therefore n - 9$ is even.

Also $n \geq 12 \Rightarrow n - 9 \geq 3$, so that $n - 9$ is an even number greater than 2 and consequently $n - 9$ is a composite number. Also 9 is a composite number. Thus, n is sum of two composite numbers. If n is even, $n = (n - 4) + 4$

Since $n \geq 12$, $n - 4$ are both composite even numbers. Thus, n is the sum of two composite even numbers.

Example 51. *Prove that $n^4 + 4$ is a composite number for each natural number n is greater than 1.*

Solution
$$n^4 + 4 = (n^4 + 4n^2 + 4) - 4n^2$$
$$= (n^2 + 2)^2 - (2n)^2$$
$$= (n^2 + 2 + 2n)(n^2 + 2 - 2n)$$

Showing that $n^4 + 4$ is a composite number.

Example 52. *Prove that $n^4 + 4^n$ is a composite number for all integer values of $n > 1$.*

Solution If n is even, $n^4 + 4^n$ is divisible by 4

$\therefore$ It is composite number

If n is odd, suppose $n = 2p + 1$, where p is positive integer

Then,
$$n^4 + 4^n = n^4 + 4.4^{2p} = n^4 + 4(2^p)^4$$

which is of the form $n^4 + 4b^4$, where b is a positive integer $(= 2^p)$

$$n^4 + 4b^4 = (n^4 + 4b^2 + 4b^4) - 4b^2$$
$$= (n^2 - 2b^2)^2 - (2b)^2$$
$$= (n^2 + 2b + 2b^2)(n^2 - 2b + 2b^2)$$

We find that $n^4 + 4b^4$ is a composite number consequently $n^4 + 4^n$ is composite when n is odd.

Hence, $n^4 + 4^n$ is composite for all integer values of $n > 1$.

Example 53. *Prove that there is an infinitely of numbers m with the property that $n^4 + m$ is composite for every natural number n.*

Solution If $m = 4k^4$, where k is any natural number whatever, then $n^4 + m = n^4 + 4k^4$

$$= n^4 + 2.n^2(2k^2) + (2k^2)^2 - (2nk)^2$$
$$= (n^2 + 2k^2)^2 - (2nk)^2$$
$$= (n^2 + 2k^2 + 2nk)(n^2 + 2k^2 - 2nk)$$

Showing that $n^4 + m$ is a perfect square.

Example 54. *Find the sum of all integers n that satisfy the properties*

(*a*) $1 \le n \le 1994$ (*b*) 30 *divides* $n^3 + 15n^2 + 50n$

Solution Let $P - n(n^2 + 15n + 590)$

$$= n(n^2 + 3n + 2 + a \text{ multiple of } 6)$$
$$= n(n + 1)(n + 2) + a \text{ multiple of } 6$$

Since the product of any three consecutive natural numbers is divisible by 6, it follows that P is a multiple of 6. Also P is a multiple of 5 iff n^3 is a multiple of 5, which is true iff n is a multiple of 5 (as 5 is prime). Thus, we find that P is a multiple of 30 iff P is a multiple of 5.

We have to find the sum of all multiple of 5 not exceeding 1995.

There are 398 such numbers, namely

$$5, 10, 16, \dots, 1990$$

Sum of all these numbers is $\frac{398}{2}(2.5 + 397.5) = \frac{5.398.399}{2} = 397005$

Example 55. Show that, if the difference of two consecutive cubes is a square, then it is the square of the sum of two successive squares.

Solution Suppose $a \in Z$ such that

$$a^2 = (n + 1)^3 - n^3$$

Then, $$a^2 = 3n(n + 1) + 1 \quad \dots(i)$$

Now, $$4a^2 - 1 = 12n(n + 1) + 3 = 3(4n^2 + 4n + 1)$$

$\Rightarrow$ $$(2a - 1)(2a + 1) = 3(2n + 1)^2$$

$\therefore$ $$3 | 2a - 1 \text{ or } 2a + 1$$

If $3 | 2a - 1$, then $2a - 1 = 3m$ for some $m \in Z$, then

$$3m(2a + 1) = 3(2n + 1)^2$$

$\Rightarrow$ $$n(2a + 1) = (2n + 1)^2$$

Since, m and $2a + 1$ are relatively prime.

[$\because (2a+1, 2a-1) = 1$], $2a+1$ is an odd square, say $(2r+1)^2$. Then $a = 2r^2 + 2r$ is even, a contradiction because by Eq. (i), a^2 is odd.

$\therefore$ $$3 \mid 2a+1.$$

Now proceeding as above we see that $2a - 1 = (2r+1)^2$ for some $r \in Z$

i.e., $$a = r^2 + (r+1)^2$$

Example 56. *Show that the sum of any two consecutive odd primes is the product of at least three primes which need not be distinct.*

Solution Let p_1 and p_2 are two consecutive odd primes, since $p_1 + p_2$ is even, $\frac{1}{2}(p_1 + p_2) \in Z$. Further $\frac{1}{2}(p_1 + p_2)$ being in between the consecutive primes p_1 and p_2, is a composite number.

$$p_1 + p_2 = 2\,\frac{p_1 + p_2}{2} \text{ has at least three prime factors.}$$

Example 57. *Prove that if p and $8p-1$ are prime numbers then $8p+1$ is a compsite number.*

Solution When $p = 3$, $8p + 1 = 25$, a composite number otherwise $3 \mid (8p-1)8p$.

$\Rightarrow$ $$3 \mid 8p+1 \qquad [\because 3 \mid (8p-1)8p(8p+1)]$$

Hence, $8p+1$ is not prime.

Example 58. *Let $p(x) = x^2 + 40$. Show that for any two integers a, b either $p(a) + p(b)$ or $p(a) - p(b)$ is composite.*

Solution Since, $p(x) = p(-x)$ for any real x, we can assume that a and b are non negative and also that $a > b$. Since, $p(a) - p(b) = (a-b)(a+b)$ if $a - b > 1$, then $p(a) - p(b)$ is composite. So, assume $a - b = 1$.

Now, $$p(a) - p(b) = (b+1) + b = 2b + 1$$

and $$p(a) + p(b) = (b+1)^2 + 40 + b^2 + 40$$
$$= 2b^2 + 2b + 81 = 2b(b+1) + 81$$

If $3 \mid b(b+1)$, then $p(a) + p(b)$ is composite otherwise $3 \mid b - 1$ and hence $3 \mid 2(b-1) + 3$

i.e., $$3 \mid p(a) - p(b)$$

$\therefore$ $p(a) - p(b)$ is composite if $b \neq 1$; when $b = 1$

$p(a) + p(b) = 85$ and so it is composite.

Example 59. *Find the smallest integer> 1 which is simultaneously a square, a cube, a fourth power and a fifth power (of certain integers).*

Solution If a positive integer $N > 1$ is to be simultaneously a square a cube, a fourth power and a fifth power (of certain integers) it must be a kth power of some integer where k = LCM of 2, 3, 4, 5 = 60. If N is to be the smallest such integer, then N must be 2^{60}.

Example 60. *Prove that ten's digit of any power of 3 is even.*

Solution For any $n \in N$, let

$$v_n = \begin{Bmatrix} 3 \text{ if } n \equiv 1 \bmod 4 \\ 9 \text{ if } n \equiv 2 \bmod 4 \\ 7 \text{ if } n \equiv 3 \bmod 4 \\ 1 \text{ if } n \equiv 4 \bmod 4 \end{Bmatrix}$$

Note that $3u_n = u_{n+1} \bmod 20$

Since, $20 | (3 - u_1)$ and for any $n \in N$

$$20 | (3^n - u_n) \Rightarrow 20 | (3^{n+1} - 3u_n)$$

$$\Rightarrow \qquad 20 | (3^{n+1} - u_{n+1})$$

It follows by induction, $20 | 3^n - u_n$ for all $n \in N$. Hence, ten's digit of 3^n is even.

(Note that unit's digit of 3^n is u_n)

Example 61. *The number of positive integral values of n of which $n^3 - 8n^2 + 20n - 13$ represents a prime number.*

Solution If $\qquad y = n^3 - 8n^2 + 20n - 13$

Then, $\qquad y = (n^2 - 7n + 13)(n - 1)$

for $n = 1, y = 0$ not admissible
for $n = 2, y = 3$, a prime number
for $n = 4, y = 3$, a prime number
for $n > 4, n - 1 > 3$ is a factor of y
Hence, y is not prime.
Number of integral values of n for which y is prime is 3.

Example 62. *Find the sum of all the digits of the result of the subtraction* $10^{99} - 99$.

Solution

$$\begin{array}{r} 1000000........0000 \\ -\ 99 \\ \hline 999999..........9901 \\ \hline \end{array}$$

(1 followed by 99 zeros)
The result is a 99 digit number having 97 nines followed by a zero and a 1.
$\therefore$ The sum of the digits $= (97 \times 9) + 0 + 1 = 874$

Example 63. *Find all integers n for which* $\dfrac{n^2 + 13n + 2}{3n + 5}$ *also is an integer.*

Solution On dividing $n^2 + 13n + 2$ by $3n + 5$, we get $n|3 + 34|9$ as quotient and remainder $-152|9$. So if

$$\frac{(n^2 + 13n + 2)}{3n + 5} = m$$

Then, $\qquad 9m(3n + 5) = 9(n^2 + 13n + 2) = (3n + 34)(3n + 5) - 152$

$$\Rightarrow \qquad 152 = (3n + 5)(3n + 34 - 9m)$$

If m is an integer, then $3n + 5$ is a divisor of $152 = 8 \times 19$.
The divisors of 152 are $\pm 1, \pm 2, \pm 4, \pm 8, \pm 19, \pm 38, \pm 76$ and ± 152.

$$3n + 5 = \pm 1, \pm 2, \pm 4, \pm 8, \pm 19, \pm 38, \pm 76, \pm 152$$

$$3n + 34 - 9m = \pm 152, \pm 76, \pm 38, \pm 19, \pm 8\ \pm 4, \pm 2, \pm 1$$

Considering the '+' signs first.

$$9m - 29 = -151, -74, -34, -11, 11, 34, 74, 151$$

$$9m = -122, -45, -5, 18, 40, 63, 103, 180$$

So the possible values of m are -5, 2, 7 and 20; these correspond to $3n + 5 = 2, 8, 38$ and 152.

Thus, $$n = -1, 1, 11, 49.$$

Next consider '-' signs.

$$9m - 29 = 151, 74, 34, 11, -11, -34, -74, -151$$

These give the same possible values of m, namely 20, 7, 2, and -5 these corresponds to $3n + 5 = -1, -4, -19$ and -76. Thus, $n = -2, -3, -8$ and -27.

Thus, desired values of n are 1, 11, 49, -1, -2, - 3, - 8 and - 27. Possible values of m are -5, 2, 7 and 20.

Example 64. *Each of the numbers* $x_1, x_2, \ldots, x_n$ *equal* 1 *or* -1

and
$$x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + \ldots + x_{n-1}x_nx_1x_2 + x_nx_1x_2x_3 = 0.$$

Prove that n is divisible by 4.

Solution $y_i = x_i + 1x_i + 2x_{i+3}$ for $i = 1, 2, \ldots, n.$ where $x_{n+1} = x_1, x_{n+2} = x_2$ and $x_{n+3} = x_3$

Then, $$y_1 + y_2 + \ldots + y_n = 0$$

$y_i = \pm 1$ for each i.

Suppose that $y_i = 1$ for n_1 values of i and $y_i = -1$ for n_2 values of i.

Then, $$n_1 + n_2 = n$$

$$0 = y_1 + y_2 + \ldots + y_n = n_1(1) + n_2(-1) = n_1 - n_2$$

Hence, $$n_1 = n_2, n = 2n_2$$

and $$y_1y_2 \ldots y_n = (1)^{n_1}(-1)^{n_2} = (-1)^{n_2}$$

But $$y_1y_2 \ldots y_n = x_1^4x_2^4x_3^4 \ldots x_n^4 = 1$$

This shows that n_2 is even.

Hence, $n = 2n_2$ is divisible by 4.

Example 65. *Which of the numbers* 101, 10101, 1010101, ... *with alternating* 0's *and* 1's *beginning and ending with* 1 *can be primes?*

Solution Let $N_2 = 101, N_3 = 10101$ and in general

$$N_k = 1010\ldots01 = 1 + 100 + \ldots + 100^{k-1} = \frac{100^k - 1}{100 - 1}$$

Here N_k involves k 1's.

If k is odd, then $$N_k = \left(\frac{10^k - 1}{10 - 1}\right)\left(\frac{10^k + 1}{10 + 1}\right)$$

This is composite since $10^k + 1$ is divisible by $10 + 1$ if k is odd.

If k is even and $k = 2r$, then $N_k = \dfrac{100^{2r} - 1}{99} = \left(\dfrac{100^r - 1}{99}\right)(100^r + 1)$

This is composite if $r > 1$. Thus,only one number in the list, namely $N_2 = 101$, is prime, the remaining are composite.

Example 66. Find all integers n (positive, negative or zero) such that $n^2 + 73$ is divisible by $n + 73$.

Solution Let $n + 73 = m$.

Then, $$n^2 + 73 = (m - 73)^2 + 73 = m^2 - 146m + 73^2 + 73$$

This is divisible by m if and only if $73^2 + 73$ is divisible by m. Since $73^2 + 73 = 73 \times 37 \times 2$.

We must have $\pm m = 1, 2, 37, 2.37, 73, 2.73, 37.73$ or $2.37.73$.

These give 16 values for m and hence 16 values for n given below.

$$-72, -71, -36, \quad 1, \quad 0, \quad 73, \quad 2628, \quad 5329$$
$$-74, -75, -110, -147, -146, -219, -2774, -5475$$

Example 67. *Find all positive integers which can be expressed as the sum of three distinct composite numbers.*

Solution If n is the sum of three distinct composite number, then $n \geq 4 + 6 + 8 = 18$

Let $$n \geq 18$$

If n is even then $n - 10$ is an even number > 6 and $n = 4 + 6 + (n - 10)$. If n is odd then $n - 13$ is an even number greater than 4 and $n = 4 + 9 + (n - 13)$. So the desired positive integers are all $n \geq 18$.

Example 68. *If p is a prime other than* 2 *and* 5, *prove that p divides an infinite number of the numbers* 1, 11, 111, 1111, ...

Solution Let N_k the kth number in the sequence.

Then, $$N_k = (10^k - 1)|9.$$

Given that 10 is not divisible by p.

Hence, by Fermat's theorem $10^{p-1} \equiv 1 \pmod p$

Thus, $$10^{(p-1)r} \equiv 1 \bmod p \text{ for } r = 1, 2, 3, \ldots$$

Let $$10^{(p-1)r} - 1 = mp. \text{ If } p \neq 3, \text{ then } (9, p) = 1$$

Since $10^k - 1$ is divisible by 9 for all k, we see that $9|m$. Therefore $N^k = \dfrac{10^k - 1}{9} = \left(\dfrac{m}{9}\right)p$

when $$k = (p-1)r, r = 1, 2, \ldots$$

Since the sum of digit in N_k is k we see that N_k is divisible by 3 when $k = 3r, r = 1, 2, \ldots$

Example 69. *Find the remainder obtained, when the number* $10^{10} + 10^{(10^2)} + \ldots + 10^{(10^{10})}$ *is divided by* 7.

Solution By Fermat's theorem $10^6 \equiv 1 \bmod 7$

Hence, $$10^{6m} \equiv 1 \bmod 7 \text{ for all } m.$$

Now $$10 \equiv 4 \bmod 6, 10^2 \equiv 40 \equiv 4 \bmod 6$$

By induction $10^n \equiv 4 \bmod 6$ for all n.

Thus, $$10^n = 6m + 4$$

and $$10^{(10n)} = 10^{6m} . 10^4 \equiv 10^4 \bmod 7 \equiv 4 \bmod (7)$$

consequently $$10^{10} + 10^{(10^2)} + \ldots + 10^{(10^{10})} \equiv 4 \times 10 \bmod 7 \equiv 5 \bmod 7$$

The remainder is 5.

Example 70. *Find a positive integer n such that* $7n^{25} - 10$ *is divisible by* 83.

Solution Since, $7 \times 37 = 259 \equiv 10 \bmod 83$

We have to find a value of n such that $7n^{25} \equiv 7 \times 37 \bmod 83$

This is equivalent to $n^{25} \equiv 37 \equiv 2^{20} \bmod 83$

By Fermat's theorem $2^{82k} \equiv 1 \bmod 83$ for all k. So it is enough, if we choose n such that $n^{25} \equiv 2^{82k+20} \bmod 83$ If $k = 15$, this will be satisfied if $n^{25} \equiv 2^{1250} \bmod 83$ and so if $n = 2^{50}$.

This gives one value of n.

Example 71. *If a and b are positive integers, then prove that* $\frac{(2a)!(2b+1)!}{(a!)(b!)(a+b+1)!}$ *is an integer.*

Solution Let p be any prime, we will show that highest power of p that divides the numerator is $\geq$ the highest power of p that divides the denominator thus showing that entire denominator divides numerator exactly .

Let
$$D = (a)!(b)!\,(a+b+1)!$$
$$N = (2a)!(2b+1)!$$

p^a the highest power of prime p which divides D p^b the highest power of p which divides N. Then, D divides N if for every prime p, $a \leq b$.

Now the greatest exponent α such that p^α divides k' is $\alpha = \sum_{j=1}^{\infty}\left[\frac{k}{p^j}\right]$ $[x]=$ integral part of x.

Suppose during factorization of D and N, the prime p occurs a and b times respectively.

$$a = \sum_{j=1}^{\infty}\left(\left[\frac{a}{p^j}\right]+\left[\frac{b}{p^j}\right]+\left[\frac{a+b+1}{p^j}\right]\right)$$

$$b = \sum_{j=1}^{\infty}\left(\left[\frac{2a}{p^j}\right]+\left[\frac{2b+1}{p^j}\right]\right)$$

In order to prove $a \leq b$ it is sufficient to prove

$$\left[\frac{2a}{p^j}\right]+\left[\frac{2b}{p^j}\right] \geq \left[\frac{a}{p^j}\right]+\left[\frac{b}{p^j}\right]+\left[\frac{a+b+1}{p^j}\right]$$

let $p^j = c$, then

$$\left[\frac{2a}{c}\right]+\left[\frac{2b}{c}\right] \geq \left[\frac{a}{c}\right]+\left[\frac{b}{c}\right]+\left[\frac{b}{c}\right]+\left[\frac{a+b+1}{c}\right]$$

we have proved earlier $\frac{(2a)!(2b+1)!}{(a!)(b!)(a+b+1)!}$ is an integer.

Example 72. *Determine the largest* 3 *digit prime factor of the integer* ${}^{2000}C_{1000}$.

Solution If p is any 3 digit prime, then $p^2 > 2000$. Thus, the highest power of p that divides 2000!, is $\left[\frac{2000}{p}\right]$. Highest power of p that divides 1000! is $\left[\frac{1000}{p}\right]$. Since ${}^{2000}C_{1000} = \frac{2000!}{(1000!)^2}$.

The highest power of p that divides ${}^{2000}C_{1000}$ is $\left[\frac{2000}{p}\right] - 2\left[\frac{1000}{p}\right]$. If $p > 666$, then

$$\left[\frac{2000}{p}\right] - 2\left[\frac{1000}{p}\right] = 2 - 2(1) = 0$$

i.e., p does not divide ${}^{2000}C_{1000}$.

The required prime is the largest one such that is it less than 666.

$\therefore$ $p = 661$, when $p = 661$

$$\left[\frac{2000}{p}\right] - \left[\frac{1000}{p}\right] = 3 - 2(1) = 1$$

$\therefore$ 661 is the largest 3 digit prime that divide ${}^{2000}C_{1000}$.

Example 73. *Prove that inradius of a right angled triangle with integer sides is an integer.*

Solution Let ABC be right Δ with $\angle B = 90°$, O be its incentre. L, M, N the points of contact of the in circle with the sides a, b, c respectively.

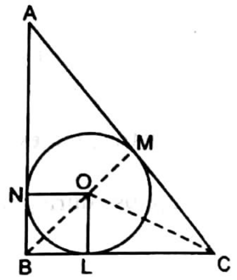
Suppose that the inradius is r. Now as $\angle ABC = 90°$, the quadrilateral $NBLO$ is a square. So $NB = BL = r$. Also as the two tangents drawn from an external point to a circle are of equal length, we have

$$AM = AN = AB - NB = c - r$$

and $$CM = CL = BC - BL = a - r$$

so, $$b = AC = AM + CM = c - r + a - r$$

$$= c + a - 2r \Rightarrow r = \frac{b - (c + a)}{2}$$



As $\angle B = 90° \Rightarrow b^2 = c^2 + a^2$ we have (i) if c and a are both odd or both even, $c^2 + a^2$ is even $\Rightarrow b^2$ is given $\Rightarrow b$ is even $\Rightarrow b - (c + a)$ is even (ii) if one of c and a is even and the other odd, $c^2 + a^2$ is odd $\Rightarrow b^2$ is odd $\Rightarrow b$ is odd $\Rightarrow b - (c + a) =$ an even number.

So, in any case if a, b, c are integers, we have $r = \frac{b - (c + a)}{2}$ = an integer.

Example 74. *Find the number of positive integers which divide* 10^{999} *but not* 10^{998}.

Solution All the positive divisors of p^r (p is a positive prime, $r \in N$) are $1, p, p^2, p^3, \dots, p^r$. That is the number of positive divisors of p^r is $r + 1$. Similarly, the number of the positive divisors of q^s (q is a positive prime, $s \in N$) is $s + 1$, so the number of positive divisors of $p^r q^s$ is $(r + 1)(s + 1)$.

Now, $10^{999} = 2^{999} \times 5^{999}$ where 2 and 5 are positive primes. So the number of positive divisors of 10^{999} is $(999 + 1)(999 + 1) = 1000^2$. For similar reasons, the number of positive divisors of 10^{998} is 999^2. So the number of positive numbers which divides 10^{999} but not 10^{998} is $1000^2 - 999^2$

$$= (1000 + 999)(1000 - 999)$$

$$= 1999 \times 1 = 1999$$

Example 75. *Let* $n = 1983$, *find the least positive integer* k *such that*

$$k(n^2)(n^2 - 1^2)(n^2 - 2^2)(n^2 - 3^2)\dots(n^2 - (n - 1)^2) = r!$$

for some positive integer r.

Solution Factorising and rearranging the resulting factors of the left side gives

$$kn(2n - 1)(2n - 2)\dots(n + 3)(n + 2)$$

$$(n + 1)(n)(n - 1)(n - 2)(n - 3)\dots(2)(1),$$

which equals $kn(2n - 1)!$. Thus, $k = 2$ is the smallest k that makes this a factorial $[(2n)!]$.

Example 76. *Find all positive integers* n *less than* 17 *for which* $n! + (n + 1)! + (n + 2)!$ *is an integral multiple of* 49.

Solution The given expression equals to

$$n!\{1 + (n + 1) + (n + 1)(n + 2)\} = n!(n + 2)^2.$$

Either 7 divides $n + 2$ or 49 divides $n!$

$\therefore$ $$n = 5, 12, 14, 15, \text{and } 16.$$

Example 77. *Prove that* $n = \dfrac{5^{125}-1}{5^{25}-1}$ *is a composite number.*

Solution Let $x = 5^{25}$, then

$$\begin{aligned}5^{125}-1 = x^5 - 1 &= (x-1)(x^4+x^3+x^2+x+1)\\ &= (x^4+9x^2+1+6x^3+6x+2x^2-5x^3-10x^2-5x)(x-1)\\ &= ((x^2+3x+1)^2-5x(x+1)^2)(x-1)\\ &= ((x^2+3x+1)^2-(5^{13}(x+1))^2(x-1)\\ &= \{x^2+3x+1+5^{13}(x+1)\}\\ &\quad \{x^2+3x+1-5^{13}(x+1)\}(x-1)\end{aligned}$$

Thus, the given number is composite.

Example 78. *Find all numbers p such that the number* p^2+11 *has exactly* 6 *divisors.*

Solution $p = 2$ does not work as $2^2+11 = 15 = 3\times 5$ has 4 divisors but not 6 divisors. So p must be an odd prime.

$$\Rightarrow \qquad p^2+11 \text{ is even}$$

$\therefore$ p^2+11 contains the prime 2 as factor. We now use the formula for $d(n)$, the divisor function. Since $6 = 3\times 2$ there are precisely two categories of numbers with 6 divisors, those of the kind q^5 and those of the kind q^2r (with q, r unequal primes)

If p^2+11 has 6 divisors then $p^2+11 = q^5$ or q^2r where q, r are primes, $q \neq r$.

1st case is ruled out due to earlier observation so $p^2+11 = q^2r$. Here $p = 3$ works. Which has 6 divisor we need to only consider the case when $p > 3$.

Since p is prime, it is indivisible by 3, so $p \equiv \pm 1$. which means $p^2 \equiv 1 \pmod 3$

$$\therefore \qquad p^2+11 \equiv 0 \pmod 3.$$

3 is a divisor of p^2+11.

2 is also divisor of p^2+11.

It means that q_1, r are 2, 3 in some order.

For neither $2^2 3 = 12$ nor $3^2 2 = 18$ is of the form p^2+11 for any prime p.

Neither possibility works.

Example 79. *Show that the quantity* $(n+1)(n+2)\ldots(2n-1)(2n)$ is divisible by 2^n.

Solution We can write $a_n = (n+1)(n+2)\ldots(2n-1)(2n)$, then $a_1 = 2$, we have $2^1|2$.

$$a_2 = 3\times 4 = 12, \text{ we have } 2^2|12$$

So, assertion does hold good, when $n = 1$ and 2.

Now $$\frac{a_n}{a_{n-1}} = \frac{(n+1)(n+2)\ldots(2n-1)(2n)}{n(n+1)\ldots(2n-3)(2n-2)} = 2(2n-1)$$

So $a_n = 2(2n-1)a_{n-1}$, a_n contains precisely one more 2 in its prime factorization than a_{n-1}.

$\therefore$ if $2^{n-1} | a_{n-1}$ then, $2^n | a_n$

we have $2^1|a_1$ which leads to $2^2|a_2$ which leads to $2^3|a_3$ which leads to $2^4|a_4$ and so on. All the way to infinity.

$$\therefore \qquad 2^n | a_n \text{ for all integers } n > 0.$$

Example 80. *Find all ordered triples (x, y, z) such that x, y and z are positive primes and $x^{y+1} = 2$.*

Solution If y is odd, then $x^{\text{odd}} + 1$ would be divisible by $x + 1$ and would not be prime.

So y is even and must be 2.

Also since z must be odd, x must be even.

So, $x = 2$ which make $z = 5$

so only answer $(2, 2, 5)$.

Example 81. *The integers x, y and z each are perfect square and $x > y > z > 0$. If x, y and z form an AP find smallest possible value of x.*

Solution Let
$$x = a^2, y = b^2, z = c^2$$
$$c^2 - b^2 = b^2 - a^2$$
So
$$2b^2 = a^2 + c^2$$
Since b is at least 2, consider values of b from 2 onwards and find $2b^2$.

The first such value *i.e.*, the sum of two squares occurs when $b = 5$ $(a = 7, c = 1)$, then $x = 49$

Example 82. *Prove that the expression $\dfrac{21n+4}{14n+3}$ is irreducible for every +ve integer n.*

Solution If d divides two integers, then it divides their multiples, their sum and difference,
$$\because \quad 3(14n + 3) - 2(21n + 4) = 1.$$
A common divisor of numerator and the denominator also divides 1; for this reason the fraction cannot be multiplied.

Alternative Solution If $\dfrac{a}{b} = e + \dfrac{c}{b}$ $(a, b, c, e$ integers) then $c = a - be$ and $a = c + be$.

Consequently the common divisor of a and b also divides c; common divisors of c and b divide a.

$\therefore$ $\dfrac{a}{b}$ can be simplified if and only if $\dfrac{c}{b}$ can be simplified. Same holds for $\dfrac{a}{b}$ and $\dfrac{b}{a}$.
$$\because \quad \frac{21n+4}{14n+3} = 1 + \frac{7n+1}{14n+3} \quad \text{and} \quad \frac{14n+3}{7n+1} = 2 + \frac{1}{7n+1}$$
and the last term cannot be simplified, the same is true for our original expressions.

Example 83. *Determine all 3 digit numbers which are equal to 11 times the sum of the squares of their digits.*

Solution Denote the digits by a, b and c $(a \neq 0)$ According to the assumption
$$100a + 10b + c = 11(a^2 + b^2 + c^2) \quad \text{...(i)}$$
$$\Rightarrow \quad (99a + 11b) + (a - b + c) = 11(a^2 + b^2 + c^2)$$
RHS and the first term on LHS is divisible by 11.

Hence, so is $a - b + c$

$\because$ $-8 \le a - b + c \le 18$, we conclude that $a - b + c$ is equal either to 0 or 11.

Now $b = a + c$ substituting this expression in Eq. (i), we get
$$100a + 10(a + c) + c = 11(a^2 + (a + c)^2 + c^2)$$
After ordering the quadratic equation
$$2a^2 + (2c - 10)a + (2c^2 - c) = 0 \quad \text{...(ii)}$$
$\because$ The first two terms of this expression are even third term should be even as well.

$\Rightarrow c$ is even.

Eq. (ii) admits integer solution if and only if its discriminant $4(-3c^2 - 8c + 25)$ is a square. This is possible only in case $c = 0$, substitute $c = 0$ in Eq. (ii) we get $2a^2 - 10a = 0$

$\because a \neq 0$, we have $a = 5 \Rightarrow b = a + c = 5$. Hence we get 550 for the integer we are seeking for.

Now, when $b = a + c - 11$. After substituting and ordering Eq. (i) we get the expression

$$2a^2 + (2c - 32)a + (2c^2 - 23c + 131) = 0 \qquad \ldots\text{(iii)}$$

Now c cannot be even. The discriminant admits the form $4(-3c^2 + 14c - 16)$; for odd c it is a square only in case $c = 3$. Substituting it in Eq. (iii), we get $2a^2 - 26a + 80 = 0$, on solving this quadratic equation, we get $a = 5$ or $a = 8 \Rightarrow b = -3$ and $b = 0$. Only the latter figure might serve as a solution of the original problem. This provides the second solution 803 and an easy check shows that this number satisfies the assumption of the problem.

Two solutions are 550 and 803.

Example 84. *Prove that there are infinitely many +ve integers a such that $z = n^4 + a$ is not a prime for any positive integer n.*

Solution Let $a = 4b^4$ where $b > 1$ is an arbitrary integer, we show that z is not prime.

$$z = n^4 + 4b^4 + 4n^2b^2 - 4n^2b^2 = (n^2 + 2b^2)^2 - (2nb)^2$$
$$= (n^2 + 2b^2 + 2nb)(n^2 + 2b^2 - 2nb) = ((n + b)^2 + b^2)((n - b)^2 + b^2)$$

$\because b > 1$, both factors are > 1

So, z is not prime.

Example 85. *Let p and q be +ve integers such that* $\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319}$. *Prove that* 1979 *divides p*.

Solution Applying the following transformation

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1319}$$
$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{1318}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1319} - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{659}\right)$$
$$= \frac{1}{660} + \frac{1}{661} + \ldots + \frac{1}{1319}$$
$$= \left(\frac{1}{660} + \frac{1}{1319}\right) + \left(\frac{1}{661} + \frac{1}{1318}\right) + \ldots + \left(\frac{1}{989} + \frac{1}{990}\right)$$
$$= 1979\left(\frac{1}{660.1319} + \frac{1}{661.1318} + \ldots + \frac{1}{989.990}\right)$$

The sum in parentheses is $\frac{a}{b}$, where a is a positive integer and $b = 660.661 \ldots 1319$

As 1979 is a prime and every factor of b is smaller than 1979, b and 1979 are coprime.

Thus, $\frac{p}{q} = \frac{1979a}{b}$ implies that $pb = 1979\,aq$

So, 1979 divides p.

Remark : The problem can be generalised for primes of the form $3k + 2$. If

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2k} + \frac{1}{2k + 1}$$

then $3k + 2$ divides p.

Example 86. *Find one pair of positive integers a, b such that $ab(a+b)$ is not divisible by 7 and $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.*

Solution Using the binomial theorem and observing that both $(a+b)^7$ and $a^7 + b^7$ is divisible by $(a+b)$.

$$a(a+b) \quad \ldots(i)$$

and

$$(a+b)^7 - a^7 - b^7 \quad \ldots(ii)$$

$$= 7ab(a+b)(a^2+ab+b^2)^2 \quad \ldots(iii)$$

$\because$ $ab(a+b)$ is not divisible by 7 we must choose a and b such that 7^6 divides $(a^2+ab+b^2)^2$ or a^2+ab+b^2 is divisible by $7^3 = 343$.

We try to choose a pair a, b such that $b = 1$ *i.e.*, for some integer k

$$a^2 + a + 1 = 343k$$

i.e.,

$$a^2 + a + (1 - 343k) = 0 \quad \ldots(iv)$$

This is possible, when discriminant of this equation of degree 2 in a is a perfect square. *i.e.*, $1 - 4(1 - 343k) = 1372k - 3$, is a perfect square.

It is true for $k = 1$ as $1369 = 37^2$.

Hence Eq. (iv) becomes $a^2 + a - 342 = 0$; $a = 18, b = 1$ and the pair $(a, b) = (18, 1)$ satisfies the condition 7 does not divide 18.1.19.

Example 87. *Let p be a prime number. Prove that there exist a prime number q such that for every integer n the number $n^p - p$ is not divisible by q.*

Solution Since

$$\frac{p^p - 1}{p - 1} = 1 + p + p^2 + \ldots + p^{p-1}$$

$$\equiv (p+1) \bmod p^2,$$

We can get at least one prime divisor of $\frac{(p^p - 1)}{p - 1}$ which is not congruent to 1 modulo p^2. Denote such a prime divisor by q. This q is what we wanted.

Assume that there exist an integer n such that $n^p \equiv p \bmod (q)$.

Then, we have $n^{p^2} \equiv p^p \equiv 1 \bmod q$.

On the other hand from Fermat's little theorem $n^{q-1} \equiv 1 \bmod q$ because q is prime. Since, $(p^2, (q-1))$, we have $(p^2, q-1) | p$ which leads to $n^p \equiv 1 \bmod q$.

Hence, we have $p \equiv 1 \pmod q$

However this implies $1 + p + p^2 + \ldots + p^{p-1} \equiv p \bmod q$

From the definition of q, this leads to $p \equiv 0 \bmod q$ which is a contradiction.

Let us Practice

Let us Practice

Level 1

1. Find GCD of 595 and 252 and express n in the form $252\,m + 595\,n$.
2. Find the greatest common divisor d of the numbers 275 and 200 and then find integers m and n such that $d = m\,275 + n\,200$
3. If $a|c$ and $b|c$, then is it true that $ab|c$.
4. If a, m, n are non zero integers, then $(a, mn) = 1$ if and only if $(a, m) = 1$ and $(a, n) = 1$.
5. If $(a, b) = 1$ and $c|a$, then $(c, b) = 1$
6. If $(a, b) = 1$; then $(ac, b) = (c, b)$.
7. If $a|b$, $c|d$ and $(b, d) = 1$, then prove that $(a, c) = 1$.
8. Show that $(a, b) = (a + b, b)$.
9. If $(a, m) = 1$, prove that $(m - a, m) = 1$
10. Prove that the product of two odd numbers is always an odd number.
11. Show that product of two numbers of the form $(6n + 1)$ is of the same form.
12. If n is an integer, prove that $n(n-1)(2n-1)$ is divisible by 6.
13. If n is odd, show that $n(n^2 - 1)$ is divisible by 24.
14. If x and y are positive integers and if $(x - y)$ is even, show that $x^2 - y^2$ is divisible by 4.
15. Prove that $2^{4n} - 1$ is divisible by 15.
16. Prove that $3^{2n} + 7$ is a multiple of 8.
17. Show that $2^{2n} + 1$ is divisible by 5 for a positive odd integer n.
18. Prove that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3.
19. Show that there are infinitely many primes of the form
    (i) $6n + 5$ (ii) $6n - 1$ (iii) $4n - 1$
20. If each of the two primes p and q is a factor of 'a', show that the product pq is also a factor of 'a'.
21. Find the highest power of 7, which is contained in 50!.
22. Find the highest power of 3 which is contained in 500!.
23. Find the number of zeros at the end of $(5^n)!$.
24. Find the number of zeros at the end of $(5^n + 1)!$
25. Find all n such that $n!$ has 1998 zeros at the end of $n!$.
26. Find exponent of 12 in 100!.
27. Show that 100! is divisible by 15^{20}.
28. Show that 33! is not divisible by 2^{40}.
29. Find highest power of 3 in ${}^{200}C_{100}$.
30. What is the largest integer n such that 33! is divisible by 2^n.
31. Find the number of zeros at the end of ${}^{200}C_{100}$.
32. Find the number of zeros at the end of $2 \cdot {}^{10n}C_{10^n}$.
33. Find the number of zeros at the end of ${}^{10^n}C_{5^n}$.
34. If $x + 2 = (18)(18)(18)\ldots n$ digits, find the number of zeros at the end of ${}^{x}C_{x/2}$.
35. Show that there is a positive integer n such that $n!$ when written in decimal notation ends with exactly 1993 zeros. **(INMO 1993)**
36. Find the number of zeros at the end of
    (i) $\left(\dfrac{10^3 - 1}{9}\right)!$ (ii) $(5^{20} - 1)!$ (iii) $\sum_{n=1}^{20} (5n)!$
    (iv) Product of 1, 2, 3, ..., 1994.
    (RMO 1994 Delhi)
37. If $x = 6\sqrt{(\underbrace{11\ldots1}_{2n}) - (\underbrace{22\ldots2}_{n})} + 2$, find number of zeros at end of ${}^{x}C_{x/2}$.
38. Show that the highest power of 2 contained in $(2^r - 1)!$ is $2^r - r - 1$.

39. Show that $3.7^{800} + 7.3^{800} \ldots 210$ is divisible by 10.

40. Show that
$$3(7^{204} + 7^{202} + 7^{200}) + 7(3^{204} + 3^{200}) - (210)$$
is divisible by 10.

41. Prove that $A = (2903)^n - (803)^n - (464)^n + (261)^n$ is divisible by 1897.

42. Show that $1^{99} + 2^{99} + 3^{99} + 5^{99}$ is divisible by 5.

43. Show that $(6! + 1)$ is divisible by 7.

44. Find the remainder when
$$(2222)^{5555} + (5555)^{2222}$$ is divided by 7.

45. Which is the largest $444^4, 44^{44}, 4^{444}$?

46. Find number of digits in $2^{2^{22}}$.

47. Show $\left(\sum_{n=1}^{100} n!\right)^7$ is of the form of $7k + 4$.

48. When $3^{1994} + 2$ is divided by 11. Find the remainder.

49. Show that $2^{55} + 1$ is divisible by 11.

50. Show that the difference of the squares of any two odd primes greater than 3 is divisible by 24.

51. Prove that 8th power of any number is of the form $17n$ or $17n \pm 1$.

52. Show that $a^{18} - b^{18}$ is divisible by 133 if a and b are coprime to 133.

53. Show that $n^{16} - a^{16}$ is divisible by 85 if n and a are coprime to 85.

54. Show that $n^5 - n$ is divisible by 30.

55. Prove that $a^{12} - b^{12}$ is divisible by 13 if a and b are coprime to 13.

56. Prove that $n^{11} - n$ is divisible by 11 for any integer n.

57. If $n > 1$ is always odd, then $n^3 - 2n^2 + n$ is divisible by 4. Prove.

58. Let m and n be two integers such that $m = n^2 - n$. Show that $m^2 - 2m$ is divisible by 24.

59. Show that there are infinitely many pairs (*a*, b) of relatively prime integers (not necessarily positive) such that both quadratic equations $x^2 + 2ax + b = 0$ and $x^2 + ax + b = 0$ have integral roots. **(INMO 1995)**

60. (i) Consider two positive integers a and b which are such that $a^a b^b$ is divisible by 2000. What is the least possible value of the product ab?

(ii) Consider two positive integers a and b which are such that $a^b b^a$ is divisible by 2000. What is the least possible value of the product ab? **(RMO 2000)**

61. Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer. **(RMO 2001)**

62. Consider an $n \times n$ array of numbers
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$
Suppose each row consists of the n numbers $1, 2, 3, \ldots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are $1, 2, 3, \ldots, n$ in some order. **(RMO 2001)**

63. Prove that the product of the first 200 positive even integers differs from the product of the first 200 positive odd integers by a multiple of 401. **(RMO 2001)**

64. Let a, b, c be positive integers such that a divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a + b + c)^7$. **(RMO 2002)**

65. Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint non-empty subsets A and B of X such that

(a) $A \cup B = X$;

(b) prod(A) is divisible by prod(B), where for any finite set of numbers C, prod(C) denotes the product of all numbers in C;

(c) the quotient prod(A)/prod(B) is as small as possible. **(RMO 2003)**

66. Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T. **(RMO 2004)**

67. Let $\langle p_1, p_2, p_3, \ldots, p_n, \ldots \rangle$ be a sequence of primes defined by $p_1 = 2$ and for $n \geq 1$, p_{n+1} is the largest prime factor of $p_1 p_2 \ldots p_n + 1$. (Thus $p_2 = 3, p_3 = 7$). Prove that $p_n \neq 5$ for any n. **(RMO 2004)**

68. If x, y are integers and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$. **(RMO 2005)**

69. Prove that there are infinitely many positive integers n such that $n(n+1)$ can be expressed as a sum of two positive squares in *at least* two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.) **(RMO 2006)**

70. Find the *least* possible value of $a + b$, where a, b are positive integers such that 11 divides $a + 13b$ and 13 divides $a + 11b$. **(RMO 2006)**

71. A 6×6 square is dissected in to 9 rectangles by lines parallel to its sides such that all these rectangles have integer sides. Prove that there are always two congruent rectangles. **(RMO 2006)**

72. Let a, b, c be three natural numbers such that $a < b < c$ and $\text{GCD}(c - a, c - b) = 1$. Suppose there exists an integer d such that $a + d, b + d, c + d$ form the sides of a right-angled triangle. Prove that there exist integers l, m such that $c + d = l^2 + m^2$. **(RMO 2007)**

73. In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off ? **(RMO 2009)**

74. Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289. **(RMO 2009)**

75. Show that $3^{2008} + 4^{2009}$ can be written as product of two positive integers each of which is larger than 2009^{182}. **(RMO 2009)**

Level 2

1. Find all positive integers n for which $2^n - 1$ is divisible by 7.

2. Compute the smallest positive integer N such that $\dfrac{N!}{12^{12}}$ is an integer.

3. Find one pair of positive integers a and b such that

(i) $ab(a + b)$ is not divisible by 7.

(ii) $(a + b)^7 - a^7 - b^7$ is divisible by 7^7.

Justify your answer

4. Factor the number $5^{1985} - 1$ into a product of three integers each of which is larger than 5^{100}.

5. If $(1 + x + x^2 + x^3 + x^4)^{496} = a_0 + a_1 x + \ldots + a_{1984} x^{1984}$.

(i) Determine GCD of the coefficients $a_3, a_8, a_{13}, \ldots, a_{1983}$.

(ii) Show that $10^{347} > a_{992} > 10^{340}$.

6. Find the largest positive integer n with the property that $n + 10 \mid n^3 + 100$.

7. Find the number of positive integers n less than 1991 for which $6/n^2 + 3n + 2$.

(INMO 1991)

8. Show that positive integers that have an odd number of divisors are the squares.

9. Prove that there is no positive integer n for which $2^n + 1$ is divisible by 7.

10. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime.

11. For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $\dfrac{d(n^2)}{d(n)} = k$ for some n.

12. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides, $a^2 b + a + b$.

13. Find all pairs (n, p) of positive integers such that p is prime, $n \leq 2p$ and $(p-1)^n + 1$ is divisible by n^{p-1}.

14. Compute the unique positive integer n such that

$$2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + 5 \cdot 2^5 + \ldots + n \cdot 2^n = 2^{(n+10)}.$$

15. Given two odd integers a and b. Prove that $a^2 - b^2$ is divisible by 2^n, if and only if $a - b$ is divisible by 2^n.

16. Prove that $[5x]+[5y]\geq[3x+y]+[3y+x]$ where $x,y\geq 0$ and $[u]$ denotes the greatest integer $\leq u$.

17. Find all 3 digit numbers equal to the sum of the factorials of their digits.

18. Find all whole numbers equal to the sum of the squares of their digits.

19. Prove that the expressions $2x+3y$ and $9x+5y$ are divisible by 17 for the same set of integral values of x and y.

20. Prove that there are no positive integral numbers which increase twice when their initial digits are carried to the end of the numbers.

21. Find all integers n (positive, negative or zero) for which n^2+n+41 is a perfect square.

22. Given any nine integers show that it is possible to choose, from among them, four integers a,b,c,d such that $a+b-c-d$ is divisible by 20. Further show that such a selection is not possible, if we start with eight integers instead of nine. **(INMO 2001)**

23. Do there exist three distinct positive real numbers a,b,c such that the numbers $a,b,c,b+c-a,\ c+a-b,\ a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order? **(INMO 2002)**

24. Determine the least positive value taken by the expression $a^3+b^3+c^3-3abc$ as a,b,c vary over all positive integers. Find also all triples (a,b,c) for which this least value is attained. **(INMO 2002)**

25. Find all primes p and q, and even numbers $n>2$, satisfying the equation
$$p^n+p^{n-1}+\ldots+p+1=q^2+q+1.$$
(INMO 2003)

26. Let S denote the set of all 6-tuples (a,b,c,d,e,f) of positive integers such that $a^2+b^2+c^2+d^2+e^2=f^2$. Consider the set
$$T=\{abcdef:(a,b,c,d,e,f)\in S\}.$$
Find the greatest common divisor of all the members of T. **(INMO 2004)**

27. Let x_1 be a given positive integer. A sequence $\langle x_n\rangle_{n=1}^{\infty}=\langle x_1,x_2,x_3,\ldots\rangle$ of positive integers is such that x_n, for $n\geq 2$, is obtained from x_{n-1} by adding some non-zero digit of x_{n-1}. Prove that
(a) the sequence has an even number;
(b) the sequence has infinitely many even numbers. **(INMO 2005)**

28. Let n be a natural number such that $n=a^2+b^2+c^2$, for some natural numbers a,b,c.
Prove that
$$9n=(p_1a+q_1b+r_1c)^2+(p_2a+q_2b+r_2c)^2+(p_3a+q_3b+r_3c)^2,$$
where p_j's, q_j's, r_j's are all non-zero integers. Further, if 3 does not divide at least one of a,b,c, prove that $9n$ can be expressed in the form $x^2+y^2+z^2$, where x,y,z are natural numbers none of which is divisible by 3. **(INMO 2007)**

29. Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that a^2+bc is a rational number for all distinct numbers a,b,c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A. **(INMO 2008)**

30. Define a sequence $\langle a_n\rangle_{n=1}^{\infty}$ as follows :
$$a_n=\begin{cases}0, & \text{if the number of positive divisors of } n \text{ is } odd.\\ 1, & \text{if the number of positive divisors of } n \text{ is } even.\end{cases}$$
(The positive divisors of n include 1 as well as n). Let $x=0.a_1a_2a_3\ldots$ be the real number whose decimal expansion contains a_n in the nth place, $n\geq 1$. Determine, with proof, whether x is rational or irrational. **(INMO 2009)**

31. Define a sequence $\langle a_n\rangle_{n\geq 0}$ by $a_0=0, a_1=1$ and $a_n=2a_{n-1}+a_{n-2}$, for $n\geq 2$.
(a) For every $m>0$ and $0\leq j\leq m$, prove that $2a_m$ divides $a_{m+j}+(-1)^j a_{m-j}$.
(b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n. **(INMO 2010)**

32. Find all natural numbers $n>1$ such that n^2 does not divide $(n-2)!$. **(INMO 2010)**

33. Call a natural number n *faithful*, if there exist natural numbers $a<b<c$ such that a divides b, b divides c and $n=a+b+c$.
(i) Show that all but a finite number of natural numbers are faithful.
(ii) Find the sum of all natural numbers which are not faithful. **(INMO 2011)**

Solutions

Solutions

Level 1

1. $595 = 2.252 + 91$ ...(i)

[Dividing 595 by 252]

$252 = 2.91 + 70$ ...(ii)

[Dividing 252 by 91]

$91 = 1.70 + 21$ ...(iii)

[Dividing 91 by 70]

$70 = 3.21 + 7$ ...(iv)

[Dividing 70 by 21]

$\therefore$ GCD of 595 and 252 is $d = 7$

From Eq. (iv), $d = 7 = 70 - 3.21$

$= 70 - 3(91 - 1.70)$

[Putting value of 21 from Eq. (iii)]

$= 70 - 3.91 + 3.70 = 4.70 - 3.91$

$= 4(252 - 2.91) - 3.91$

[Putting value of 70 from Eq. (ii)]

$= 4.252 - 11.91$

$= 4.252 - 11(595 - 2.252)$

[Putting value of 91 from Eq. (i)]

$= 4.252 - 11.595 + 22.252$

$= 26.252 - 11.595$

$\Rightarrow \quad d = 252m + 595n$

Here, $\quad m = 26, n = -11$

2. $d = 25 = 3(275) + (-4)(200)$

3. It is not true.

For example, $a = 3 \mid c = 12$

$b = 6 \mid c = 12$

But $\quad a.b = 3.6 = 18 \times c = 12$

4. Let $(a, m) = 1$ and $(a, n) = 1$

$\because \quad (a, m) = 1$

$\therefore$ $\exists$ integers h and k such that

$ah + mk = 1$ ...(i)

$\because \quad (a, n) = 1$

$\therefore$ $\exists$ integers r and t such that

$ar + nt = 1 \Rightarrow ar + nt.1 = 1$ ...(ii)

Putting the value of 1 from Eq. (i) in LHS of Eq. (ii), we have

$ar + nt(ah + mk) = 1$

$\Rightarrow \quad ar + ahnt + mntk = 1$

$\Rightarrow \quad a(r + hnt) + mn(tk) = 1$

$\Rightarrow \quad (a, mn) = 1$

Now, let $(a, mn) = 1$

$\therefore$ $\exists$ integers u and v such that

$au + mnv = 1$

$a(u) + m(nv) = 1$

$(a, m) = 1$

Similarly, $\quad (a, n) = 1$

5. $\because \quad (a, b) = 1$

$\because$ $\exists$ integers x and y such that

$ax + by = 1$ ...(i)

$\because \quad c \mid a$

$\therefore$ $\exists$ an integer m such that $a = cm$

Putting $a = cm$ in Eq. (i), we get

$cmx + by = 1$

or $\quad c(mx) + b(y) = 1$

$\therefore \quad (c, b) = 1$

6. Let $(ac, b) = d$ and $(c, b) = e$

$(ac, b) = d$

d is a common division of ac and b.

$\therefore \quad d \mid c$

i.e., d is a common divisor of b and c.

$\therefore \quad d \mid c$ ...(i)

$\because \quad (c, b) = e$

Again $\quad (c, b) = e$

$e \mid c$ and $e \mid b$

$e \mid ac$ and $e \mid b$

i.e., e is a common divisor of ac and b.

$\therefore \quad e \mid d \quad [\because (ac, b) = d]$ ...(ii)

From Eqs. (i) and (ii) $d = e$ i.e., $(ac, b) = (c, b)$

7. $\because$ $a \mid b$, so there exists an integer m, such that

$b = am$ ...(i)

$\because$ $c \mid d$, so there exits an integer n, such that

$d = cn$ ...(ii)

$\because (b, d) = 1$, so there exist integers x and y such that

$$bx + dy = 1 \quad \text{...(iii)}$$

Putting the values of b and d in Eq. (iii)

$$amx + cny = 1$$
$$a(mx) + c(ny) = 1$$
$$\therefore \quad (a, c) = 1.$$

8. Let $(a, b) = d$ and $(a + b, b) = e$

$\because \quad (a, b) = d$

$\therefore \quad d \mid a$ and $d \mid b$

$\therefore \quad d \mid a + b$

Now, $\quad d \mid (a + b)$ and $d \mid b$

$\therefore \quad d \mid (a + b), b$

[Since d is a common divisor of $a + b$ and b so d also divides gcd of $a + b, b$].

or $\quad d \mid e \quad$ ...(i)

Now $\because \quad (a + b, b) = e$

$\therefore \quad e \mid (a + b)$ and $e \mid b$.

$\therefore \quad e \mid (a + b) - b$ or $e \mid a$.

Now $e \mid a$ and $e \mid b$

$\therefore \quad e\,(a, b)$

i.e., $\quad e \mid d \quad$ ...(ii)

From Eqs. (i) and (ii), $d = e$

i.e., $\quad (a, b) = (a + b, b)$.

9. Let $(m - a, m) = d$

$\therefore \quad d \mid m - a$ and $d \mid m$

$\therefore \quad d \mid [m - (m - a)]$ or $d \mid a$

Now, $\quad d \mid m$ and $d \mid a$

$\therefore\ d \mid (a, m)$ [As d is a common divisor of a and m and $d \mid$ GCD of a, m]

But $\quad (a, m) = 1$

$\therefore \quad d \mid 1$

$\therefore \quad d = 1$ [only divisor of 1 is 1]

so $\quad (m - a, m) = 1$

10. Let $a = 2k + 1$ and $b = 2k' + 1$ be two odd numbers.

$$\therefore \quad ab = (2k + 1)(2k' + 1)$$
$$= 4kk' + 2k + 2k' + 1$$
$$= 2(2kk' + k + k') + 1 = 2q + 1$$

where $\quad q = 2kk' + k + k'$

$\therefore\ ab$ is an odd number

11. Let $a = 6k + 1$ and $b = 6k' + 1$ be two numbers of the form $(6n + 1)$

$$\therefore \quad ab = (6k + 1)(6k' + 1)$$
$$= 36kk' + 6k + 6k' + 1$$
$$= 6(6kk' + k + k') + 1 = 6l + 1$$

where $l = 6kk' + k + k'$, which is of the form.

12. $n(n - 1)(2n - 1) = n(n - 1)\{(n + 1) + (n - 2)\}$

$$= n(n - 1)(n + 1) + n(n - 1)(n - 2)$$
$$= (n - 1)n(n + 1) + (n - 2)(n - 1)n$$

Each of the two $(n - 1)n(n + 1)$ and $(n - 2)(n - 1)n$ being the product of three consecutive integers is divisible by $3! = 6$

$\therefore$ Their sum $= n(n - 1)(2n - 1)$ is also divisible by 6.

13. $\because n$ is odd, so let $n = 2m + 1$

Putting $n = 2m + 1$ in $n(n^2 - 1)$, we have

$$n(n^2 - 1) = n(n - 1)(n + 1)$$
$$= (2m + 1)(2m + 1 - 1)(2m + 1 + 1)$$
$$= (2m + 1)\,2m(2m + 1)$$
$$= 4m(m + 1)(2m + 1)$$

Now proceed yourself.

14. As $(x - y)$ is even.

$\therefore \quad x - y = 2m$ or $x = 2m + y$

Putting $x = 2m + y$ in $x^2 - y^2$, we have

$$x^2 - y^2 = (2m + y)^2 - y^2$$
$$= 4m^2 + 4my + y^2 - y^2$$
$$= 4m^2 + 4my$$
$$= 4(m^2 + my)$$

so $(x^2 - y^2)$ is divisible by 4.

15. $2^{4n} - 1 = (2^4)^n - 1 = 16^n - 1^n$

$$= (16 - 1)[(16)^{n-1} + (16)^{n-2}.1 + \ldots + 1^{n-1}]$$

$(\because x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \ldots + y^{n-1})\ \forall$ all n

or $\quad 2^{4n} - 1 = 15[(16)^{n-1} + (16)^{n-2} + \ldots + 1]$

$\therefore\ 2^{4n} - 1$ is divisible by 15.

[by definition of divisibility]

16. $3^{2n} + 7 = (3^2)^n + 7 = 9^n + 7 = (9^n - 1) + 8$

$$= (9^n - 1^n) + 8$$
$$= (9 - 1)(9^{n-1} + 9^{n-2} + \ldots + 1) + 8$$
$$= 8(9^{n-1} + 9^{n-2} + \ldots + 1) + 8$$
$$= 8(9^{n-1} + 9^{n-2} + \ldots + 1 + 1)$$

$\therefore\ 3^{2n} + 7$ is a multiple of 8.

17. $2^{2n} + 1 = (2^2)^n + 1 = 4^n + 1 = 4^n + 1^n$

$$= (4 + 1)(4^{n-1} - 4^{n-2} + 4^{n-3} - \ldots + 1)$$

[$\because$ for odd]

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots + y^{n-1})$$

$$= 5(4^{n-1} - 4^{n-2} + 4^{n-3} \ldots + 1)$$

$\therefore$ $2^{2n} + 1$ is divisible by 5.

19. (i) If possible let the number of primes of the form $(6n + 5)$ be finite.

These primes are 5, 11, 17, ... (putting $n = 0$, 1, 2, 3...) let q be the greatest prime of the form $(6n + 5)$. Let $a = 5.11.17\ldots.\ q$ be the product of all primes of the form $(6n + 5)$.

Let $\quad b = 6a - 1 \quad \ldots$(i)

$\therefore \quad b > 1 \quad [\because a \ge 5 \Rightarrow b = 6a - 1 \ge 29]$

$\therefore$ By Fundamental Theorem, b can be expressed as a product of primes say $p_1, p_2, p_3 \ldots p_r$.

i.e., $\quad b = p_1.p_2.p_3 \ldots p_r \quad \ldots$(ii)

Now, $\quad b = 6a - 1$ is odd.

Hence 2 can't be a factor of b.

Again 3 is also not a factor of b.

$\therefore$None of the prime factors in RHS of Eq. (ii) is 2 or 3.

i.e., Every prime factor in RHS of (2) is an odd number > 3.

$\therefore$ Each of $p_1, p_2, \ldots pr$ is of the form $(6n + 1)$ or $(6n + 5)$.

$\therefore$ Again all of f $p_1, p_2, \ldots, p_r$ can't be primes of the form $(6n + 1)$.

$\therefore$ At least one of $p_1, p_2, p_3 \ldots p_r$ say p is (a prime factor of b) of the form $(6n + 5)$ *i.e.,* p/b

Also $p|a$ [$\because$ p is one prime of the form $(6n + 5)$ and a is the product of all such primes.]

$\therefore \quad p|6a$

$\therefore \quad p|6a - b$ *i.e.,* $p|1$

[$\because$ from Eq. (i), $6a - b = 1$]

Which is impossible.

[$\because$ p being a prime of the form $6n + 5$ is ≥ 5]

$\therefore$ Our supposition is wrong.

$\therefore$ Number of primes of the form $(6n + 5)$ is infinite.

(ii) Try yourself with the help of part (i)

(iii) Try yourself

20. $\because$ p/a $\therefore$ $\exists$ an integer m such that

$$a = pm \quad \ldots\text{(i)}$$

$\because$ q/a $\therefore$ $\exists$ an integer n such that

$$a = qn \quad \ldots\text{(ii)}$$

As p and q are both primes

$\therefore$ They are coprime *i.e.,* $(p, q) = 1$

$\therefore$ $\exists$ integers x and y such that

$$px + qy = 1 \quad \ldots\text{(iii)}$$

Multiplying both sides of Eq. (iii) by a

$\therefore \quad apx + aqy = a$

Putting $a = qn$ from Eq. (ii) in apx and $a = pm$ from Eq. (i) in aqy, we have

$$qnpx + pmqy = a$$

$$pq(nx + my) = a$$

$\therefore \quad pq|a.$

21. 8

22. 247

23. $\frac{5}{4}(5^{n-1} - 1)$ zeros

24. $\frac{5}{4}(5^{n-1} - 1)$ zeros

25. The greatest power of $a > 1, a \in N$ dividing n is given by $\sum_{i=1}^{\infty}\left[\frac{n}{a^i}\right] \quad \ldots$(i)

but $\quad \sum_{i=1}^{\infty}\left[\frac{n}{a^i}\right] < \sum_{i=1}^{\infty}\frac{n}{a^i} = n\left(\frac{1}{a-1}\right) \quad \ldots$(ii)

We want to find n such that

$$\sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right] = 1998$$

from Eq. (ii) $\sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right] < n\left(\frac{1}{5-1}\right) = \frac{n}{4}$

So, $\quad \frac{n}{4} > 1998$ so $n > 7992$

By trial and error, we take $n = 7995$ and then find the correct value.

If $n = 7995$, then number of zeroes at the end of 7995 is from Eq. (i)

$$\frac{7995}{5} + \frac{7995}{5^2} + \ldots$$

$$= 1599 + 319 + 63 + 12 + 2 = 1995$$

so true for $n = 8000$, we get the number of zeroes at the end of 8000!

$$= 1600 + 320 + 64 + 12 + 2 = 1998$$

26. 48

29. 3

30. 31

31. 1 zero

32. 1 zero

33. No zeros

34. 1 zero

35. For any $k, n \in N$ the highest power of k which divides n is given by

$$\psi_k(n) = \left[\frac{n}{k}\right] + \left[\frac{n}{k^2}\right] + \left[\frac{n}{k^3}\right] + \dots$$

The number of zeros with which n ends when written in decimal system is

$$\min \{\psi_2(n), \psi_5(n)\} = \psi_5(n)$$

Hence we have to find n such that

$$\psi_5(n) = 1993$$

Since, $\psi_5(n) \le \frac{n}{5} + \frac{n}{5^2} + \dots = \frac{n}{5}\left(1 + \frac{1}{5} + \dots\right)$

$$= \frac{n}{5} \cdot \frac{5}{4} = \frac{n}{4}.$$

We have $\frac{n}{4} \ge 1993 \Rightarrow n > 7970$

We find that $\psi_5(7975) = 1991$

$\therefore \quad \psi(7985) = \psi_5(1975) + 2 = 1993$

Hence, 7985 is one solution.

Other numbers are 7986, 7987, 7988, 7989.

36. (i) 26 zeros.

(ii) $\frac{5^{20} - 4^{20} - 1}{4}$ zeros.

(iii) 300 zeros.

(iv) 495 zeros.

37. 1

44. 0

45. 4^{444}

46. 14

48. 6

50. Let x and y be two odd prime > 3

$\because$ x is a prime greater than 3,

$\therefore \quad (x, 3) = 1$

Also 3 is prime

$\therefore$ By Fermat theorem, $x^2 \equiv 1 \bmod 3$ ...(i)

similarly $y^2 \equiv 1 \bmod 3$ ...(ii)

Subtracting congruences (i) and (ii), we have

$$x^2 - y^2 \equiv 0 \bmod 3$$

$\therefore$ $x^2 - y^2$ is divisible by 3 ...(iii)

Again, as x and y are odd primes > 3

$\therefore$ x and y are odd numbers > 3

Let $x = 2k + 1$; $y = 2k' + 1$, where $k > 1, k' > 1$

$$\therefore \quad x^2 - y^2 = (x - y)(x + y)$$
$$= (2k + 1 - 2k' - 1)(2k + 2k' + 2)$$
$$= (2k - 2k')(2k + 2k' + 2)$$
$$= 4(k - k')(k + k')$$
$$= 4[(k + k') - 2k'](k + k' + 1)$$
$$= 4(k + k')(k + k' + 1) - 8k'(k + k' + 1)$$
$$= 8l - 8k'(k + k' + 1)$$

[where $= (k + k')(k + k' + 1)$ being the product of two consecutive integers is divisible by $2! = 2$]

$$= 8[l - k'(k + k' + 1)]$$

$\therefore$ $x^2 - y^2$ is divisible by 8. ...(iv)

From Eqs. (iii) and (iv)

$x^2 - y^2$ is divisible by 24 $[\because (3, 8) = 1]$

51. Let a be any integer

$\because$ 17 is prime and a is any number.

$\therefore$ Either $(a, 17) = 1$ or $17 | a$

Case I $(a, 17) = 1$

Also 17 is prime.

$\therefore$ By Fermat theorem

$$a^{16} \equiv 1 \bmod 17$$

$\therefore \quad 17 | (a^{16} - 1)$ or $17 | (a^8 - 1)(a^8 + 1)$

$\therefore$ Either $17 | (a^8 - 1)$ or $17 | (a^8 + 1)$

$[\because 17$ is prime$]$

$\therefore \quad a^8 - 1 = 17n$ or $a^8 + 1 = 17n$

i.e., $\quad a^8 = 17n + 1$ or $a^8 - 17n - 1$ ...(i)

Case II $17 | a$

$\therefore \quad 17 | a^8$

$\therefore \quad a^8 = 17n$ ...(ii)

From Eqs. (i) and (ii) we can say that 8th power of any number is of the form $17n$ or $17n \pm 1$

52. $\because$ a and b are both coprime to 133

$\therefore$ a and b are coprime to 19.

($\because$ 19 is a factor of 133)

Also 19 is a prime.

$\therefore$ By Fermat theorem

$$a^{18} \equiv 1 \bmod 19$$
$$b^{18} \equiv 1 \bmod 19$$

Subtracting these congruences

$$a^{18} - b^{18} \equiv 0 \bmod 19$$

$\therefore$ $a^{18} - b^{18}$ is divisible by 19 ...(i)

Again if a and b are both coprime to 133.

$\therefore$ a and b are coprime to 7

($\because$ 7 is a factor of 133)

Also 7 is prime

$\therefore$ $a^6 \equiv 1 \bmod 7$

$b^6 \equiv 1 \bmod 7.$

Raising both congruences to power 3.

$$a^{18} \equiv 1 \bmod 7$$
$$b^{18} \equiv 1 \bmod 7$$

Subtracting these congruences

$$a^{18} - b^{18} \equiv 0 \bmod 7$$

$\therefore$ $(a^{18} - b^{18})$ is divisible by 7. ...(ii)

From Eqs. (i) and (ii)

$a^{18} - b^{18}$ is divisible by $133 (= 19 \times 7)$

$[\because (7, 19) = 1]$

53. $\because$ $(n, 85) = 1$ and $(a, 85) = 1$

$\therefore$ $(n, 17) = 1$ and $(a, 17) = 1$

($\because$ 17 is a divisor of 85)

But 17 is a prime number.

$\therefore$ By Fermat theorem, $n^{16} \equiv 1 \bmod 17$

$a^{16} \equiv 1 \bmod 17$

Subtracting these congruences $n^{16} - a^{16}$ is divisible by 17 ...(i)

Again, as n and a are both coprime to 85

$\therefore$ n and a are both coprime to 5.

($\because$ 5 is a factor of 85)

$\therefore$ By Fermat theorem $n^4 \equiv 1 \bmod 5$

and $a^4 \equiv 1 \bmod 5.$

Raising both congruences to power 4.

($\because$ If $a \equiv b \bmod m$, then $a^n \equiv b^n \bmod m$)

$$n^{16} \equiv 1 \bmod 5$$
$$a^{16} \equiv 1 \bmod 5$$

Subtracting these congruences

$$n^{16} - a^{16} \equiv 0 \bmod 5$$

$\therefore$ $(n^{16} - a^{16})$ is divisible by 5 ...(ii)

From Eqs. (i) and (ii)

$n^{16} - a^{16}$ is divisible by $85 (= 17 \times 5)$

$[\because (17, 5) = 1]$

54. $\because$ $(a^p \equiv a \bmod p)$

$\therefore$ $n^5 \equiv n \bmod 5$

$n^5 - n$ is divisible by 5 ...(i)

$\because$ 5 is a prime number)

Again, $n^5 - n = n(n^4 - 1)$

$= n(n^2 - 1)(n^2 + 1)$

$= n(n - 1)(n + 1)(n^2 + 1)$

Now $(n - 1)n(n + 1)$ being the product of three consecutive integers is divisible by $3! = 6$ and hence

$n^5 - n = n(n - 1)(n + 1)(n^2 + 1)$ is divisible by 6. ...(ii)

$\therefore$ From Eqs. (i) and (ii)

$n^5 - n$ is divisible by 30 $(= 6 \times 5)$

$[\because (6, 5) = 1]$

55. $\because$ 13 is prime and $(a, 13) = 1$

By Fermat theorem, $a^{13-1} \equiv 1 \bmod 13$

i.e., $a^{12} \equiv 1 \bmod 13$ ...(i)

$b^{12} \equiv 1 \bmod 13$ ...(ii)

Subtracting congruences Eqs. (i) and (ii)

$$a^{12} - b^{12} \equiv 0 \bmod 13$$

$\therefore$ $(a^{12} - b^{12})$ is divisible by 13.

56. $\because$ 11 is a prime number.

$n^{11} \equiv n \bmod 11$ $(a^p \equiv a \bmod p)$

$\therefore$ $n^{11} - n$ is divisible by 11.

57. $\because n(n^2 - 2n + 1) \Rightarrow n(n - 1)^2$

$\Rightarrow$ $n(n - 1)(n - 1)$

Since, n is odd, $(n - 1)$ is even.

We know $(n - 1)n = 2k$

$(n - 1)$ is even $= 2q$

$\therefore$ $(n - 1)n(n - 1) = 4qk = 4m$

$\therefore$ It is of the form $4m$

$\therefore$ $n^3 - 2n^2 + n$ is divisible by 4.

58. $m^2 - 2m = m(m-2)$

$$= (n^2 - n)(n^2 - n - 2)$$
$$= n(n-1)(n-2)(n+1)$$
$$= (n-2)(n-1)(n)(n+1)$$

which is the product of four consecutive integers since the product of four consecutive integers is divisible by $4!\ (= 24)$.

$\therefore\ m^2 - 2m$ is divisible by 24.

60. We have $2000 = 2^4 5^3$.

(i) Since 2000 divides $a^a b^b$, it follows that 2 divides a or b and similarly 5 divides a or b. In any case 10 divides ab. Thus the least possible value of ab for which $2000 \mid a^a b^b$ must be a multiple of 10. Since 2000 divides $10^{10} 1^1$, we can take $a = 10, b = 1$ to get the least value of ab equal to 10.

(ii) As in (i) we conclude that 10 divides ab. Thus the least value of ab for which $2000 \mid a^b b^a$ is again a multiple of 10. If $ab = 10$, then the possibilities are $(a, b) = (1, 10), (2, 5), (5, 2), (10, 1)$. But in all these cases it is easy to verify that 2000 does not divide $a^b b^a$. The next multiple of 10 is 20. In this case we can take $(a, b) = (4, 5)$ and verify that 2000 divides $4^5 5^4$. Thus the least value here is 20.

61. Let p, q be primes such that $p^2 + 7pq + q^2 = m^2$ for some positive integer m. We write

$5pq = m^2 - (p+q)^2 = (m + p + q)(m - p - q)$.

We can immediately rule out the possibilities $m + p + q = p, m + p + q = q$ and $m + p + q = 5$ (In the last case $m > p, m > q$ and p, q are at least 2).

Consider the case $m + p + q = 5p$ and $m - p - q = q$. Eliminating m, we obtain $2(p + q) = 5p - q$. It follows that $p = q$. Similarly, $m + p + q = 5q$ and $m - p - q = p$ leads to $p = q$. Finally taking $m + p + q = pq$, $m - p - q = 5$ and eliminating m, we obtain $2(p + q) = pq - 5$. This can be reduced to $(p-2)(q-2) = 9$. Thus $p = q = 5$ or $(p, q) = (3, 11), (11, 3)$. Thus the set of solutions is

$\{(p, p) : p \text{ is a prime}\} \cup \{(3, 11), (11, 3)\}$.

62. Let us see how many times a specific term, say 1, occurs in the matrix. Since 1 occurs once in each row, it occurs n times in the matrix.

Now consider its occurrence off the main diagonal. For each occurrence of 1 below the diagonal, there is a corresponding occurrence above it, by the symmetry of the array. This accounts for an even number of occurrences of 1 off the diagonal. But 1 occurs exactly n times and n is odd.

Thus 1 must occur at least once on the main diagonal. This is true of each of the numbers $1, 2, 3, \ldots, n$. But there are only n numbers on the diagonal. Thus each of $1, 2, 3, \ldots, n$ occurs exactly once on the main diagonal. This implies that $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ is a permutation of $1, 2, 3, \ldots, n$.

63. We have to prove that

401 divides $2 \cdot 4 \cdot 6 \cdot \ldots \cdot 400 - 1 \cdot 3 \cdot 5 \cdot \ldots \cdot 399$.

Write $x = 401$. Then, this difference is equal to $(x-1)(x-3)\ldots(x-399) - 1 \cdot 3 \cdot 5 \cdot \ldots \cdot 399$.

If we expand this as a polynomial in x, the constant terms get canceled as there are even number of odd factors $((-1)^{200} = 1)$. The remaining terms are integral multiples of x and hence the difference is a multiple of x. Thus 401 divides the above difference.

64. Consider the expansion of $(a + b + c)^7$. We show that each term here is divisible by abc. It contains terms of the form $r_{klm} a^k b^l c^m$, where r_{klm} is a constant (some binomial coefficient) and k, l, m are non-negative integers such that $k + l + m = 7$. If $k \geq 1, l \geq 1, m \geq 1$, then abc divides $a^k b^l c^m$.

Hence we have to consider terms in which one or two of k, l, m are zero. Suppose for example $k = l = 0$ and consider c^7. Since b divides c^2 and a divides c^4, it follows that abc divides c^7.

A similar argument gives the result for a^7 or b^7. Consider the case in which two indices are non-zero, say for example, bc^6. Since a divides c^4, here again abc divides bc^6. If we take b^2c^5, then also using a divides c^4 we obtain the result. For b^3c^4, we use the fact that a divides b^2. Similar argument works for b^4c^3, b^5c^2 and b^6c. Thus each of the terms in the expansion of $(a + b + c)^7$ is divisible by abc.

65. The prime factors of the numbers in set $\{1, 2, 3, \ldots, 9, 10\}$ are 2, 3, 5, 7. Also only $7 \in X$ has the prime factor 7. Hence, it cannot appear in B. For otherwise, 7 in the denominator would not get canceled. Thus $7 \in A$.

Hence, $\text{prod}(A)/\text{prod}(B) \geq 7$.

The numbers having prime factor 3 are 3, 6, 9. So, 3 and 6 should belong to one of A and B, and 9 belongs to the other. We may take $3, 6 \in A, 9 \in B$.

Also 5 divides 5 and 10. We take $5 \in A$, $10 \in B$. Finally we take $1, 2, 4 \in A$, $8 \in B$. Thus,

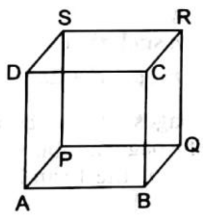
$A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{8, 9, 10\}$,

so that $\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{8 \cdot 9 \cdot 10} = 7$

Thus, 7 is the minimum value of $\frac{\text{prod}(A)}{\text{prod}(B)}$.

There are other possibilities for A and B: *e.g.*, 1 may belong to either A or B. We may take $A = \{3, 5, 6, 7, 8\}$, $B = \{1, 2, 4, 9, 10\}$.

66. Let $ABCDPQRS$ be a cube, and the numbers a, b, c, d, e, f be written on the faces $ABCD$, $BQRC$, $PQRS$, $APSD$, $ABQP$, $CRSD$ respectively. Then, the products written at the corners A, B, C, D, P, Q, R, S are respectively $ade, abe, abf, aaf, cde, bce, bcf, caf$. The sum of these 8 numbers is



$$= (e + f)(ab + bc + cd + ad)$$
$$= (e + f)(a + c)(b + d).$$

This is given to be equal to $2004 = 2^2 \cdot 3 \cdot 167$.

Observe that none of the factors $a + c, b + d, e + f$ is equal to 1. Thus $(a + c)(b + d)(e + f)$ is equal to $4 \cdot 3 \cdot 167$, $2 \cdot 6 \cdot 167$, $2 \cdot 3 \cdot 334$ or $2 \cdot 2 \cdot 501$. Hence the possible values of $T = a + b + c + d + e + f$ are $4 + 3 + 167 = 174$, $2 + 6 + 167 = 175$, $2 + 3 + 334 = 339$, or $2 + 2 + 501 = 505$.

Thus, there are 4 possible values of T and they are 174, 175, 339, 505.

67. By data $p_1 = 2$, $p_2 = 3$, $p_3 = 7$. It follows by induction that $p_n, n \geq 2$ is odd. [For if $p_2, p_3, \ldots, p_{n-1}$ are odd, then $p_1 p_2 \ldots p_{n-1} + 1$ is also odd and nor 3. This also follows by induction. For if $p_3 = 7$ and if $p_3, p_4, \ldots, p_{n-1}$ are neither 2 nor 3, then $p_1 p_2 p_3 \ldots p_{n-1} + 1$ are neither by 2 nor by 3. So, p_a is neither 2 nor 3.

68. Observe that

$$x^2 - 3xy + 2y^2 + x - y = (x - y)(x - 2y + 1)$$

Thus, 17 divides either $x - y$ or $x - 2y + 1$. Suppose that 17 divides $x - y$. In this case $x \equiv y$ (mod 17) and hence

$$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 2y^2 + y^2 - 5y + 7y \equiv 2y \pmod{17}.$$

Thus, the given condition that 17 divides $x^2 - 2xy + y^2 - 5x + 7y$ implies that 17 also divides $2y$ and hence y itself. But then $x \equiv y$ (mod 17) implies that 17 divides x also. Hence, in this case 17 divides $xy - 12x + 15y$.

Suppose on the other hand that 17 divides $x - 2y + 1$. Thus, $x \equiv 2y - 1$ (mod 17) and hence

$$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 5y + 6 \pmod{17}.$$

Thus, 17 divides $y^2 - 5y + 6$. But $x \equiv 2y - 1$ (mod 17) also implies that

$$xy - 12x + 15y \equiv 2(y^2 - 5y + 6) \pmod{17}.$$

Since, 17 divides $y^2 - 5y + 6$, it follows that 17 divides $xy - 12x + 15y$.

69. Let $Q = n(n + 1)$. It is convenient to choose $n = m^2$, for then Q is already a sum of two squares : $Q = m^2(m^2 + 1) = (m^2)^2 + m^2$. If further m^2 itself is a sum of two squares, say $m^2 = p^2 + q^2$, then

$$Q = (p^2 + q^2)(m^2 + 1) = (pm + q)^2 + (p - qm)^2.$$

Note that the two representation for Q are distinct. Thus, for example, we may take $m = 5k$, $p = 3k$, $q = 4k$, where k varies over natural numbers. In this case $n = m^2 = 25k^2$, are

$$Q = (25k^2)^2 + (5k)^2 = (15k^2 + 4k)^2 + (20k^2 - 3k)^2.$$

As we vary k over natural numbers, we get infinitely many number of the from $n(n + 1)$ each of which can be expressed as a sum of two squares in two distinct ways.

70. Since 13 divides $a + 11b$, we see that 13 divides $a - 2b$ and hence it also divides $6a - 12b$. This in turn implies that $13 \mid (6a + b)$.

Similarly $11 \mid (a + 13b) \Rightarrow 11 \mid (a + 2b)$

$\Rightarrow \quad 11 \mid (6a + 12b) \Rightarrow 11 \mid (6a + b)$.

Since GCD $(11, 13) = 1$, we conclude that $143 \mid (6a + b)$.

Thus, we may write $6a + b = 143k$ for some natural number k. Hence,

$$6a + 6b = 143k + 5b = 144k + 6b - (k + b).$$

This shows that 6 divides $k+b$ and hence $k+b \geq 6$. We therefore obtain

$$6(a+b) = 143k + 5b = 138k + 5(k+b)$$
$$\geq 138 + 5 \times 6 = 168$$

If follows that $a+b \geq 28$. Taking $a = 23$ and $b = 5$, we see that the conditions of the problem are satisfied. Thus the minimum value of $a+b$ is 28.

71. Consider the dissection of the given 6×6 square in to non-congruent rectangles with least possible areas. The only rectangle with area 1 is an 1×1 rectangle. Similarly, we get $1 \times 2, 1 \times 3$ rectangles for areas 2, 3 units. In the case of 4 units we may have either a 1×4 rectangle or a 2×2 square.

Similarly, there can be a 1×5 rectangle for area 5 units and 1×6 or 2×3 rectangle for 6 units. Any rectangle with area 7 units must be 1×7 rectangle, which is not possible since the largest side could be 6 units. And any rectangle with area 8 units must be a 2×4 rectangle. If there is any dissection of the given 6×6 square in to 9 non-congruent rectangles with areas $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \leq a_9$, then we observe that

$$a_1 \geq 1, a_2 \geq 2, a_3 \geq 3, a_4 \geq 4, a_5 \geq 4, a_6 \geq 5,$$
$$a_7 \geq 6, a_8 \geq 6, a_9 \geq 8,$$

and hence the total area of all the rectangles is

$$a_1 + a_2 + \ldots + a_9 \geq 1 + 2 + 3 + 4 + 4 + 5$$
$$+ 6 + 6 + 8 = 39 > 36$$

which is the area of the given square. Hence, if a 6×6 square is dissected in to 9 rectangles as stipulated in the problem, there must be two congruent rectangles.

72. We have

$$(c+d)^2 = (a+d)^2 + (b+d)^2.$$

This reduces to

$$d^2 + 2d(a+b-c) + a^2 + b^2 - c^2 = 0.$$

Solving the quadratic equation for d, we obtain

$$d = -(a+b-c) \pm \sqrt{(a+b-c)^2 - (a^2+b^2-c^2)}$$
$$= -(a+b-c) \pm \sqrt{2(c-a)(c-b)}.$$

Since d is an integer, $2(c-a)(c-b)$ must be a perfect square; say $2(c-a)(c-b) = x^2$, But GCD $(c-a, c-b) = 1$. Hence we have

$$c - a = 2u^2, c - b = v^2$$

or $\quad c - a = u^2, c - b = 2v^2,$

where $u > 0$ and $v > 0$ and GCD $(u, v) = 1$. In either of the cases $d = -(a+b-c) \pm 2uv$. In the first case

$$c + d = 2c - a - b \pm 2uv$$
$$= 2u^2 + v^2 \pm 2uv$$
$$= (u \pm v)^2 + u^2$$

We observe that $u = v$ implies that $u = v = 1$ and hence $c - a = 2, c - b = 1$. Hence a, b, c are three consecutive integers. We also see that $c + d = 1$. forcing $b + d = 0$, contradicting that $b + d$ is a side of a triangle. Thus $u \neq v$ and hence $c + d$ is the sum of two non-zero integer squares.

Similarly, in the second case we get $c + d = v^2 + (u \pm v)^2$. Thus, $c + d$ is the sum of two squares.

Aliter

One may use characterisation of primitive Pythagorean triples. Observe that GCD $(c-a, c-b) = 1$ implies that $c+d, a+d, b+d$ are relatively prime. Hence, there exist integers $m > n$ such that

$$a + d = m^2 - n^2, b + d + 2mn, c + d = m^2 + n^2.$$

73. Suppose r pages of the book are torn off. Note that the page numbers on both the sides of a page are of the form $2k-1$ and $2k$, and their sum is $4k-1$. The sum of the numbers on the torn pages must be of the form

$$4k_1 - 1 + 4k_2 - 1 + \ldots + 4k_r - 1$$
$$= 4(k_1 + k_2 + \ldots + k_r) - r.$$

The sum of the numbers of all the pages in the untron book is

$$1 + 2 + 3 + \ldots + 100 = 5050$$

Hence, the sum of the numbers on the torn pages is $5050 - 4949 = 101$.

We therefore have

$$4(k_1 + k_2 + \ldots + k_r) - r = 101$$

This shows that $r \equiv 3 \pmod 4$. Thus $r = 4l + 3$ for some $l \geq 0$.

Suppose $r \geq 7$, and suppose $k_1 < k_2 < k_3 < \ldots < k_r$. Then we see that

$$4(k_1 + k_2 + \ldots + k_r) - r \geq 4(k_1 + k_2 + \ldots + k_7) - 7$$
$$\geq 4(1 + 2 + \ldots + 7) - 7$$
$$= 4 \times 28 - 7 = 105 > 101$$

Hence, $r = 3$. This leads to $k_1 + k_2 + k_3 = 26$ and one can choose distinct positive integers k_1, k_2, k_3 in several ways.

74. We write

$$a^2 - 3a - 19 = a^2 - 3a - 70 + 51$$
$$= (a - 10)(a + 7) + 51$$

Suppose 289 divides $a^2 - 3a - 19$ for some integer a. Then, 17 divides it and hence 17 divides $(a - 10)(a + 7)$. Since 17 is a prime, it must divide $(a - 10)$ or $(a + 7)$. But $(a + 7) - (a - 10) = 17$. Hence, whenever 17 divides one of $(a - 10)$ and $(a + 7)$, it must divide the other also. Thus, $17^2 = 289$ divides $(a - 10)(a + 7)$. It follows that 289 divides 51, which is impossible. Thus, there is no integer a for which 289 divides $a^2 - 3a - 19$.

75. We use the standard factorisation :

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

We observe that for any integers x, y,

$$x^2 + 2xy + 2y^2 = (x + y)^2 + y^2 \geq y^2,$$

and $x^2 - 2xy + 2y^2 = (x - y)^2 + y^2 \geq y^2$.

We write

$$3^{2008} + 4^{2009} = 3^{2008} + 4(4^{2008})$$
$$= (3^{502})^4 + 4(4^{502})^4.$$

Taking $x = 3^{502}$ and $y = 4^{502}$, we see that $3^{2008} + 4^{2009} = ab$, where

$$a \geq (4^{502})^2, b \geq (4^{502})^2.$$

But we have

$$(4^{502})^2 = 2^{2008} > 2^{2002} = (2^{11})^{182} > (2009)^{182},$$

since $2^{11} = 2048 > 2009$

Level 2

1. We look for powers of 2 congruent to 1 modulo 7 and find that $2^1 \equiv 2 \pmod 7$

$$2^2 \equiv 4 \pmod 7;\ 2^3 \equiv 1 \pmod 7$$

It follows that every natural numbers k

$$2^{3k} = (2^3 \mid k \equiv 1^k \equiv 1 \pmod 6)$$

Hence, every number of the form $2^{3k} - 1$ divisible by 7, when n is not a multiple of 3.

It is of the form $3k + 1$ or $3k + 2$.

$\because \quad 2^{3k} \equiv 1 \pmod 7$

we have

$$2^{3k+1} = 2 \cdot 2^{3k} \equiv 2 \pmod 7$$
$$2^{3k+2} = 4 \cdot 2^{3k} \equiv 4 \pmod 7$$

From which it follows that multiples of 3 are the only exponents n such that $2^n - 1$ is divisible by 7. From relation it follows that

$$2^{3k+1} + 1 \equiv 3 \pmod 7$$
$$2^{3k+2} + 1 \equiv 5 \pmod 7$$

Moreover $2^{3k} + 1 \equiv 2 \pmod 7$

So, $2^n + 1$ leaves a remainder of 2, 3 or 5 when divided by 7 and hence is not divisible by 7.

2. It is necessary that 2^{34} and 3^{12} each divide $N!$.

If $\left[\frac{N}{3}\right] = 8$, then $\left[\frac{N}{3^2}\right] = 2$ and $\left[\frac{N}{3^3}\right] = 0$

So, $3^{10} | N!$ (not enough). If $\left[\frac{N}{3}\right] = 9$, then

$$\left[\frac{N}{3^2}\right] = 3 \text{ and } \left[\frac{N}{3^3}\right] = 1$$

$\therefore \quad \left[\frac{N}{3}\right] \geq 9$ and $N \geq 27$

Now, $\left[\frac{27}{2}\right] + \left[\frac{27}{2^2}\right] + \left[\frac{27}{2^3}\right] + \left[\frac{27}{2^4}\right]$

$$= 13 + 6 + 3 + 1 = 23$$

So, $2^{23} | N!$ (not enough)

But $N = 28$ works for both 2 and 3.

3. We have

$$(a + b)^7 - a^7 - b^7 = 7ab(a + b)(a^2 + ab + b^2)^2$$

Since, 7 does not divide $ab(a + b)$, we must choose a, b so that 7^3 divides $a^2 + ab + b^2$

i.e., $\quad a^2 + ab + b^2 \equiv 0 \pmod{7^3} \qquad \ldots(i)$

$\because \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2) \qquad \ldots(ii)$

Eq. (ii) is equivalent to

$$a^3 \equiv b^3 \pmod{7^3} \qquad \ldots(iii)$$

For any number c relatively prime to n, we have $e^{\phi(n)} \equiv 1 \pmod n$;

Now, $\quad \phi(7^3) = (7 - 1)7^2 = 3.98$

so $c^{3.08} \equiv 1 \pmod{7^3}$ for any $c \not\equiv 0 \pmod 7$

for *e.g.,* $\quad c = 2$, set $b = 1, a = 2^{98}$

Then, $\quad (2^{98})^3 \equiv 1 \pmod{7^3}$.

$2^{98} \equiv 4 \pmod 7$, $a + b = 2^{98} + 1 \equiv 5 \pmod 7$

$$a - b = 2^{98} - 1 \equiv 3 \pmod 7.$$

7 does not divide $ab(a+b)$ nor $(a-b)$.

Now, 2^{98} is terribly large and can be reduced to size [mod $7^3 = 343$]

for *e.g.*,

$$2^{10} = 1024 = 3.7^3 - 5, \text{ so } 2^{10} \equiv -5 \pmod{7^3}$$

so $2^{20} \equiv 25 \pmod{7^3}, 2^{40} \equiv 61, 2^{80} \equiv -52$

$$2^{90} \equiv -83, 2^8 \equiv -87, 2^{98} \equiv 18 \pmod{7^3}$$

so $a = 18, b = 1$ is a solution.

4. First consider

$$x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$$

where $x = 5^{397}$. We can verify that

$$x^4 + x^3 + x^2 + x + 1 = (x^2 + 3x + 1)^2 - 5x(x+1)^2 \quad \text{...(i)}$$

Since, $x = 5^{397}$ the RHS of Eq. (i) is the difference of two squares and can be factored. It is easy to verify that each of the three factors of $5^{1985} - 1$ exceeds 5^{100}.

5. (i) Denote the five distinct roots of $x^5 - 1 = 0$ by $1, \omega, \omega^2, \omega^3, \omega^4$.

If k is +ve integer, then

$$1 + \omega^k + \omega^{2k} + \omega^{3k} + \omega^{4k} = \begin{cases} 5 & \text{if } k \text{ is multiple of } 5 \\ 0 & \text{otherwise} \end{cases}$$

The first result is obvious and second follows by substitution of $x = \omega^k$ in the identity

$$x^5 - 1 = (x-1)(1 + x + x^2 + x^3 + x^4)$$

Now, replace x successively by $1, \omega, \omega^2, \omega^3, \omega^4$ in the identity

$$x^2(1 + x + x^2 + x^3 + x^4)^{496} = x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots).$$

(We multiplied the given identity by x^2 in order that $a_3, a_8, \dots, a_{5n+3}$ be coefficients of fifth powers of x). Add the resulting five equations and divide by 5 to get

$$5^{495} = a_3 + a_8 + a_{13} + \dots + a_{1983}$$

It shows that any common divisor of $a_3, a_8, \dots, a_{1983}$ is a power of 5.

Now, $a_{1983} = 496$ which is not divisib'

It follows that *gcd* we seek is 1.

(ii) $A \to$ setting $x = 1$, we get $5^{496} > a_{992}$.

Since $a_i \geq 0$, to show that $10^{347} > a_{992}$ it suffices to show that

$$10^{347} > 5^{496} \text{ or } 347 > 496\,(1 - \log 2)$$

An easy way to verify the last inequality is to recall that

$$0.3010 < \log 2 < 0.3011$$

Then,

$$496(1 - \log 2) < 496(1 - 0.3010) = 346.704$$

Now, $2^{10} = 1024 > 10^3$, multiply our inequality by 2^{496} and raise it to the tenth power.

It reduces to

$$2^{4960} > 10^{1490} \text{ or } 1.024^{496} > 100$$

or $$(1.024)^{124} > \sqrt{10}$$

In binomial expansion.

$$(1 + 0.024)^{124} = 1 + 124(0.024) + \dots$$

the sum of first two term is already $> \sqrt{10}$.

(iii) $B \to$ It follows inductively that the coefficients $a_0, a_1, \dots$ are unimodal and symmetric a_{992} being the largest.

Thus, $1985\, a_{992} > 5^{496}$. To show that $a_{992} > 10^{340}$, it suffices to show that

$$5^{496} > 2000.\,10^{340} \text{ or } 497 - 343 > 497 \log 2$$

or $$153 > 149.647 \quad \text{...(ii)}$$

Alternatively we can use the fact that $5^4 > 2^9$ to conclude first that $(5^4)^{39} > (2^9)^{39}$ or $5^{156} > 2^{351}$

$$\because \quad 2^{351} = 2^{11} \cdot 2^{340} > 2 \cdot 10^3 \cdot 2^{340}$$

We have that $5^{156} > 2000.2^{340}$ which is equivalent to Eq. (ii).

6. Let $m = n + 10$, then $n = m - 10$

$$n^3 + 100 = (m-10)^3 + 100$$
$$= m^3 - 30m^2 + 300m - 900.$$

Condition $(n+10)/(n^3+100)$ now translates into

$$m \mid m^3 - 30m^2 + 300m - 900$$

Since, m is a divisor of each of the quantities $m^3, 30m^2, 300m$.

It follows that $m \mid 900$. The largest m for which this is true is $m = 900$, so it follows that the largest n with the given property is $n = 890$.

7. $n^2 + 3n + 2 = (n+1)(n+2)$ and $6 = 2 \cdot 3$

So if 6 is to be divisor of $n^2 + 3n + 2$ then either

(a) 6 is a divisor of $n+1$.

(b) 6 is a divisor of $n+2$.

(c) 3 is a divisor of $n+1$ and 2 is a divisor of $n+2$ or

(d) 2 is a divisor of $n+1$ and 3 is a divisor of $n+2$

Possibility (a) holds for $n = 5, 11, 17, \ldots, 1991$ or 332 values in all. Possibility (b) holds for $n = 4, 10, 16, \ldots, 1990$, another 332 values. Possibility (c) holds for $n = 2, 8, 14, \ldots\ldots, 1981$ another 332 values and possibility (d) holds for $n = 1, 7, 13, \ldots\ldots, 1987$ yet another 332 values. So, there are $4 \times 332 = 1328$ values of n between 1 and 1991 for which $n^2 + 3n + 2$ is divisible by 6.

8. Let n be a given +ve integer, let d be any divisor of n.

Then, n/d is an integer. It is also a divisor of n (as $n = d \times n/d$). If n is not a square, then $n \neq d^2$, so d and n/d are unequal. If we pair up d and n/d then each divisor acquires one and precisely one match.

The divisors now get grouped into pairs and this tell us that the number of divisor is even (for it is twice the number of pairs).

9. $\because$ $$2^3 \equiv 1 \mod 7$$

$2^{3n} \equiv 1 \mod 7$ for any +ve integer n.

$\therefore$ $$2^{3n+1} \equiv 2 \mod 7$$
$$2^{3n+2} \equiv 4 \mod 7$$

So, remainder $(2^n \div 7)$ cycles through the values 1, 2 and 4 in that order.

We never have $2^n \equiv -1$, this means that $2^n + 1$ is never a multiple of 7.

10. Suppose $ab + cd$ is prime.

$$ab + cd = (a+d)c + (b-c)a$$
$$= m\,\text{GCD}\,(a+d, b-c)$$

for some +ve integer m. By assumption either $m = 1$ or $gcd\,(a+d, b-c) = 1$

Case I $m = 1$, then

$$\text{GCD}\,(a+d, b-c)$$
$$= ab + cd > ab + cd - (a - b + c + d)$$
$$= (a+d)(c-1) + (b-1)(a+1)$$
$$\geq \text{GCD}\,(a+d, b-c)$$

which is false.

Case II $\text{GCD}\,(a+d, b-c) = 1$

substituting $ac + bd = (a+d)b - (b-c)a$

for LHS of $ac + bd = (b + d + a - c)$
$$(b + d - a + c)$$

we get $(a+d)(a-c-d) = (b-c)(b+c+d)$

There exist a +ve integer k such that

$$a - c - d = k(b-c);$$
$$b + c + d = k(a+d)$$

Adding these we obtain

$$a + b = k(a + b - c + d) \text{ and thus,}$$
$$k(c-d) = (k-1)(a+b)$$

Recall that $a > b > c > d$

If $k = 1$, then $c = d$, a contradiction.

If $k \geq 2$, then $2 \geq \dfrac{k}{k-1} = \dfrac{a+b}{c-d} > 2$

a contradiction

So, $ab + cd$ is not prime.

11. Let $n = p_1^{a_1} \ldots p_r^{a_r}$

Then, $$d(n) = (a_1+1)(a_2+1)\ldots(a_r+1)$$

and $$d(n^2) = (2a_1+1)(2a_2+1)\ldots(2a_r+1)$$

so, a_i must be chosen so that

$$(2a_1+1)(2a_2+1)\ldots(2a_r+1)$$
$$= k(a_1+1)(a_2+1)\ldots(a_r+1)$$

$\because$ $(2a_i + 1)$ are odd

$\Rightarrow$ k must be odd we show that conversely given any odd k, we can find a_i.

We use a form of induction on k.

It is true for $k = 1$ (Take $n = 1$).

If it is not true for $2^m k - 1$.

That is sufficient since any odd number has the form $2^m k - 1$ for some smaller odd number k. Take $a_i - 2^i(2^m - 1)k - 1$ for $i = 0, 1, \ldots, m-1$

Then, $$2a_i + 1 = 2^{i+1}(2^m - 1)k - (2^{i+1} - 1)$$

and $$a_i + 1 = 2^i(2^m - 1)k - (2^i - 1)$$

So, the product of the $(2a_i + 1)$, divided by the product of the $(a_i + 1)$'s is $2^m(2^m - 1)k - (2^m - 1)$ divided by $(2^m - 1)k$

or $$(2^m k - 1)/k.$$

Thus, if we take these a_i's together with those giving k, we get $2^m k - 1$ which completes induction.

12. $(a, b) = (11, 1), (49, 1)$ or $(7k^2, 7k)$.

If $a < b$, then $b \geq a + 1$

So, $ab^2 + b + 7 > ab^2 + b \geq (a+1)(ab+1)$

$$= a^2b + a + ab \geq a^2b + a + b$$

So, there can be no solutions with $a < b$. Assume that $a \geq b$ let $k =$ the integer $(a^2b + a + b) + (ab^2 + b + 7)$. We have

$$(a/b + 1/b)(ab^2 + b + 7) = ab^2 + a$$

$$+ ab + 7a/b + 7/b + 1 > ab^2 + a + b$$

So, $k < a/b + 1/b$. Now, if $b \geq 3$,

Then, $b(b - 7/b) > 0$

$\therefore$ $(a/b - 1/b)(ab^2 + b + 7) = ab^2 + a - a$

$(b - 7/b) - 1 - 7/b < ab^2 + a < ab^2 + a + b$

Either $b = 1$ or 2 or $k > a/b - 1/b$

If $a/b - 1/b < k < a/b + 1/b$

Then, $a - 1 < kb < a + 1$

Hence, $a = kb$. It gives solution $(a, b) = (7k^2, 7k)$

It remains to consider $b = 1$ and 2.

If $b = 1$, then $a + 8$ divides $a^2 + a + 1$

Also, $a(a + 8) - (a^2 + a + 1) = 7a - 1$

Also, $7(a + 8) - (7a - 1) = 57$

Only factors bigger than 8 are 19 and 57.

So $a = 11$ or 49. It is easy to check

$(a, b) = (11, 1)$ and $(49, 1)$ are indeed solution.

If $b = 2$, then $4a + 9$ divides $2a^2 + a + 2$

Also $a(4a + 9) - 2(2a^2 + a + 2) = 7a - 4$

Also $7(4a + 9) - 4(7a - 4) = 79$

Only factor greater than 9 is 79.

But that gives $a = 35/2$ which is not integral. Hence, there are no solutions for $b = 2$.

13. $(1, p)$ is a solution for every prime p.

Assume $n > 1$ and take q to be the smallest prime divisor of n.

We first show that $q = p$ let x be the smallest +ve integer for which $(p-1)^x = -1 \bmod q$.

y the smallest +ve integer for which

$$(p-1)^y = 1 \bmod q.$$

Certainly y exists and indeed $y < q$.

Since $(p-1)^{q-1} = 1 \bmod q$

We know that $(p-1)^n = -1 \bmod q$

So, x exists also.

$$n = sy + r \text{ with } 0 \leq r < y$$

So, $(p-1)^r = -1 \bmod q$

Hence, $x \leq r < y$ (r cannot be zero).

[$\because$ 1 is not -1 mod q]

write $n = hx + k$ with $0 \leq k < x$

Then, $-1 = (p-1)^n = (-1)^h (p-1)^k \bmod q$ h cannot be even, then

$(p-1)^k = -1 \pmod q$, contradicting the minimality of x, so h is odd.

Hence, $(p-1)^k = 1 \bmod q$ with $0 \leq k < x < y$.

This contradicts the minimality of y unless $k = 0$

So, $n = hx$ but $x < q$ so $x = 1$.

So, $(p-1) = -1 \bmod q$ p and q are primes.

$\therefore$ $q = p$

So, p is the smallest prime divisor of n.

We are also given that $n \leq 2p$ so either.

$p = n$ or $p = 2, n = 4$. The latter does not work so we have shown that $n = p$.

Evidently $n = p = 2$ and $n = p = 3$ work

Now, $p > 3$ we show that there are no solutions of this type.

Expand $(p-1)^p + 1$ by the binomial theorem to get

$$\because \quad (-1)^p = (-1)1 + (-1) + p^2 - \frac{1}{2}p(p-1)p^2 + \frac{p(p-1)(p-2)}{6p^3}$$

The terms of the form p^i with $i \geq 3$ are obviously divisible by p^3.

$\because$ Binomial coefficients are all integral. Hence, sum is

$p^2 +$ (a multiple of p^3) so the sum is not divisible by p^3. But for $p > 3$, p^{p-1} is divisible by p^3 so it cannot divide $(p -)^p + 1$

There are no more solutions.

14. $(1 \cdot 2^1) + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + n \cdot 2^n$

$$= 2^{(n+10)} + (2)$$

LHS can be summed as

$$2^1 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 2$$
$$2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 2^2$$
$$2^3 + \dots + 2^n = 2^{n+1} - 2^3$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$+ 2^n = \frac{2^{n+1} - 2^n}{n(2^{n+1}) - (2^{n+1} - 2)}$$
$$= 2^{n+1}(n-1) + 2$$

$\because$ This equals

$$2^{n+10} + 2,$$
$$n - 1 = \frac{2^{n+10}}{2^{n+1}} = 27$$

So, $\quad n = 2^9 + 1 = 513$

15. Set $\quad A = a^2 + ab + b^2$

$$B = a - b$$

Then, $a^3 - b^3 = AB$. If B is divisible by 2^n, then product AB is divisible by 2^n

$\because$ a and b are odd.

$A = a^2 + ab + b^2$ is a sum of 3 odd numbers and is therefore also odd.

So, A is relatively prime to 2^n.

Thus, 2^n divides AB only if it divides B.

16. Let $\quad x = x_1 + u$

$$y = y_1 + v$$

where x_1 and y are non negative integers and $0 \le u < 1, 0 \le v < 1$

Now, $x_1 + y_1 + [5u] + [5v]$

$$\ge [3u + v] + [3v + u] \quad \dots\text{(i)}$$

We prove that,

$$[5u] + [5v] \ge [3u + v] + [3v + u] \quad \dots\text{(ii)}$$

which implies Eq. (i).

In view of the symmetric roles of u and v we assume $u \ge v$ which gives inequality $[5u] \ge [3u + v]$.

If $u \le 2v$ we also have

$$[5v] \ge [3v + u]$$

So Eq. (ii) is established in this case finally, we prove that Eq. (ii) hold if $u > 2v$

Let $5u = a + b$ and $5v = c + d$. where a and c are non -ve integers.

$$0 \le b < 1, 0 \le d < 1$$

Then, Eq. (ii) can be written as

$$a + c \ge [(3a + c + 3b + d)/5]$$
$$+ [3c + a + 3d + b)/5] \quad \dots\text{(iii)}$$

Now, $1 > u > 2v$, so that $5 > 5u > 10v$ or $5 > a + b > 2c + 2d$. The first inequality here gives. $5 > a$ or $4 \ge a$.

IInd inequality gives $a \ge 2c$ as $a < 2c$ would imply that $a \le 2c - 1$, $a + 1 - 2c \le 0$ and $a + b - 2c < 0$

Thus, we have $4 \ge a \ge 2c$ and hence only cases we need to consider are

a	4	4	4	3	3	2	2	1	0
c	2	1	0	1	0	1	0	0	0

It is easy to verify Eq. (iii) in these 9 cases

as $\quad 3a + d$ and $3d + b < 4$

17. Let us denote the digits in tens, hundreds and ones place of the sought for number N. as x, y, z respectively

$$N = 100x + 10y + z$$
$$100x + 10y + z = x! + y! + z!$$

Now, $\quad 7! = 5040$

So none of digits of N exceeds 6.

N itself does not exceed 700.

Its digit can exceed 5 (as $6! = 720 > 700$)

At least one digit of $N = 5$. N is a 3 digit number and $3 \cdot 4! = 72 < 100$.

$\therefore$ x cannot be equal to 5

$\because \quad 3 \cdot 5! = 360 < 500$

x cannot exceed 3.

x does not exceed 2 since $3! + 2 \cdot 5! = 246 < 300$

But 255 does not satisfy.

x cannot exceed 1 as $2! + 5! + 4! = 146 < 200$

Now, $\quad 1! + 5! + 4! = 145 < 150$

y cannot exceed 4.

$z = 5$ [at least one value of N must be 5]

We have $x = 1$, $4 \ge y \ge 0$ and $z = 5$.

So, $\quad N = 145$

18. $N = 1$

19.

$$2x + 3y = k \quad \dots\text{(i)}$$

We obtain $x = \dfrac{k - 3y}{2} = -y + \dfrac{k - y}{2} \quad \dots\text{(ii)}$

From Eq. (ii), we find that x is an integer only if $(k - y)/2$ is an integer.

Hence, $(k - y)/2 = s$

Hence, $y = k - 2s$

equation 2 yields

$$x = -y + s = -(k - 2s) + 3s - k$$

If integers x and y satisfy Eq. (i) they are of form

$$x = -k + 3s, y = k - 2s \quad \text{...(iii)}$$

where s is some integer.

for an arbitrary integer s, Eq. (iii) defines integers x and y which satisfied Eq. (i).

In a same way we can determine integral solution of $9x + 5y = l$ ...(iv)

where l is a given integer.

$$y = \frac{l - 9x}{5} = -2x + \frac{l + x}{5} \quad \text{...(v)}$$

y is an integer only if $(l + x)/5$ is an integer say $\frac{l + x}{5} = t$.

Hence, $x = 5t - l$ and substitute in Eq. (v)

$$y = -2x + t = -2(5t - l) + t = -9t + 2l$$

If x and y satisfy Eq. (iv) they are of form

$$x = 5t - l, y = -9t + 2l \quad \text{...(vi)}$$

where t is an integer.

Conversely, for an arbitrary integer t

Eq. (vi) defines integers x and y which satisfy Eq. (iv)

If $2x + 3y =$ some multiple of 17 say $17.n$.

then $x = -17n + 3.3s\ (x, y)$

$y = 17n - 3s$ (n, s) are integers

then $9x + 5 = 9(-17n + 35) + 5(17n - 2s)$

$= 17(-4n + s)$

i.e., if $2x + 3y$ is divisible by 17 so $9x + 5y$.

For all integers x and y for which $9x + 5y$ is divisible by 17, $2x + 3y$ is also divisible by 17.

20. $\because$ Product of the sought for number by 2 has same number of digits as the original number. Initial digit of that number cannot exceed 4. When initial digit carried to end the resultant number must be even (it is equal to the duplicated original number). So, initial digit of the sought for number is equal to 2 or 4.

Let us suppose that the initial digit of the sought for number is equal to 2 or 4 on denoting x the number obtained from sought for number by discading its initial digit.

We can write by analogy with first of equality

$$(2 \cdot 10^m + x) \cdot 2 = 10x + 2$$

$$x = \frac{4 \cdot 10^m - 2}{8} = \frac{2 \cdot 10^m - 1}{2}$$

or $(4 \cdot 10^m + x) \cdot 2 = 10 \times + 4$

$$x = \frac{8 \cdot 10^m - 4}{8} = \frac{2 \cdot 10^m - 1}{2}$$

Neither of the formulas for no. x hold as a whole no. can not be equal to a fraction whose numerator is odd and denominator is even.

21. Let $n^2 + n + 41 = m^2$, then

$$4m^2 = 4n^2 + 4n + 164 = (2n + 1)^2 + 163$$

$$(2m + 2n + 1)(2m - 2n - 1) = 163$$

$\because$ 163 is prime we must have

$2m + 2n + 1 = \pm 1, \pm 163.$

Correspondingly $2m - 2n - 1 = \pm 163, \pm 1$.

Subtracting $4n + 2 = \mp 162$

Hence, $n = -41$ or 40.

22. Suppose there are four numbers a, b, c, d among the given nine numbers which leave the same remainder modulo 20. Then, $a + b \equiv c + d \pmod{20}$ and we are done.

If not, there are two possibilities :

1. We may have two disjoint pairs $\{a, c\}$ and $\{b, d\}$ obtained from the given nine numbers such that $a \equiv c \pmod{20}$ and $b \equiv d \pmod{20}$. In this case we get $a + b \equiv c + d \pmod{20}$.
2. Or else there are at most three numbers having the same remainder modulo 20 and the remaining six numbers leave distinct remainders which are also different from the first remainder (*i.e.*, the remainder of the three numbers). Thus there are at least 7 distinct remainders modulo 20 that can be obtained from the given set of nine numbers. These 7 remainders give rise to $\binom{7}{2} = 21$ pairs of numbers. By pigeonhole principle, there must be two pairs $(r_1, r_2), (r_3, r_4)$ such that $r_1 + r_2 \equiv r_3 + r_4 \pmod{20}$. Going back we get four numbers a, b, c, d such that $a + b \equiv c + d \pmod{20}$.

If we take the numbers 0, 0, 0, 1, 2, 4, 7, 12, we check that the result is not true for these eight numbers.

23. We show that the answer is NO. Suppose, if possible, let a, b, c be three distinct positive real numbers such that $a, b, c, b + c - a$, $c + a - b, a + b - c$ and $a + b + c$ form a 7-term arithmetic progression in some order. We may assume that $a < b < c$. Then, there are only two cases we need to check:

(I) $a + b - c < a < c + a - b < b < c < b + c - a < a + b + c$

and (II) $a + b - c < a < b < c + a - b < c < b + c - a < a + b + c$.

Case I Suppose the chain of inequalities $a + b - c < a < c + a - b < c < b + c - a < a + b + c$ holds good. Let d be the common difference. Thus we see that

$c = a + b + c - 2d, b = a + b + c - 3d,$
$a = a + b + c - 5d.$

Adding these, we see that $a + b + c = 5d$. But then $a = 0$ contradicting the positivily of a.

Case II Suppose the inequalities

$$a + b - c < a < b < c + a - b < c < b + c - a < a + b + c$$

are true. Again we see that

$c = a + b + c - 2d, b = a + b + c - 4d,$
$a = a + b + c - 5d.$

We thus obtain $a + b + c = (11/2)d$. This gives

$$a = \frac{1}{2}d,\ b = \frac{3}{2}d, c = \frac{7}{2}d.$$

Note that $a + b - c = a + b + c - 6d = -(1/2)d$. However we also get

$$a + b - c = [(1/2) + (3/2) - (7/2)]d = -(3/2)d.$$

It follows that $3e = e$ giving $d = 0$. But this is impossible.

Thus there are no three distinct positive real numbers a, b, c such that $a, b, c, b + c - a$, $c + a - b, a + b - c$ and $a + b + c$ form a 7-term arithmetic progression in some order.

24. We observe that

$$Q = a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

Since, we are looking for the least positive value taken by Q, it follows that a, b, c are not all equal.

Thus, $a + b + c \geq 1 + 1 + 2 = 4$ and $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 1 + 1 + 0 = 2.$

Thus we see that $Q \geq 4$. Taking $a = 1, b = 1$ and $c = 2$, we get $Q = 4$. Therefore, the least value of Q is 4 and this is achieved only by $a + b + c = 4$ and $(a - b)^2 + (b - c)^2 + (c - a)^2 = 2$. The triples for which $Q = 4$ are therefore given by

$$(a, b, c) = (1, 1, 2), (1, 2, 1), (2, 1, 1).$$

25. Obviously $p \neq q$. We write this in the form

$$p(p^{n-1} + p^{n-2} + \ldots + 1) = q(q + 1).$$

If $q \leq p^{n/2} - 1$, then $q < p^{n/2}$ and hence we see that $q^2 < p^n$. Thus, we obtain

$$q^2 + q < p^n + p^{n/2} < p^n + p^{n-1} + \ldots + p$$

Since $n > 2$. It follows that $q \geq p^{n/2}$. Since, $n > 2$ and is an even number, $n/2$ is a natural number larger than 1. This implies that $q \neq p^{n/2}$ by the given condition that q is a prime. We conclude that $q \geq p^{n/2} + 1$. We may also write the above relation in the form

$$p(p^{n/2} - 1)(p^{n/2} + 1) = (p - 1)q(q + 1).$$

This shows that q divides $(p^{n/2} - 1)(p^{n/2} + 1)$. But $q \geq p^{n/2} + 1$ and q is a prime. Hence, the only possibility is $q = p^{n/2} + 1$. This gives

$$p(p^{n/2} - 1) = (p - 1)(q + 1) = (p - 1)(p^{n/2} + 2).$$

Simplification leads to $3p = p^{n/2} + 2$. This shows that p divides 2. Thus $p = 2$ and hence $q = 5, n = 4$. It is easy to verify that these indeed satisfy the given equation.

26. We show that the required GCD is 24. Consider an element $(a, b, c, d, e, f) \in S$. We have $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$.

We first observe that not all a, b, c, d, e can be odd. Otherwise, we have

$$a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 8$$

and hence $f^2 \equiv 5 \pmod 8$, which is impossible because to square can be congruent to 5 modulo 8. Thus, at least one of a, b, c, d, e is even.

Similarly, if none of a, b, c, d, e, is divisible by 3, then $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 3$ and hence $f^2 \equiv 2 \pmod 2$ which again is impossible because no square is congruent to 2 modulo 3. Thus, 3 divides $abcdef$.

There are several possibilities for a, b, c, d, e.

Case I Suppose one of them is even and the other four are odd; say a is even, b, c, d, e are

odd. Then, $b^2 + c^2 + d^2 + e^2 \equiv 4 \pmod 8$. If $a^2 \equiv 4 \pmod 8$, then $f^2 \equiv 0 \pmod 8$ and hence $2 \mid a$, $4 \mid f$ giving $8 \mid af$. If $a^2 \equiv 0 \pmod 8$, then $f^2 \equiv 4 \pmod 8$ which again gives that $4 \mid a$ and $2 \mid f$ so that $8 \mid af$. It follows that $8 \mid abcdef$ and hence $24 \mid abcdef$.

Case II Suppose a, b are even and c, d, e are odd. Then, $c^2 + d^2 + e^2 \equiv 3 \pmod 8$. Since $a^2 + b^2 \equiv 0$ or 4 modulo 8, it follows that $f^2 \equiv 3$ or $7 \pmod 8$ which is impossible. Hence, this case does not arise.

Case III If three of a, b, c, d, e are even and two odd, then $8 \mid abcdef$ and hence $24 \mid abcdef$.

Case IV If four of a, b, c, d, e are even, then again $8 \mid abcdef$ and $24 \mid abcdef$. Here, again for any six tuple (a, b, c, d, e, f) in S, we observe that $24 \mid abcdef$. Since

$$1^2 + 1^2 + 1^2 + 2^2 + 3^2 = 4^2.$$

We see that $(1, 1, 1, 2, 3, 4) \in S$ and hence $24 \in T$. Thus, 24 is the GCD of T.

27. (a) Let us assume that there are no even numbers in the sequence. This means that x_{n+1} is obtained from x_n, by adding a non-zero even digit of x_n to x_n, for each $n \geq 1$.

Let E be the left most even digit in x_1 which may be taken in the form

$$x_1 = O_1O_2 \ldots O_kED_1D_2 \ldots D_l$$

where $O_1, O_2, \ldots, O_k$ are odd digits $(k \geq 0)$; $D_1, D_2, \ldots, D_{l-1}$ are even or odd; and D_l odd, $l \geq 1$.

Since, each time we are adding at least 2 to a term of the sequence to get the next term, at some stage, we will have a term of the form

$$x_r = O_1O_2 \ldots O_kE999\ldots9\,F$$

where $F = 3, 5, 7$ or 9. Now we are forced to add E to x_r to get x_{r+1}, as it is the only even digit available. After at most four steps of addition, we see that some next term is of the form

$$x_s = O_1O_2 \ldots O_kG000 \ldots M$$

where G replaces E of x_r, $G = E + 1$, $M = 1, 3, 5$ or 7. But x_s has no non-zero even digit contradicting our assumption. Hence, the sequence has some even number as its term.

(b) If there are only finitely many even terms and x_t is the last term, then the sequence $\langle x_n \rangle_{n=t+1}^{\infty} = \langle x_{t+1}, x_{t+2}, \ldots \rangle$ is obtained in a similar manner and hence must have an even term by (a), a contradiction. Thus, $\langle x_n \rangle_{n=1}^{\infty}$, has infinitely many even terms.

28. It can be easily seen that

$$9n = (2b + 2c - a)^2 + (2c + 2a - b)^2 + (2a + 2b - c)^2.$$

Thus, we can take $p_1 = p_2 = p_3 = 2$, $q_1 = q_2 = q_3 = 2$ and $r_1 = r_2 = r_3 = -1$. Suppose 3 does not divide GCD (a, b, c). Then, 3 does divide at least one of a, b, c, say 3 does not divide a. Note that each of $2b + 2c - a$, $2c + 2a - b$ and $2a + 2b - c$ is either divisible by 3 or none of them is divisible by 3, as the difference of any two sums is always divisible by 3. If 3 does not divide $2b + 2c - a$, then we have the required representation. If 3 divides $2b + 2c - a$, then 3 does not divide $2b + 2c + a$. On the other hand, we also note that

$$9n = (2b + 2c + a)^2 + (2c - 2a - b)^2 + (-2a + 2b - c)^2 = x^2 + y^2 + z^2,$$

where $x = 2b + 2c + a$, $y = 2c - 2a - b$ and $z = -2a + 2b - c$. Since, $x - y = 3(b + a)$ and 3 does not divide x, it follows that 3 does not divide y as well. Similarly, we conclude that 3 does not divide z.

29. Suppose $0 \in A$. Then, $a^2 = a^2 + 0 \times b$ is rational and $ab = 0^2 + ab$ is also rational for all a, b in A, $a \neq 0, b \neq 0, a \neq b$. Hence, $a = a_1\sqrt{M}$ for some rational a_1 and natural number M. For any $b \neq 0$, we have

$$b\sqrt{M} = \frac{ab}{a_1},$$

which is a rational number.

Hence, we may assume 0 is not in A. If there is a number a in A such that $-a$ is also in A, then again we can get the conclusion as follows. Consider two other elements c, d in A. Then, $c^2 + da$ is rational and $c^2 - da$ is also rational. It follows that c^2 is rational and da is rational. Similarly, d^2 and ca are also rationals. Thus, $d/c = (da)/(ca)$ is rational. Note that we can vary d over A with $d \neq c$ and $d \neq a$. Again c^2 is rational implies that $c = c_1\sqrt{M}$ for some rational c_1 and natural number M. We observe that $c\sqrt{M} = c_1 M$ is rational, and

$$a\sqrt{M} = \frac{ca}{c_1},$$

so that $a\sqrt{M}$ is a rational number. Similarly is the case with $-a\sqrt{M}$. For any other element d,

$$b\sqrt{M} = Mc_1 \frac{d}{c}$$

is a rational number.

Thus, we may now assume that 0 is not in A and $a + b \neq 0$ for any a, b in A. Let a, b, c, d be four distinct elements of A. We may assume $|a| > |b$. Then, $d^2 + ab$ and $d^2 + bc$ are rational numbers and so is their difference $ab - bc$. Writting $a^2 + ab = a^2 + bc + (ab - bc)$, and using the facts $a^2 + bc, ab - bc$ are rationals, we conclude that $a^2 + ab$ is also a rational number. Similarly, $b^2 + ab$ is also a rational number.

Consider $q = \frac{a}{b} = \frac{a^2 + ab}{b^2 + ab}$

Note that $a^2 + ab > 0$. Thus, q is a rational number and $a = bq$. This gives $a^2 + ab = b^2(q^2 + q)$. Let us take $b^2(q^2 + q) = l$.

Then, $|b| = \sqrt{\frac{l}{q^2 + q}} = \sqrt{\frac{x}{y}}$,

where x and y are natural number. Take $M = xy$. Then, $|b|\sqrt{M} = x$ is a rational number. Finally, for any c in A, we have

$$c\sqrt{M} = b\sqrt{M}\frac{c}{b},$$

is also a rational number.

30. We show that x is irrational. Suppose that x is rational. Then, the sequence $\langle a_n \rangle_{n=1}^{\infty}$ is periodic after some stage; there exist natural numbers k, l such that $a_n = a_{n+l}$ for all $n \geq k$. Choose m such that $ml \geq k$ and ml is a perfect square. Let

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}, l = p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r},$$

be the prime decompositions of m, l so that $\alpha_j + \beta_j$ is even for $1 \leq j \leq r$. Now take a prime p different from $p_1, p_2, \dots, p_r$. Consider ml and pml. Since $pml - ml$ is divisible by l, we have $a_{pml} = a_{ml}$. Hence, $d(pml)$ and $d(ml)$ have same parity. But $d(pml) = 2d(ml)$, since $\text{GCD}(p, ml) = 1$ and p is a prime. Since ml is a square, $d(ml)$ is odd. It follows that $d(pml)$ is even and hence $a_{pml} \neq a_{ml}$. This contradiction implies that x is irrational.

Aliter

As earlier, assume that x is rational and choose natural numbers k, l such that $a_n = a_{n+l}$ for all $n \geq k$. Consider the numbers $a_{m+1}, a_{m+2}, \dots, a_{m+l}$, where $m \geq k$ is any number. This must contain at least one 0. Otherwise $a_n = 1$ for all $n \geq k$. But $a_r = 0$ if and only if r is a square. Hence, it follows that there are no squares for $n > k$, which is absurd. Thus every l consecutive terms of the sequence $\langle a_n \rangle$ must contain a 0 after certain stage. Let $t = \max\{k, l\}$, and consider t^2 and $(t+1)^2$. Since, there are no squares between t^2 and $(t+1)^2$, we conclude that $a_{t^2+j} = 1$ for $1 \leq j \leq 2t$. But then, we have $2t(> l)$ consecutive terms of the sequence $\langle a_n \rangle$ which miss 0, contradicting our earlier observation.

31. (a) Consider $f(j) = a_{m+j} + (-1)^j a_{m-j}$, $0 \leq j \leq m$, where m is a natural number. We observe that $f(0) = 2a_m$ is divisible by $2a_m$.

Similarly,

$$f(1) = a_{m+1} - a_{m-1} = 2a_m$$

is also divisible by $2a_m$. Assume that $2a_m$ divides $f(j)$ for all $0 \leq j < l$, where $l \leq m$. We prove that $2a_m$ divides $f(l)$. Observe

$$f(l-1) = a_{m+l-1} + (-1)^{l-1} a_{m-l+1}.$$

$$f(l-2) = a_{m+l-2} + (-1)^{l-2} a_{m-l+2}.$$

Thus, we have

$$\begin{aligned} a_{m+l} &= 2a_{m+l-1} + a_{m+l-2} \\ &= 2f(l-1) - 2(-1)^{l-1} a_{m-l+1} \\ &\quad + f(l-2) - (-1)^{l-2} a_{m-l+2} \\ &= 2f(l-1) + f(l-2) \\ &\quad + (-1)^{l-1}(a_{m-l+2} - 2a_{m-l+1}) \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} a_{m-l}. \end{aligned}$$

This gives

$$f(l) = 2f(l-1) + f(l-2).$$

By induction hypothesis $2a_m$ divides $f(l-1)$ and $f(l-2)$. Hence, $2a_m$ divides $f(l)$. We conclude that $2a_m$ divides $f(j)$ for $0 \leq j \leq m$.

(b) We see that $f(m) = a_{2m}$. Hence, $2a_m$ divides a_{2m} for all natural number m. Let $n = 2^k l$ for some $l \geq 1$. Taking $m = 2^{k-1} l$, we see that $2a_m$ divides a_n. Using an easy induction, we conclude that $2^k a_l$ divides a_n. In particular 2^k divides a_n.

32. Suppose $n = pqr$, where $p < q$ are primes and $r > 1$. Then, $p \geq 2, q \geq 3$ and $r \geq 2$, not necessarily a prime. Thus we have

$$n - 2 \geq n - p = pqr - p \geq 5p > p,$$
$$n - 2 \geq n - q = q(pr - 1) \geq 3q > q,$$
$$n - 2 \geq n - pr = pr(q - 1) \geq 2pr > pr,$$
$$n - 2 \geq n - qr = qr(p - 1) \geq qr.$$

Observe that p, q, pr, qr are all distinct. Hence, their product divides $(n-2)!$. Thus, $n^2 = p^2q^2r^2$ divides $(n-2)!$ in this case. We conclude that either $n = pq$ where p, q are distinct primes or $n = p^k$ for some prime p.

Case I Suppose $n = pq$ for some primes p, q where $2 < p < q$. Then, $p \geq 3$ and $q \geq 5$. In this case

$$n - 2 > n - p = p(q - 1) \geq 4p,$$
$$n - 2 > n - q = q(p - 1) \geq 2q$$

Thus $p, q, 2p, 2q$ are all distinct numbers in the set $\{1, 2, 3, \ldots, n-2\}$. We see that $n^2 = p^2q^2$ divides $(n-2)!$. We conclude that $n = 2q$ for some prime $q \geq 3$. Note that $n - 2 = 2q - 2 < 2q$ in this case so that n^2 does not divide $(n-2)!$.

Case II Suppose $n = p^k$ for some prime p. We observe that $p, 2p, 3p, \ldots, (p^{k-1} - 1)p$ all lie in the set $\{1, 2, 3, \ldots, n-2\}$. If $p^{k-1} - 1 \geq 2k$, then there are at least $2k$ multiples of p in the set $\{1, 2, 3, \ldots, n-2\}$. Hence, $n^2 = p^{2k}$ divides $(n-2)!$. This $p^{k-1} - 1 < 2k$.

If $k \geq 5$, then $p^{k-1} - 1 \geq 2^{k-1} - 1 \geq 2k$, which may be proved by an easy induction. Hence, $k \leq 4$. If $k = 1$, we get $n = p$, a prime. If $k = 2$, then $p - 1 < 4$ so that $p = 2$ or 3, we get $n = 2^2 = 4$ or $n = 3^2 = 9$. For $k = 3$, we have $p^2 - 1 < 6$ giving $p = 2$; $n = 2^3 = 8$ in this case. Finally, $k = 4$ gives $p^3 - 1 < 8$. Again $p = 2$ and $n = 2^4 = 16$. However $n^2 = 2^8$ divides $14!$ and hence is not a solution.

Thus, $n = p, 2p$ for some prime p or $n = 8, 9$. It is easy to verify that these satisfy the conditions of the problem.

33. (i) Suppose $n \in N$ is faithful. Let $k \in N$ and consider kn. Since $n = a + b + c$, with $a > b > c$, $c \mid b$ and $b \mid a$, we see that $kn = ka + kb + kc$ which shows that kn is faithful.

Let $p > 5$ be a prime. Then, p is odd and $p = (p-3) + 2 + 1$ shows that p is faithful. If $n \in N$ contains a prime factor $p > 5$, then the above observation shows that n is faithful. This shows that a number which is not faithful must be of the form $2^\alpha 3^\beta 5^\gamma$. We also observe that $2^4 = 16 = 12 + 3 + 1$, $3^2 = 9 = 6 + 2 + 1$ and $5^2 = 25 = 22 + 2 + 1$, so that $2^4, 3^2$ and 5^2 are faithful. Hence, $n \in N$ is also faithful if it contains a factor of the form 2^α where $\alpha \geq 4$; a factor of the form 3^β where $\beta \geq 2$; or a factor of the form 5^γ where $\gamma \geq 2$. Thus the numbers which are not faithful are of the form $2^\alpha 3^\beta 5^\gamma$, where $\alpha \leq 3, \beta \leq 1$ and $\gamma \leq 1$. We may enumerate all such numbers.

1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.

Among these $120 = 112 + 7 + 1$, $60 = 48 + 8 + 4$, $40 = 36 + 3 + 1$, $30 = 18 + 9 + 3$, $20 = 12 + 6 + 2$, $15 = 12 + 2 + 1$, and $10 = 6 + 3 + 1$. It is easy to check that the other numbers cannot be written in the required form. Hence, the only numbers which are not faithful are 1, 2, 3, 4, 5, 6, 8, 12, 24.

Their sum is 65.

(ii) If $n = a + b + c$ with $a < b < c$ is faithful, we see that $a \geq 1, b \geq 2$ and $c \geq 4$. Hence, $n \geq 7$. Thus 1, 2, 3, 4, 5, 6 are not faithful. As observed earlier, kn is faithful whenever n is. We also notice that for odd $n \geq 7$, we can write $n = 1 + 2 + (n-3)$ so that all odd $n \geq 7$ are faithful. Consider $2n, 4n, 8n$, where $n \geq 7$ is odd. By observation, they are all faithful. Let us list a few of them :

$2n$: 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, ...

$4n$: 28, 36, 44, 52, 60, 68, ...

$8n$: 56, 72, ...

We observe that $16 = 12 + 3 + 1$ and hence it is faithful. Thus all multiples of 16 are also faithful. Thus we see that 16, 32, 48, 64, ... are faithful. Any even number which is not a multiple of 16 must be either an odd multiple of 2 or that of 4, or that of 8. Hence, the only numbers not covered by this process are 8, 10, 12, 20, 24, 40. Of these, we see that

$$10 = 1 + 3 + 6, 20 = 2 \times 10, 40 = 4 \times 10,$$

so that 10, 20, 40 are faithful. Thus, the only numbers which are not faithful are

1, 2, 3, 4, 5, 6, 8, 12, 24.

Their sum is 65.

Unit 2 Theory of Equations

Unit 2

Theory of Equations

Polynomial

An expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

where n is a whole number and $a_0, a_1, a_2, \ldots, a_n$ belong to some number system F, is called a polynomial in the variable x over the number system F. A *polynomial* is denoted by $f(x)$ or $g(x)$ etc.

Real Polynomial

A polynomial is called a *real polynomial* if all the coefficients are real numbers.

Leading Coefficient and Leading Term

If $a_0 \neq 0$, then a_0 (the coefficient of highest degree term) is called the *leading coefficient* and a_0x^n is called the *leading term.*

Degree of Polynomial

The highest index of the variable x occuring in the polynomial $f(x)$ is called the *degree of the polynomial.*

Zero Degree Polynomial

The constant $c = cx^0$ is called a polynomial of *degree zero.*

Linear Polynomial

The polynomial $f(x) = ax + b, a \neq 0$ is of degree one and is called a *linear polynomial.*

Quadratic Polynomial

The polynomial $f(x) = ax^2 + bx + c, a \neq 0$ is of degree two and is called a *quadratic polynomial.*

Cubic Polynomial

The polynomial $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$ is of degree three and is called a *cubic polynomial.*

Biquadratic Polynomial

The polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$ is of degree four and is called a *biquadratic polynomial.*

Zero Polynomial

A polynomial, all of whose coefficients are zero, is called a *zero polynomial.*

Equality of Two Polynomials

Two polynomials

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$
$$g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$$

are said to be equal if the coefficients of like powers of x in the two polynomials are equal.

When $m \geq n$, if $f(x) = g(x)$, then $a_0 = b_0, a_1 = b_1, \ldots, a_n = b_n, b_{n+1} = b_{n+2} = \ldots = b_m = 0$

When $m \leq n$, if $f(x) = g(x)$, then $a_0 = b_0, a_1 = b_1, \ldots, a_m = b_m$, $a_{m+1} = a_{m+2} = \ldots = a_n = 0$

Complete and Incomplete Polynomials

A polynomial of degree n, which contains all powers of the variable from 0 to n, is called a complete polynomial. Otherwise it is said to be incomplete polynomial.

For example, the incomplete polynomial $2x^5 + 7x^3 - 5$ can be made complete by writing it as $2x^5 + 0x^4 + 7x^3 + 0x^2 + 0x - 5$.

Theorem 1 *If (atleast) one of the two polynomials $f(x)$ and $g(x)$ is the zero polynomial, then the product $f(x) \cdot g(x)$ is also the zero polynomial.*

Proof Let

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_ix^i + \ldots$$
$$g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_ix^i + \ldots$$

be any two given polynomials.

Since, atleast one of the two polynomials $f(x), g(x)$ is zero polynomial.

$\therefore$ Without loss of generality

Let $f(x) =$ zero polynomial

Now, coefficient of x^i in $f(x) \cdot g(x)$

$$= a_0b_i + a_1b_{i-1} + \ldots + a_ib_0 = 0 \cdot b_i + 0 \cdot b_{i-1} + \ldots + 0 \cdot b_0 = 0 \; \forall \, i \geq 0$$
$$f(x) \cdot g(x) = 0$$

Thus, the product $f(x) \cdot g(x)$ is a zero polynomial.

Theorem 2 *Let $f(x), g(x)$ be two non-zero polynomials, then*

(i) $f(x) \cdot g(x$ is a non-zero polynomials.

(ii) $deg\,\{f(x) \cdot g(x)\} = deg\, f(x) + deg\, g(x)$

Proof Let $\quad f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_i\,x^i + \ldots$

and $\quad g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_i\,x^i + \ldots$

be any two non-zero polynomials.

Let $\quad \deg f(x) = m$

$\therefore \quad a_m \neq 0$ and $a_i = 0$

$\forall\, i > m$ and $\deg g(x) = n$

$\therefore \quad b_n \neq 0$ and $b_i = 0, \forall\, i > n$

(i) Coefficient of x^{m+n} in $f(x)\cdot g(x)$

$$= a_0b_{m+n} + a_1b_{m+n-1} + \ldots + a_m\, b_n + \ldots + a_{m+n}\, b_0$$

$$= a_0\cdot 0 + a_1\cdot 0 + \ldots + a_m\, b_n + \ldots + 0\cdot b_0 \qquad [\because a_i = 0\ \forall\, i > m,\ b_i = 0\ \forall\, i > n]$$

$$= a_m b_n \neq 0 \qquad [\because a_m \neq 0, b_n \neq 0 \therefore a_m b_n \neq 0]$$

$\therefore$ $f(x)\cdot g(x)$ is not a zero polynomial.

(ii) Coefficient of x^{m+n+1} in $f(x)\cdot g(x)$

$$= a_0b_{m+n+1} + a_1b_{m+n} + \ldots + a_mb_{n+1} + a_{m+1}b_n + \ldots + a_{m+n+1}\, b_0$$

$$= a_0\cdot 0 + a_1\cdot 0 + \ldots + a_m\cdot 0 + 0\cdot b_n + \ldots + 0\cdot b_0 \qquad [\because b_i = 0\ \forall\, i > n \text{ and } a_i = 0\ \forall\, i > m]$$

$$= 0$$

$\therefore$ Coefficient of x^{m+n+1} in $f(x)\cdot g(x) = 0$

Similarly, coefficient of x^{m+n+2} in $f(x)\cdot g(x) = 0$ and so on

Also coefficient of x^{m+n} in $f(x)\cdot g(x) \neq 0$

$$\therefore \quad \deg\{f(x)\cdot g(x)\} = m + n$$

$$= \deg f(x) + \deg g(x)$$

Division Algorithm

If $f(x)$ and $g(x)$ are two non-zero polynomials, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x)\cdot g(x) + r(x)$$

where either $r(x) = 0$

$$\deg r(x) < \deg g(x)$$

The polynomial $q(x)$ is called the quotient and $r(x)$ the remainder.

When $f(x)$ is divided by $g(x)$, then degree of $q(x) = \deg f(x) - \deg g(x)$

Particular Case : When $g(x) = ax + b$, a linear polynomial, then either $r(x) = 0$ or

$$\deg r(x) < \deg g(x) = 1$$

i.e., $\deg r(x) = 0$

So that $r(x)$ is a constant.

Root of an Equation

A number 'α' is called a root of equation

$$f(x) = 0, \text{ iff } f(\alpha) = 0$$

α is a root of $f(x) = 0$

$\Rightarrow$ $f(\alpha) = 0$

Conversely $f(\alpha) = 0$

$\Rightarrow$ α is a root of $f(x) = 0$

Theorem 3 *Remainder Theorem : If $f(x)$ is a polynomial, then $f(h)$ is the remainder when $f(x)$ is divided by $x - h$.*

Proof Let $Q(x)$ be the quotient R the remainder.

When $f(x)$ is divided by $x - h$.

Then, $f(x) = (x - h)\,Q(x) + R$

Putting $x = h, f(h) = R,$

i.e., $R = f(h)$

Theorem 4 *Factor Theorem : If h is a root of equation $f(x) = 0$, then $(x - h)$ is a factor of $f(x)$ and conversely.*

Proof Let $Q(x)$ be the quotient and R, the remainder when $f(x)$ is divided by $x - h$.

Then, $\quad f(x) = (x - h)Q(x) + R$

Putting $\quad x = h, f(h) = R,$

But $\quad f(h) = 0$

$\because h$ is a root of $f(x) = 0$

$\therefore \quad R = 0$

$\therefore \quad f(x) = (x - h)Q$

which shows that $x - h$ is a factor of $f(x)$.

Conversely, if $x - h$ is a factor of $f(x)$, h must be a root of $f(x) = 0$.

Divide $f(x)$ by $x - h$ and let $Q(x)$ be the quotient.

Then, $\quad f(x) = (x - h)Q(x)$

Putting $\quad x = h, f(h) = 0,$

which shows that h is a root of $f(x) = 0$

Theorem 5 *Fundamental Theorem of Algebra : Every polynomial function of degree ≥ 1 has atleast one zero in the complex number.*

Proof Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ with $n \geq 1$, then there exists atleast one $h \in C$ such that

$$a_n h^n + a_{n-1}h^{n-1} + \dots + a_1 h + a_0 = 0$$

from this it is easy to deduce that a polynomial function of degree 'n' has exactly n zeros.

Theorem 6 *Every equation of the nth degree has n roots and no more.*

Proof Let the given equation be

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$$

Since, this equation must have a root, real or imaginary (by *fundamental theorem of algebra*).

Let us denote it by α_1 so that $x - \alpha_1$ is a factor of $f(x)$.

Thus, $\quad f(x) = (x - \alpha_1)\phi_1(x) \quad$ ...(i)

where $\phi_1(x)$ is a polynomial of degree $(n - 1)$.

Again, the equation $\phi_1(x) = 0$ must also have a root. Let us denote it by α_2 so that $x - \alpha_2$ is a factor of $\phi_1(x)$.

Thus, $\quad \phi_1(x) = (x - \alpha_2)\phi_2(x) \quad$ ...(ii)

where $\phi_2(x)$ is a polynomial of degree $(n - 2)$.

Putting this value of $\phi_1(x)$ in Eq. (i), we get

$$f(x) = (x - \alpha_1)(x - \alpha_2)\phi_2(x)$$

Proceeding in this way, we shall obtain n factors of $f(x)$.

$\therefore \quad f(x) = (x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n)\phi_n(x)$

$\phi_n(x)$ is a polynomial of degree $(n - n)$ *i.e.*, 0 so that $\phi_n(x)$ is a constant polynomial.

Let $\quad \phi_n(x) = K$

Thus, $\quad a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$

$$= K(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n)$$

Comparing coefficients of x^n on both sides

$$a_0 = K$$

$\therefore \quad f(x) = a_0(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n)$

so that the given equation may be written as

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)\ldots(x - a_n) = 0 \qquad \ldots\text{(iii)}$$

This is satisfied by n values of x *viz.*, $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Hence, equation $f(x) = 0$ has n roots.

If possible let $x = \beta$ be any other root distinct from $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Then, $$f(\beta) = a_0(\beta - \alpha_1)(\beta - \alpha_2)\ldots(\beta - \alpha_n)$$

Since, $a_0 \neq 0$ and β is distinct from $\alpha_1, \alpha_2, \ldots, \alpha_n$

$\therefore$ $$\beta - \alpha_1 \neq 0, \beta - \alpha_2 \neq 0, \ldots, \beta - \alpha_n \neq 0$$

$\therefore$ $f(\beta)$ can never vanish.

$\therefore$ β is not a root of $f(x) = 0$

Hence, $f(x) = 0$ has exactly n roots.

Theorem 7 In an equation with real coefficients, non-real complex roots occur in conjugate pairs.

Proof Let the given equation be

$$f(x) = 0 \qquad \ldots\text{(i)}$$

Let $\alpha + i\beta$ (where $\beta \neq 0$) be a root of $f(x) = 0$,

then $$f(\alpha + i\beta) = 0 \qquad \ldots\text{(ii)}$$

We have to prove that $\alpha - i\beta$ is also a root of (1)

Divide $f(x)$ by $$[x - (\alpha + i\beta)][x - (\alpha - i\beta)]$$

Let $Q(x)$ be the quotient and if there is any remainder, since it must be a linear polynomial in x, take it as $Rx + R'$.

$\therefore$ $$f(x) = [x - (\alpha + i\beta)][x - (\alpha - i\beta)]Q(x) + Rx + R' \qquad \ldots\text{(iii)}$$

Putting $x = \alpha + i\beta$,

$$f(\alpha + i\beta) = R(\alpha + i\beta) + R'$$

But $$f(\alpha + i\beta) = 0 \qquad [\because \text{Eq. (ii)}]$$

$\therefore$ $$R(\alpha + i\beta) + R' = 0 \text{ or } (R\alpha + R') + iR\beta = 0$$

Equating the real and imaginary parts on both sides.

$$R\alpha + R' = 0 \qquad \ldots\text{(iv)}$$

$$R\beta = 0 \qquad \ldots\text{(v)}$$

from Eq. (v), either $$R = 0 \text{ or } \beta = 0$$

Now, $$\beta \neq 0$$

[$\because$ in case $\beta = 0$ even the complex root $\alpha + i\beta$ becomes real]

$\therefore$ $$R = 0$$

Putting this in Eq. (iv), we have $R' = 0$

$\therefore$ from Eq. (iii)

$$f(x) = [x - (\alpha + i\beta)][x - (\alpha - i\beta)]Q(x)$$

i.e., $$[x - (\alpha + i\beta)][x - (\alpha - i\beta)]$$

divides $f(x)$ exactly.

Hence, $\alpha - i\beta$ is also a root of the given equation $f(x) = 0$.

Corollary Every equation of an odd degree having real coefficients, has atleast one real root, because complex roots occur in pairs.

Theorem 8 *In an equation with rational coefficients irrational roots occur in conjugate pairs.*

Proof Let the given equation be

$$f(x) = 0 \quad \text{...(i)}$$

Let $\alpha + \sqrt{\beta}$ be a root of Eq. (i) [where α and β are rational, β is +ve but not a perfect square]

$$\therefore \quad f(\alpha + \sqrt{\beta}) = 0 \quad \text{...(ii)}$$

We have to prove that $\alpha - \sqrt{\beta}$ is also a root of Eq. (i)

Divide

$$f(x) \text{ by } [x - (\alpha + \sqrt{\beta})][x - (\alpha - \sqrt{\beta})]$$

Let $Q(x)$ be the quotient and if there is a remainder, since it must be a linear polynomial in x, take it as $Rx + R'$

$$\therefore \quad f(x) = [x - (\alpha + \sqrt{\beta})][x - (\alpha - \sqrt{\beta})]\, Q(x) + Rx + R' \quad \text{...(iii)}$$

Putting $x = \alpha + \sqrt{\beta}$

$$f(\alpha + \sqrt{\beta}) = R(\alpha + \sqrt{\beta}) + R'$$

But $f(\alpha + \sqrt{\beta}) = 0$ [$\because$ Eq. (ii)]

$$\therefore \quad R(\alpha + \sqrt{\beta}) + R' = 0$$

i.e., $(R\alpha + R') + R\sqrt{\beta} = 0$

Now, rational and irrational number cannot destroy one another.

$\therefore$ If there sum vanishes, each must vanish separately.

$$\Rightarrow \quad R\alpha + R' = 0 \quad \text{...(iv)}$$

$$R\sqrt{\beta} = 0 \quad \text{...(v)}$$

Since, $\beta \neq 0$, $\therefore$ from Eq. (v), $R = 0$

Putting $R = 0$ in Eq. (iv), $R' = 0$

$\therefore$ Eq. (iii) becomes

$$f(x) = [x - (\alpha + \sqrt{\beta})][x - (\alpha - \sqrt{\beta})]\, Q(x)$$

This shows that

$[x - (\alpha + \sqrt{\beta})][x - (\alpha - \sqrt{\beta})]$ is a factor of $f(x)$.

Hence, $\alpha - \sqrt{\beta}$ is also root of $f(x)$.

Theorem 9 *If the rational number $\frac{p}{q}, q \neq 0, (p, q) = 1$ (i.e., p and q are relatively prime) is a root of the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$, where $a_0, a_1, a_2, \ldots, a_n$ are integers and $a_n \neq 0$, then p is a divisor of a_0 and q that of a_n.*

Proof Since, $\frac{p}{q}$ is a root of given equation, so we have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_1 \frac{p}{q} + a_0 = 0$$

$$\Rightarrow \quad a_n p^n + a_{n-1} q p^{n-1} + \ldots + a_1 q^{n-1} p + a_0 q^n = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad a_{n-1} p^{n-1} + a_{n-2} p^{n-2} q + \ldots + a_1 q^{n-2} p + a_0 q^{n-1} = -\frac{a_n p^n}{q} \quad \text{...(ii)}$$

Since, the coefficients $a_{n-1}, a_{n-2}, \ldots, a_0$ and p, q are all integers and hence the left hand side is an integer, so that right hand side is also an integer but p and q are relatively prime to each other.

$\therefore$ q should divide a_n.

Again

$$a_n p^n + a_{n-1} q p^{n-1} + \ldots + a_1 q^{n-1} p = -a_0 q^n$$

$\Rightarrow$

$$a_n p^{n-1} + a_{n-1} q p^{n-2} + \ldots + a_1 q^{n-1}$$

This shows that P should divide a_0 since $(p, q) = 1$

$$= -\frac{a_0 q^n}{p} \qquad \ldots\text{(iii)}$$

As a consequence of the above theorem we have the following corollary.

Corollary : Every rational root of the equation $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ where each a_i $(i = 1, 2, \ldots, n)$ is an integer, must be an integer. More over every such root must be a divisor of the constant a_n.

Theorem 10 *If $f(x)$ be a polynomial of degree n and α is any complex number, show that there exist unique numbers $b_0, b_1, b_2, \ldots, b_n$ such that*

$$f(x) = b_0 + b_1(x-\alpha) + b_2(x-\alpha)^2 + \ldots + b_n(x-\alpha)^n \qquad (b_n \neq 0)$$

Proof Let us prove by induction

if $n = 0$

$\therefore$ $\deg f(x) = 0$

$\Rightarrow f(x)$ is a non-zero constant polynomial, let

$$f(x) = b_0 \Rightarrow f(x) = b_0(x-\alpha)^0$$

$\therefore$ Result is true for $n = 0$

Assume that result is true for $n = k-1$ $(k > 1)$ *i.e.*, $f(x) = b_0 + b_1(x-\alpha) + b_2(x-\alpha)^2 + \ldots + b_{k-1}(x-\alpha)^{k-1}$

Now, we shall show that result is true for $n = k$, *i.e.*, to prove that

$$f(x) = b_0 + b_1(x-\alpha) + b_2(x-\alpha)^2 + \ldots + b_k(x-\alpha)^k \qquad \ldots\text{(i)}$$

Let degree of $f(x)$ be k.

$\therefore$By Euclidean algorithm there exist a polynomial $g(x)$ of degree $k-1$ and unique remainder b_0 such that

$$f(x) = (x-\alpha)g(x) + b_0 \qquad \ldots\text{(ii)}$$

where $g(x)$ is a polynomial of degree $k-1$ [using Eq. (i)]

$$f(x) = (x-\alpha)[b_1 + b_2(x-\alpha) + b_3(x-\alpha)^2 + \ldots + b_k(x-\alpha)^{k-1}] + b_0$$

$$f(x) = b_0 + b_1(x-\alpha) + b_2(x-\alpha)^2 + \ldots + b_k(x-\alpha)^k$$

$\therefore$ Result is true for $n = k$

$\therefore$ By induction result is true.

Common Divisor

Let $f(x), g(x)$ be two polynomials over any number system F such that atleast one of them is non-zero. A non-zero polynomial $h(x)$ is said to be a common divisor of $f(x), g(x)$, if $h(x) \mid f(x)$ and $h(x) \mid g(x)$.

Theorem 11 *Let $f(x), g(x)$ be any two non-zero polynomials over k. Then their gcd exists and is unique. Further, if*

$$d(x) = (f(x), g(x))$$

then

$$d(x) = a(x)f(x) + b(x)g(x)$$

for some polynomials $a(x)$ and $b(x)$ over k.

Concept Two non-zero polynomials $f(x), g(x)$ over k are said to be relatively prime or coprime, if

$$(f(x), g(x)) = 1$$

or

$$(f(x), g(x)) = c$$

where $c \neq 0$ over k.

Theorem 12 *Two non-zero polynomials $f(x)$, $g(x)$ over k are coprime, iff there exist some polynomials $a(x)$, $b(x)$ over k such that*

$$a(x)f(x) + b(x)g(x) = 1$$

Proof Let $f(x)$, $g(x)$ be coprime

$\therefore \quad (f(x), g(x)) = 1$

$\therefore$ There exist some polynomials $a(x)$, $b(x)$ over k such that

$$a(x)f(x) + b(x)g(x) = 1$$

Conversely

Let there exist polynomials $a(x)$, $b(x)$ over k such that

$$a(x)f(x) + b(x)g(x) = 1$$

We shall prove that $(f(x), g(x)) = 1$

Let $\quad (f(x), g(x)) = d(x)$

$\therefore \quad d(x) | f(x)$ and $d(x) | g(x)$

$\therefore \quad d(x) | a(x)f(x)$ and $d(x) | b(x)g(x)$

$\therefore \quad d(x) | a(x)f(x) + b(x)g(x)$

$\therefore \quad d(x) | 1$

Hence, $\quad d | (x) = 1$

$\therefore \quad f(x)$, $g(x)$ are coprime.

Concept Let $f(x)$ be a non-constant polynomial. Then, a complex number α is called a root of multiplicity $m > 0$ of the polynomial equation $f(x) = 0$, if there exist a polynomial $g(x)$ such that $f(x) = (x - \alpha)^m g(x)$ and $g(\alpha) \neq 0$

Note 1. If $m = 0$, then α is not a root of $f(x) = 0$. If $m = 1$, then α is called a simple root of $f(x) = 0$ α is called a repeated root of $f(x) = 0$, iff $m > 1$.

2. Let $\deg f(x) = n$, where $n \geq 1$. Let α be a root of multiplicity $m > 0$. Then, there exist a polynomial $g(x)$ such that

$$f(x) = (x - \alpha)^m g(x),$$

where $g(\alpha) \neq 0$

$\therefore \quad \deg f(x) = \deg(x - \alpha)^m + \deg g(x)$

$\quad n = m + \deg g(x)$

$\therefore \quad \deg g(x) = n - m$

Since, degree of a polynomial is always a non-negative integer.

$\therefore \quad n - m \geq 0$

$\therefore \quad n \geq m$ or $m \leq n$

Hence, if α is a root of multiplicity m of the equation $f(x) = 0$ of degree n, then $\quad m \leq n$

Theorem 13 *A complex number α is a repeated root of a polynomial equation $f(x) = 0$, if and only if it is a root of both $f(x) = 0$ and $f'(x) = 0$, where $f'(x)$ denotes the derivative of $f(x)$.*

Proof Let α be a repeated root of $f(x) = 0$

$\therefore \quad (x - \alpha)^m | f(x)$, where $m \geq 2$

$\therefore \quad f(x) = (x - \alpha)^m g(x)$

for some polynomial $g(x)$

Differentiating both sides, we get

$$f'(x) = (x-\alpha)^m g'(x) + m(x-\alpha)^{m-1} g(x)$$

Putting $x = \alpha$ on both sides, we get $f'(\alpha) = 0$

$\therefore$ α is a root of $f(x) = 0$ as well as $f'(x) = 0$

Conversely let α be a root of $f(x) = 0$ as well as $f'(x) = 0$

$\therefore$ $f(\alpha) = 0$ and $f'(\alpha) = 0$

Since, α is a root of $f(x) = 0$

$\therefore$ $(x-\alpha) \mid f(x)$

$$f(x) = (x-\alpha) g(x) \quad \ldots(\text{i})$$

For some polynomial $g(x)$

Differentiating both sides, we get

$$f'(x) = (x-\alpha) g'(x) + g(x) \cdot 1$$

Putting $x = \alpha$ on both sides

$$f'(\alpha) = g(\alpha) \Rightarrow g(\alpha) = 0 \quad [\because f'(\alpha) = 0]$$

$\Rightarrow$ $(x - a) \mid g(x)$

$\Rightarrow$ $g(x) = (x-\alpha) h(x)$ $\ldots$(ii)

For some polynomial $h(x)$

From Eqs. (i) and (ii), we get

$$f(x) = (x-\alpha)^2 h(x)$$

which shows that α is a repeated root of $f(x) = 0$. Hence, the result is true.

Corollary (1) α *is a repeated roots of* $f(x) = 0$ *iff* α *is a common root of* $f(x) = 0$ *and* $f'(x) = 0$

(2) *A complex number* α *is a repeated root of the polynomial equation* $f(x) = 0$, *iff* α *is a root of* $d(x) = 0$, *where* $d(x) = (f(x), f'(x))$

Proof Let α be a repeated roots of $f(x) = 0$

$\therefore$ $f(x) = (x-\alpha)^2 g(x)$, where $g(\alpha) \neq 0$

$$f'(x) = (x-\alpha)^2 g'(x) + 2(x-\alpha) g(x)$$

$$= (x-\alpha)[(x-\alpha) g'(x) + 2g(x)] = (x-\alpha) G(x)$$

where $G(\alpha) = 0 + 2g(\alpha) = 2g(\alpha) \neq 0$

$\therefore$ α is a root of $f'(x) = 0$

$\therefore$ $x - \alpha \mid f(x)$ and $(x-\alpha) \mid f'(x)$

$x - \alpha \mid (f(x), f'(x))$

$\Rightarrow$ $x - \alpha \mid d(x)$

$\Rightarrow$ α is a root of $d(x) = 0$

Conversely

Let α be a root of $d(x) = 0$

$\therefore$ $x - \alpha \mid d(x)$

But $d(x) \mid (f(x), f'(x)) \Rightarrow d(x) \mid f(x), d(x) \mid f'(x)$

$\therefore$ $x - \alpha \mid f(x)$ and $x - \alpha \mid f'(x)$

$\therefore$ α is a root of $f(x) = 0$ as well as of $f'(x) = 0$

$\therefore$ α is a repeated root of $f(x) = 0$

Note In order to test whether a given polynomial equation $f(x)=0$ has repeated roots we find gcd of $f(x), f'(x)$.
Let $d(x)=(f(x), f'(x))$. If $d(x)=1$, then $d(x)$ is a constant polynomial.
$\therefore$ It has no roots.
$\therefore$ $f(x)=0$ has no repeated roots.
If $d(x)$ is a non-constant polynomial, then the roots of $d(x)=0$ are repeated roots of

$$f(x)=0.$$

Theorem 14 *Prove that α is a root of multiplicity m of the polynomial equation $f(x)=0$, iff* (i) $(x-\alpha)^m \mid f(x)$ *and* (ii) $(x-\alpha)^{m+1} \mid f(x)$

Proof α is a root of $f(x)=0$ of multiplicity m, iff there exist a polynomial $g(x)$ such that

$$f(x)=(x-\alpha)^m g(x),$$

where $$g(\alpha)\neq 0 \qquad \ldots(i)$$

$\therefore$ $$(x-\alpha)^m \mid f(x)$$

Next we have to prove that

$$(x-\alpha)^{m+1} \nmid f(x)$$

Let if possible

$$(x-\alpha)^{m+1} \mid f(x)$$

Then, $\exists$ a polynomial $h(x)$ such that

$$f(x)=(x-\alpha)^{m+1} h(x) \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$(x-\alpha)^m g(x)=(x-\alpha)^{m+1} h(x)$$

$\Rightarrow$ $$(x-\alpha)^m [g(x)-(x-\alpha)h(x)]=0$$

Since, $(x-\alpha)^m$ is not a zero polynomial.

$\therefore$ $g(x)-(x-\alpha)h(x)$ must be a zero polynomial.

$\therefore$ $$g(x)-(x-\alpha)h(x)=0$$

$\therefore$ $$g(x)=(x-\alpha)h(x)$$

$$g(\alpha)=0$$

which is a contradiction.

$\therefore$ Our supposition is wrong. $[\because g(\alpha)\neq 0]$

$\therefore$ $$(x-\alpha)^{m+1} \nmid f(x)$$

Example 1 *Prove that the sum of two constant polynomials is a constant polynomials.*

Solution Let $$f(x)=a_0+a_1x+a_2x^2+\ldots+a_ix^i+\ldots$$

$$g(x)=b_0+b_1x+b_2x^2+\ldots+b_ix^i+\ldots$$

be two constant polynomials.

$\therefore$ $$a_i=0 \text{ for } i>0 \text{ and } b_i=0 \text{ for } i>0$$

$\therefore$ $$f(x)=a_0 \text{ and } g(x)=b_0$$

$\therefore$ $$f(x)+g(x)=a_0+b_0=c_0$$

where $c_0=a_0+b_0=$ a constant

$\therefore$ $f(x)+g(x)$ is a constant polynomial.

[coefficient of x, coefficient of $x^2, \ldots$ are all zero]

Example 2 *Let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, find a polynomial $g(x)$ such that*

$$f(x) + g(x) = 0$$

Also, show that this polynomial $g(x)$ is unique.

Solution $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

$\therefore\ a_i = 0, \forall i > n$

Let $g(x) = b_0 + b_1x + b_2x^2 + \dots + b_ix^i + \dots$ be a polynomial such that $f(x) + g(x) = 0 =$ zero polynomial

$\therefore$ Coefficient of x^i in $[f(x) + g(x)] = 0$

[$\because$ All coefficients are zero in zero polynomial]

$$\therefore \quad a_i + b_i = 0 \ \ b_i = -a_i \quad \forall i \neq 0$$

$$b_i = -a_i \text{ for } i = 0, 1, 2, 3, \dots, n$$

$b_i = 0, \forall i > n$

$$\therefore \quad g(x) = -a_0 - a_1x - a_2x^2 - \dots - a_nx^n$$

For uniqueness,

Let $g(x)$ and $h(x)$ be two polynomials such that

$$f(x) + g(x) = 0 \text{ and } g(x) + h(x) = 0$$

$$f(x) + g(x) = f(x) + h(x)$$

$$g(x) = h(x)$$

Hence, $g(x)$ is unique.

Example 3 *Let $f(x) = x^5 + x^4 + 1$. Find a polynomial $g(x)$ such that the degree of $f(x) + g(x)$ is zero. Is it possible to find more than one such polynomial $g(x)$? If so how many.*

Solution $f(x) = x^5 + x^4 + 1$

Let $g(x) = b_0 + b_1x + b_2x^2 + \dots + b_ix^i + \dots$ be a polynomial such that degree of $\{f(x) + g(x)\} = 0$

$\therefore$ Constant term of $f(x) + g(x) \neq 0$

and coefficient of x^i in $[f(x) + g(x)] = 0, \ \forall \ i \geq 1$

$$\therefore \quad 1 + b_0 \neq 0, i.e., b_0 \neq -1$$

and $\quad b_2 = 0, b_3 = 0, b_4 = -1, b_5 = -1\, b_1 = 0,$

and $b_i = 0, \forall i > 5$

Hence, $\quad g(x) = b_0 - x^4 - x^5,$

where $\quad b_0 \neq -1$

Since, we can find infinite number of $b_0 \neq -1$ in the given number system.

Hence, there are infinitely many such polynomials $g(x)$.

Example 4 *Prove that if degree of $f(x) \cdot g(x)$ is 1, then one of $f(x), g(x)$ is a constant and the other is a linear polynomial.*

Solution Since, deg $\{f(x) \cdot g(x)\} = 1$

$$\therefore \quad \deg f(x) + \deg g(x) = 1$$

$$\therefore \quad \text{Either } \deg f(x) = 1 \text{ and } \deg g(x) = 0$$

or $\qquad \deg f(x) = 0$ and $\deg g(x) = 1$

$\therefore$ Either $f(x)$ is a linear polynomial and $g(x)$ is a constant polynomial or $f(x)$ is a constant polynomial and $g(x)$ is a linear polynomial.

Example 5 *If $(x-h)^k$ divides a polynomial $f(x)$. Prove that $(x-h)^{k-1}$ divides $f'(x)$.*

Solution $\because$ **$(x-h)^k$ divides $f(x)$**

$\Rightarrow$ There exist a polynomial $q(x)$ such that

$$f(x) = (x-h)^k q(x)$$

$$\therefore \quad \begin{aligned} f'(x) &= (x-h)^k q'(x) + q(x)k(x-h)^{k-1} \\ &= (x-h)^{k-1}[(x-h)q'(x) + kq(x)] \\ &= (x-h)^{k-1} r(x) \end{aligned}$$

[where $r(x) = (x-h)q'(x) + kq(x)$]

$\Rightarrow \qquad (x-h)^{k-1}$ divides $f'(x)$

Example 6 *Find the GCD of $f(x) = x^7 - 1$ and $g(x) = x^5 - 1$ and express it in the form $a(x)f(x) + b(x)g(x)$, where $a(x)$ and $b(x)$ are polynomials with integral coefficients.*

Solution First we find GCD of $f(x)$ and $g(x)$

$$\begin{array}{l}
x^5 - 1 \,\big)\, x^7 - 1 \,\big(\, x^2 \\
\qquad\quad x^7 - x^2 \\
\hline
x^2 - 1 \,\big)\, x^5 - 1 \,\big(\, x^3 + x \\
\qquad\quad x^5 - x^3 \\
\hline
\qquad\qquad x^3 - 1 \\
\qquad\qquad x^3 - x \\
\hline
\qquad x - 1 \,\big)\, x^2 - 1 \,\big(\, x + 1 \\
\qquad\qquad\quad x^2 - x \\
\hline
\qquad\qquad\qquad x - 1 \\
\qquad\qquad\qquad x - 1 \\
\hline
\qquad\qquad\qquad\quad \times
\end{array}$$

$$\therefore \qquad (f(x), g(x)) = (x-1) = d(x)$$

The above division can be written as

$$f(x) = x^2 g(x) + x^2 - 1$$

$$\Rightarrow \qquad x^2 - 1 = f(x) - x^2 g(x) \qquad \text{...(i)}$$

$$\text{and} \qquad g(x) = (x^2 - 1)(x^3 + x) + x - 1 \qquad \text{...(ii)}$$

$$\Rightarrow \qquad \begin{aligned} x - 1 &= g(x) - (x^2 - 1)(x^3 + x) \\ &= g(x) - (f(x) - x^2 g(x))(x^3 + x) \qquad \text{[using Eq. (i)]} \end{aligned}$$

$$\therefore \qquad \begin{aligned} d(x) &= g(x) - (x^3 + x)f(x) + x^2(x^3 + x)g(x) \\ &= -(x^3 + x)f(x) + (x^5 + x^3 + 1)g(x) \end{aligned}$$

Hence, $\qquad d(x) = a(x)f(x) + b(x)g(x)$,

where $\qquad a(x) = -(x^3 + x),\ b(x) = x^5 + x^3 + 1$

Example 7 *Find c if GCD of $f(x) = x^3 + cx^2 - x + 2c$ and $g(x) = x^2 + cx - 2$ is a linear polynomial.*

Solution

$$
\begin{array}{r}
x \\
x^2 + cx - 2 \overline{\big) x^3 + cx^2 - x + 2c} \\
\underline{x^3 + cx^2 - 2x} \\
x + 2c\ )\ x^2 + cx - 2\ (\ x - c \\
\underline{x^2 + 2cx} \\
-cx - 2 \\
\underline{-cx - 2c^2} \\
2c^2 - 2
\end{array}
$$

GCD of $f(x)$ and $g(x)$ is a linear polynomial, if $g(x)$ is exactly divisible by $x + 2c$ *i.e.*, if remainder

$$2c^2 - 2 = 0$$

$\Rightarrow$ $$c^2 = 1$$

$\Rightarrow$ $$c = \pm 1$$

Example 8 *If $f(x), g(x)$ and $h(x)$ are three polynomials such that $(f(x), g(x)) = 1$ and $f(x) | g(x) h(x)$, prove that $f(x) | h(x)$. Is the result true, if $(f(x), g(x)) \neq 1$? Give reasons.*

Solution Since, $f(x) | g(x) h(x)$

$\therefore$ There exist a polynomial $q(x)$ such that

$$g(x) h(x) = f(x) q(x)$$

Again, $\because$ $$(f(x), g(x)) = 1$$

$\therefore$ $\exists$ two polynomials $a(x), b(x)$ such that

$$a(x) f(x) + b(x) g(x) = 1$$

Multiplying both sides by $h(x)$, we get

$$a(x) f(x) h(x) + b(x) g(x) h(x) = h(x)$$

or $$a(x) f(x) h(x) + b(x) f(x) q(x) = h(x)$$

or $$f(x)[a(x) h(x) + b(x) q(x)] = h(x)$$

or $$f(x)\, Q(x) = h(x)$$

where $$Q(x) = a(x) h(x) + b(x) q(x)$$

This shows that $f(x) | h(x)$

If $(f(x), g(x)) \neq 1$. The above result may or may not be true.

For example, take $f(x) = x^2 - 1$

$$g(x) = x + 1$$

$$h(x) = x - 1$$

$$\text{GCD of } f(x), g(x) = x + 1$$

$$g(x) h(x) = x^2 - 1$$

Clearly, $f(x) | g(x) h(x)$

But $f(x)$ does not divide $h(x)$.

Example 9 *If $f(x)$, $g(x)$ are two polynomials (not both zero). Show that*

$$(f(x), g(x)) = (f(x) + g(x), f(x) - g(x))$$

Solution Let $(f(x), g(x)) = d(x)$

$\therefore$ $d(x) \mid f(x)$ and $d(x) \mid g(x)$, where $d(x)$ is a monic polynomial.

Again $d(x) \mid f(x)$ and $d(x) \mid g(x)$

$\therefore$ $\exists$ two polynomials $a(x)$ and $b(x)$ such that

$$f(x) = a(x)d(x) \quad \ldots\text{(i)}$$

$$g(x) = b(x)d(x) \quad \ldots\text{(ii)}$$

Adding Eqs. (i) and (ii), we have

$$f(x) + g(x) = [a(x) + b(x)]d(x)$$

i.e., $d(x) \mid f(x) + g(x)$ [$\because a(x) + b(x)$ is a polynomial]

Similarly, $d(x) \mid f(x) - g(x)$

$\therefore$ $d(x) \mid f(x) + g(x)$

and $d(x) \mid f(x) - g(x)$ ...(iii)

Let $d_1(x)$ be any other common divisor of $f(x) + g(x)$ and $f(x) - g(x)$

$\therefore$ $d_1(x) \mid f(x) + g(x)$

$\therefore$ $\exists$ a polynomial $p(x)$ such that

$$f(x) + g(x) = p(x)d_1(x) \quad \ldots\text{(iv)}$$

Again $\because$ $d_1(x) \mid f(x) - g(x)$

$\therefore$ $\exists$ a polynomial $q(x)$ such that

$$f(x) - g(x) = q(x)d_1(x) \quad \ldots\text{(v)}$$

Adding Eqs. (iv) and (v), we have

$$2f(x) = d_1(x)[p(x) + q(x)]$$

or

$$f(x) = d_1(x)\frac{p(x) + q(x)}{2}$$

$\therefore$ $d_1(x) \mid f(x)$ $\left[\because \dfrac{p(x) + q(x)}{2} \text{ is a polynomial}\right]$

Subtracting Eq. (v) from Eq. (iv)

$$2g(x) = d_1(x)[p(x) - q(x)]$$

$$g(x) = d_1(x)\frac{p(x) - q(x)}{2}$$

$\therefore$ $d_1(x) \mid g(x)$

Now, $d_1(x) \mid f(x)$ and $d_1(x) \mid g(x)$

$\therefore$ $d_1(x)$ divides the GCD of $f(x)$ and $g(x)$

i.e., $d_1(x) \mid d(x)$ ...(vi)

Also, $f(x) + g(x)$ and $f(x) - g(x)$ are not both zero.

$\therefore$ $f(x)$ and $g(x)$ are not both zero.

$\therefore$ $(f(x) + g(x), f(x) - g(x))$ exists.

From Eqs. (iii) and (vi), we have

$$(f(x) + g(x), f(x) - g(x)) = (f(x), g(x))$$

Example 10 *Show that the polynomial* $f(x) = x^3 + px + q$ *has a repeated zero, if*

$$4p^3 + 27q^2 = 0$$

Solution $f(x) = x^3 + px + q$ and $f'(x) = 3x^2 + p$

zeros of $f'(x)$ are given by

$$3x^2 + p = 0 \quad \Rightarrow \quad x^2 = \frac{-p}{3} \quad \Rightarrow \quad x = \pm\sqrt{\frac{-p}{3}}$$

If $\sqrt{\frac{-p}{3}}$ is a zero of $f(x)$, then $f\left(\sqrt{\frac{-p}{3}}\right) = 0$

$$\Rightarrow \qquad \left[\sqrt{\frac{-p}{3}}\right]^3 + p\left[\sqrt{\frac{-p}{3}}\right] + q = 0$$

$$\text{or} \qquad \left(\frac{-p}{3}\right)^{3/2} + p\left(\frac{-p}{3}\right)^{1/2} = -q$$

Squaring both sides

$$\left(\frac{-p}{3}\right)^3 + p^2\left(\frac{-p}{3}\right) + 2p\left(\frac{-p}{3}\right)^{3/2} \cdot \left(\frac{-p}{3}\right)^{1/2} = q^2$$

$$\Rightarrow \qquad -\frac{p^3}{27} - \frac{p^3}{3} + 2\frac{p^3}{9} = q^2$$

$$\Rightarrow \qquad -p^3 - 9p^3 + 6\,p^3 = 27q^2$$

$$\Rightarrow \qquad 4p^3 + 27q^2 = 0$$

which is the required condition.

Example 11 *If* 1 *is a twice repeated root of the equation* $ax^3 + bx^2 + bx + d = 0$. *Show that* $a = d = -b$

Solution The given equation is $f(x) = 0$,

where $$f(x) = ax^3 + bx^2 + bx + d$$

Since, 1 is a twice repeated root of $f(x) = 0$

$\therefore$ 1 is a simple root of

$$f'(x) = 3ax^2 + 2bx + b = 0$$

$$\Rightarrow \qquad f(1) = 0 \text{ and } f'(1) = 0$$

$$\Rightarrow \qquad a(1)^3 + b(1)^2 + b(1) + d = 0$$

$$\text{i.e.,} \qquad a + 2b + d = 0 \qquad \ldots\text{(i)}$$

$$\text{and} \qquad 3a(1)^2 + 2b(1) + b = 0$$

$$\text{i.e.,} \qquad 3a + 3b = 0$$

$$\Rightarrow \qquad 3a = -3b$$

$$\Rightarrow \qquad a = -b \qquad \ldots\text{(ii)}$$

From Eq. (i), we get

$$-b + 2b + d = 0 \Rightarrow b + d = 0$$

$$d = -b \qquad \ldots\text{(iii)}$$

From Eqs. (ii) and (iii), we have $a = d = -b$

Example 12 *Solve the equation*

$$x^4 + 4x^3 + 6x^2 + 4x + 5 = 0.$$

Given that its one root is i.

Solution **One root of**

$$x^4 + 4x^3 + 6x^2 + 4x + 5 = 0 \qquad \text{...(i)}$$

is i, the other root is $-i$ [coefficient of Eq. (i) are real]

$\therefore$ $(x - i)(x + i)$ is a factor of

$$x^4 + 4x^3 + 6x^2 + 4x + 5$$

$\therefore$ $x^2 + 1$ is a factor of $x^4 + 4x^3 + 6x^2 + 4x + 5$

So, $x^4 + 4x^3 + 6x^2 + 4x + 5$

$$= (x^2 + 1)(x^2 + 4x + 5)$$

$\therefore$ Other two roots are given by

$$x^2 + 4x + 5 = 0$$

i.e., $$x = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

Hence, all roots are $\pm i, -2 \pm i$.

Example 13 *Construct a real polynomial equation of degree 4 having $2 + 3i$ as a root with multiplicity 2.*

Solution Since, $2 + 3i$ is a root of multiplicity 2 of a real polynomial equation.

$\therefore$ $2 - 3i$ is also a root of multiplicity 2.

$\therefore$ The required equation is

$$(x - 2 + 3i)^2 (x - 2 - 3i)^2 = 0$$

$\Rightarrow$ $$[(x - 2)^2 - 9i^2] = 0$$

or $$(x^2 - 4x + 4 + 9)^2 = 0$$

or $$(x^2 - 4x + 13)^2 = 0$$

i.e., $$x^4 + 16x^2 + 169 - 8x^3 + 26x^2 - 104x = 0$$

i.e., $$x^4 - 8x^3 + 42x^2 - 104x + 169 = 0$$

Example 14 *Find the rational roots of $2x^3 - 3x^2 - 11x + 6 = 0$*

Solution Let the roots be of the form $\frac{p}{q}$, where $(p, q) = 1$ and $q > 0$

Since, $q | 2$, q must be 1 or 2 and $p | 6$.

$\Rightarrow$ $$p = \pm 1, \pm 2, \pm 3, \pm 6$$

By remainder theorem

$$f\left(\frac{1}{2}\right) = f\left(\frac{-2}{1}\right) = f\left(\frac{3}{1}\right) = 0$$

So, the 3 roots of equations are $\frac{1}{2}$, -2 and 3.

Example 15 (a) *Form an equation of the lowest degree with real and rational coefficients, one of its roots being* $\sqrt{2}+\sqrt{3}+i$.

(b) *Find an equation with rational coefficients which shall have for the roots the values of the expression* $\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}$, *where* $\theta_1^2=1, \theta_2^2=\theta_3^2=1$.

(c) *Use the fact that a cubic equation has atleast one real root to find the real value of* $[27+\sqrt{756}]^{1/3}+[27-\sqrt{756}]^{1/3}$.

Solution (a) $x=\sqrt{2}+\sqrt{3}+i$,

$\therefore$ $$x-i=\sqrt{2}+\sqrt{3}$$

On squaring both sides, we get

$$x^2+i^2-2xi=5+2\sqrt{6}$$
$$x^2-6=2xi+2\sqrt{6}$$

On again squaring both sides, we get

$$x^4-12x^2+36=-4x^2+24+8xi\sqrt{6}$$
$$x^4-8x^2+12=8xi\sqrt{6}$$

On squaring both sides, we get

$$(x^4-8x^2)^2+144+24(x^4-8x^2)=-64x^2(6)$$

or $$x^8-16x^6+88x^4+192x^2+144=0$$

(b) Let us suppose that $x=\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}$ square and put $\theta_1^2=\theta_2^2=\theta_3^2=1$

$\therefore$ $$x^2-(p+q+r)=2\theta_1\theta_2\sqrt{pq}+2\theta_2\theta_3\sqrt{qr}+2\theta_3\theta_1\sqrt{rp}$$

squaring again and putting

$\theta_1^2=\theta_2^2=\theta_3^2=1$, we get

$$x^4-2x^2(p+q+r)+(p+q+r)^2$$
$$=4(pq+qr+rp)+8\theta_1\theta_2\theta_3\sqrt{pqr}[\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}]$$
$$[x^4-2x^2(p+q+r)+p^2+q^2+r^2-2pq-2qr-2rp]^2$$
$$=64pqrx^2$$

as $$\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}=x \text{ and } \theta_1^2=\theta_2^2=\theta_3^2=1$$

(c) Let $x=a^{1/3}+b^{1/3}$ cube both sides

$$x^3=a+b+3a^{1/3}b^{1/3}(a^{1/3}+b^{1/3})$$
$$x^3=54+3[27^2-\sqrt{(756^2)}]^{1/3}x$$
$$x^3=54+3(-27)^{1/3}x$$

Now, real value of root of $(-27)^{1/3}$ is -3

$\therefore$ $$x^3=54+3(-3)x$$
$$x^3+9x-54=0$$

Evidently $x-3$ satisfies. It showing there by the real value of x is 3, which is therefore the required value of the expression.

Example 16 *If the roots of the equation*

$$x^n-1=0$$

are $1, \alpha, \beta, \gamma, \ldots$, *show that*

$$(1-\alpha)(1-\beta)(1-\gamma)\ldots=n$$

Solution $\because$ $1, \alpha, \beta, \gamma, \ldots$ are roots of $x^n-1=0$

$\therefore$ $$x^n-1=(x-1)(x-\alpha)(x-\beta)(x-\gamma) \quad \ldots(i)$$

Dividing both sides by $(x-1)$

$$\frac{x^n-1}{x-1}=(x-\alpha)(x-\beta)(x-\gamma)$$

or $$x^{n-1}+x^{n-2}+\ldots+x^2+x+1=(x-\alpha)(x-\beta)(x-\gamma)\ldots$$

Putting $x=1$,

$$1+1+\ldots n \text{ times}=(1-\alpha)(1-\beta)(1-\gamma)\ldots$$

Hence, $$(1-\alpha)(1-\beta)(1-\gamma)\ldots=n$$

Example 17 *Show that $x^4+qx^2+s=0$ cannot have three equal roots.*

Solution Let $f(x)=x^4+qx^2+s$

$$f'(x)=4x^3+2qx$$

Now, $f(x)=0$ has three equal roots, if

$f'(x)=0$ has two equal roots.

But $f'(x)=0$ gives $2x(2x^2+q)=0$ or $x=0, \pm\sqrt{-q/2}$

Thus, no two roots of $f'(x)=0$ are equal.

Hence, the given equation cannot have three equal roots.

Example 18 *If the equation $x^4+ax^3+bx^2+cx+d=0$ has three equal roots. Show that each of them is equal to $\frac{6c-ab}{3a^2-8b}$.*

Solution $f(x)=x^4+ax^3+bx^2+cx+d$, then

$$f'(x)=4x^3+3ax^2+2bx+c \quad \ldots(\text{i})$$

$$f''(x)=12x^2+6ax+2b \quad \ldots(\text{ii})$$

If $f(x)=0$ has three equal roots, then

$f'(x)=0$ and $f''(x)=0$ must have a common root

Multiply Eq. (i) by 3 and Eq. (ii) by x and subtracting

$$3ax^2+4bx+3c=0 \quad \ldots(\text{iii})$$

Multiplying Eq. (ii) by a and Eq. (iii) by 4 and subtracting

$$(6a^2-16b)x+(2ab-12c)=0$$

$$x=\frac{6c-ab}{3a^2-8b}$$

is the common root of $f'(x)=0$ and $f''(x)=0$

Hence, required triple root of $f(x)=0$ is $\frac{6c-ab}{3a^2-8b}$.

Concept Relation between roots and coefficients

1. If α,β be the roots of the equation $ax^2+bx+c=0$,
   then $\sigma_1\equiv\alpha+\beta=-b/a$ and $\sigma_2\equiv\alpha\beta=c/a$
2. If α,β,γ be the roots of the equation

$$ax^3+bx^2+cx+d=0,$$

then $$\sigma_1\equiv\alpha+\beta+\gamma=-b/a$$

$$\sigma_2\equiv\alpha\beta+\beta\gamma+\gamma\alpha=c/a$$

and $$\sigma_3\equiv\alpha\beta\gamma=-d/a$$

Remark

- It is convenient to write σ_2 as $\alpha(\beta+\gamma)+\beta\gamma$.

  If α, β, γ and δ be the roots of the equation
  $$ax^4 + bx^3 + cx^2 + dx + e = 0,$$
  then
  $$\sigma_1 \equiv \alpha+\beta+\gamma+\delta = -b/a$$
  $$\sigma_2 \equiv \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = c/a$$
  $$\sigma_3 \equiv \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -d/a$$
  and
  $$\sigma_4 \equiv \alpha\beta\gamma\delta = e/a$$
- It is convenient to write
  $$\sigma_2 \equiv (\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta$$
  $$\sigma_3 \equiv \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)$$

Example 1 *Find the condition which must be satisfied by the coefficients of the equation $x^3 - px^2 + qx - r = 0$ when two of its roots α, β are connected by the relation $\alpha + \beta = 0$*

Solution Let the roots of the equation be α, β, γ.

$\therefore \quad \Sigma\alpha = \alpha+\beta+\gamma = p$

But $\quad \alpha+\beta = 0$

$\therefore \quad \gamma = p$

Now, γ is a root of the given equation and as such $\gamma^3 - p\gamma^2 + q\gamma - \gamma = 0$

Put the value of γ

$\therefore\ p^3 - p\cdot p^2 + qp - r = 0$ or $pq = r$ is the required condition.

Example 2 *If α, β, γ are the roots of the equation $(a-x)(b-x)(c-x)+1=0$, then prove that a, b, c will be the roots of the equation $(x-\alpha)(x-\beta)(x-\gamma)+1=0$*

Solution $\because\ \alpha, \beta, \gamma$ are the roots of
$$(a-x)(b-x)(c-x)+1=0$$
$\therefore (a-x)(b-x)(c-x)+1 = \lambda(x-\alpha)(x-\beta)(x-\gamma)$

Equating the leading coefficients, *i.e.*, coefficients of x^3 on both sides
$$-1=\lambda \quad \text{or } \lambda = -1$$
$\therefore\ (a-x)(b-x)(c-x)+1 = -(x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow \quad -(x-a)(x-b)(x-c)+1 = -(x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow \quad (x-a)(x-b)(x-c)-1$
$$= (x-\alpha)(x-\beta)(x-\gamma)$$
$\Rightarrow \quad (x-\alpha)(x-\beta)(x-\gamma)+1$
$$= (x-a)(x-b)(x-c)$$
$\therefore \quad (x-\alpha)(x-\beta)(x-\gamma)+1=0 \qquad \ldots\text{(i)}$

$\Rightarrow \quad (x-a)(x-b)(x-c)=0$

Hence, the roots of Eq. (i) are a, b, c.

Example 3 *Find the condition that the roots of the equation $x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$ be in AP.*

Solution Let the roots be $a - 2d, a-d, a, a+d, a+2d$

Sum of roots $= 5a = -p, \ \therefore\ a = -p/5$

But a is a root of the given equation which will satisfy it. Hence, the required condition is

$$\left(\frac{-p}{5}\right)^5 + p\left(\frac{-p}{5}\right)^4 + q\left(\frac{-p}{5}\right)^3 + r\left(\frac{-p}{5}\right)^2 + s\left(\frac{-p}{5}\right) + t = 0$$

or $$4p^5 - 25qp^3 + 125rp^2 - 625\,sp + 3125t = 0$$

Example 4 *Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in AP and hence solve the equation*

$$x^3 - 12x^2 + 39x - 28 = 0.$$

Solution Let the roots of the given equation be $a - d, a, a + d$ being in AP.

$\therefore$ $$\Sigma\alpha = a - d + a + a + d = p$$

or $$3a = p \text{ or } a = \frac{p}{3}$$

But a is a root of the given equation and as such it will satisfy. The given equation

$\therefore$ $$\left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - r = 0$$

or $2p^3 - 9pq + 27r = 0$ is the required condition. Proceeding as above we find that $a = 4$ is a root of the given equation which when divided by $x - 4$ gives quadratic equation $(x^2 - 8x + 7) = 0$ other roots are 1 and 7 so, 1, 4, 7 are in AP.

Example 5 *Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ be in GP.*

Solution Let the roots of the given equation be $ap, a, \frac{a}{p}$ being in GP.

$\therefore$ $$\Sigma\alpha\beta\gamma = \alpha\beta\gamma = ap \cdot a \cdot \frac{a}{p} = r \text{ or } a^3 = r$$

But a is a root of the given equation and as such

$$a^3 - pa^2 + qa - r = 0$$

or $$-pa^2 + qa = 0 \qquad [\because a^3 = r]$$

or $$pa = q \text{ or } p^3a^3 = q^3$$

or $$p^3r = q^3$$

or $p^3r - q^3 = 0$ is the required condition.

Example 6 *Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ be in HP. Show that mean root is $3r/q$. Hence, solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$.*

Solution Let the roots be α, β, γ which are in HP.

Hence, $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in AP.

$\therefore$ $$\frac{1}{\alpha} + \frac{1}{\gamma} = \frac{2}{\beta} \text{ or } \beta\gamma + \alpha\beta = 2\gamma\alpha$$

Adding $\gamma\alpha$ to both sides, we get

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3\gamma\alpha \text{ or } q = 3\gamma\alpha \text{ or } \gamma\alpha = q/3$$

Again $\qquad \alpha\beta\gamma = r$ or $\left(\frac{q}{3}\right)\beta = r$

$\therefore \qquad \beta = \frac{3r}{q}$ *i.e.*, $\frac{3r}{q}$ is mean root.

But β is a root of the given equation and as such

$$\left(\frac{3r}{q}\right)^3 - p\left(\frac{3r}{q}\right)^2 + q\left(\frac{3r}{q}\right) - r = 0$$

or
$$27r^2 - 9pqr + 2q^3 = 0$$

which is the required condition.

Proceeding as above, we get

$$3\gamma\alpha = \Sigma\alpha\beta = -\frac{3}{6} = -\frac{1}{2}$$

$\therefore \qquad \gamma\alpha = -\frac{1}{6}$

Also, $\qquad \alpha\beta\gamma = \frac{2}{6}$

or $\qquad \left(-\frac{1}{6}\right)\beta = -\frac{2}{6}$

$\therefore$ $\beta = 2$ is a root of the given equation which when divided by $(x - 2)$, by synthetic division gives the quadratic $(6x^2 + x - 1) = 0$ giving the roots as $-\frac{1}{2}$ and $\frac{1}{3}$. Hence, roots are $-\frac{1}{2}, 2, \frac{1}{3}$ which are clearly in HP, for their reciprocals $-2, \frac{1}{2}, 3$ are in AP.

Example 7 *The distances of three points A, B, C on a straight line from a fixed origin O on the line are the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$. Find*

(a) *the condition that one of the points A, B, C should bisect the distance between the other two.*

(b) *the condition that the four points O, A, B, C should form a Harmonic division.*

Solution (a) Let $OA = \alpha$, $OB = \beta$, $OC = \gamma$, so that α, β, γ are the roots of the given equation.

By the given condition, B bisects AC

$\therefore \qquad AB = BC$ or $OB - OA = OC - OB$

$$2OB = OA + OC$$

or $\qquad 2\beta = \alpha + \gamma$

i.e., $\qquad \alpha, \beta, \gamma$ are in AP.

(b) The required condition is $2b^3 - 3abc + a^2d = 0$. The four points, O, A, B, C will form a harmonic division, if $\frac{1}{OA} + \frac{1}{OC} = \frac{2}{OB}$

i.e., $\qquad \frac{1}{OA}, \frac{1}{OB}, \frac{1}{OC}$ are in AP.

OA, OB, OC are in HP.

i.e., α, β, γ the roots of given equation be in HP and $ad^2 - 3bcd + 2c^3 = 0$.

Example 8 *Find the condition that the equation $x^4 + px^3 + qx^2 + rx + s = 0$, should have two roots connected by the relation $\alpha + \beta = 0$ and determine in that case the two quadratic which shall have for roots (α, β) and (γ, δ). Solve completely the equation*

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0.$$

Solution

$$\alpha + \beta = 0 \qquad \text{(given)}$$

$$\Sigma\alpha = \alpha + \beta + \gamma + \delta = -p$$

$$\gamma + \delta = -p \qquad [\because\ \alpha + \beta = 0] \quad \ldots\text{(i)}$$

$$\Sigma\alpha\beta = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = q$$

or
$$\alpha\beta + \gamma\delta = q \qquad \ldots\text{(ii)}$$

$$\Sigma\alpha\beta\gamma = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r$$

$\therefore$
$$\alpha\beta(-p) = -r \qquad \text{[from Eq. (i)]}$$

$\therefore$
$$\alpha\beta = \frac{r}{p} \qquad \ldots\text{(iii)}$$

$$\Sigma\alpha\beta\gamma\delta = \alpha\beta\gamma\delta = s \quad \text{or} \quad \left(\frac{r}{p}\right)\gamma\delta = s \qquad \text{[from Eq. (iii)]}$$

$\therefore$
$$\gamma\delta = \frac{ps}{r} \qquad \ldots\text{(iv)}$$

Substituting the values of $\alpha\beta$ and $\gamma\delta$ in Eq. (ii), we get the required condition as $\frac{r}{p} + \frac{ps}{r} = q$ or $pqr - p^2s + r^2 = 0$. Again, since $\alpha + \beta = 0$ and $\alpha\beta = \frac{r}{p}$. $\therefore \alpha$ and β are the roots of the quadratic $x^2 - 0x + \frac{r}{p} = 0$ or $px^2 + r = 0$. Also, $\gamma + \delta = -p$ and $\gamma\delta = \frac{ps}{r}$

$\therefore \gamma$ and δ are the roots of the equation $x^2 - (-p)x + \frac{ps}{r} = 0$

or
$$rx^2 + prx + ps = 0$$

$$f(x) = x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$$

Since,
$$\alpha + \beta + \gamma + \delta = 2 \text{ and } \alpha + \beta = 0$$

$\therefore$
$$\gamma + \delta = 2$$

Let
$$\alpha\beta = l \text{ and } \gamma\delta = m$$

$\therefore$
$$f(x) = (x^2 + l)(x^2 - 2x + m)$$

Comparing the coefficients

$$l + m = 4, -2l = 6, \Rightarrow l = -3 \text{ and } m = 7$$

$\therefore$
$$f(x) = (x^2 - 3)(x^2 - 2x + 7) = 0$$

$\therefore$
$$x = \sqrt{3}, -\sqrt{3}, 1 \pm i\sqrt{6}$$

Example 9 *Show that the roots of the biquadratic $ax^4 + 4bx^3 + 4dx + e = 0$, have only two distinct values, if $\frac{ad^2}{b^2e} = \frac{3bd}{bd - ae} = 1$.*

Solution Since, the roots are of two distinct kinds, it means that the biquadratic has two pairs of equal roots $\alpha, \alpha, \beta, \beta$.

$$\Sigma\alpha = 2(\alpha + \beta) = -\frac{4b}{a} \quad \therefore\ \alpha + \beta = -\frac{2b}{a}$$

and $$\alpha^2\beta^2 = \frac{e}{a}, \quad \therefore \quad \alpha\beta = \sqrt{\frac{e}{a}}$$

$\therefore$ $$x^4 + \frac{4b}{a}x^3 + \frac{4d}{a}x + \frac{e}{a} = \left(x^2 + \frac{2b}{a}x + \sqrt{\frac{e}{a}}\right)^2$$

Comparing the coefficients of like powers of x, we get

$$\frac{4b^2}{a^2} + 2\sqrt{\frac{e}{a}} = 0 \qquad \text{...(i)}$$

$$\frac{4b}{a}\sqrt{\frac{e}{a}} = 4\frac{d}{a}; \ \therefore \ \frac{ad^2}{b^2e} = 1,$$

$\therefore$ $$\sqrt{\frac{e}{a}} = \frac{d}{b} \qquad \text{...(ii)}$$

By Eq. (i), $2\frac{b^2}{a^2} + \frac{d}{b} = 0$ or $2b^2b + a^2d = 0$

Put $b^2 = \frac{ad^2}{e}$; $\therefore$ $2b\frac{ad^2}{e} + a^2d = 0$

or $2bd + ae = 0$

or $3db = bd - ae$

$$\frac{3bd}{bd - ae} = 1$$

and $$\frac{ad^2}{b^2e} = 1$$

Example 10 *If $z^4 + 6Hz^2 + 4Gz + (a_0^2 l - 3H^2) = 0$ has two pairs of equal roots. Prove that $G = 0$ and $a_0^2 l = 12H^2$.*

Solution Let the roots be $\alpha, \alpha, \beta, \beta$

$$\Sigma\alpha = 2(\alpha + \beta) = 0, \ \therefore \ \alpha + \beta = 0$$

Let $\alpha\beta = \lambda$; $\therefore z$ quadratic having roots α, β is $z^2 + 0z + \lambda = 0$

$\therefore$ $$f(z) = z^4 + 6Hz^2 + 4Gz + a_0^2 l - 3H^2 = (z^2 + \lambda)^2$$

Comparing $2\lambda = 6H, 4G = 0, \lambda^2 = a_0^2 l - 3H^2$

$\therefore$ $\lambda = 3H$ and $9H^2 = a_0^2 l - 3H^2$

$\therefore$ $a_0^2 l = 12H^2$

Hence, required conditions are that

$$G = 0 \text{ and } a_0^2 l = 12H^2$$

Example 11 *If $x + ay + a^2z = a^3$, $x + by + b^2z = b^3$, $x + cy + c^2z = c^3$. Find the values of x, y and z.*

Solution a, b, c satisfy the cubic $t^3 = t^2z + ty + x$ or $t^3 - t^2z - ty - x = 0$ has a, b and c as its roots

$$\Sigma a = a + b + c = z$$

$$\Sigma ab = ab + bc + ca = -y$$

$$\Sigma abc = abc = x$$

Hence the values of x, y and z are $abc, -(ab + bc + ca)$ and $(a + b + c)$ respectively.

Example 12 *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation*

$$x^n - p_1x^{n-1} + p_2x^{n-2} - p_3x^{n-3} + \ldots + (-1)^n p_n = 0,$$

find the value of $(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n)$.

Solution Now, $(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n) = 1 + s_1 + s_2 + s_3 + \ldots + s_n$

$$s_n = \text{sum of roots taken } n \text{ at a time}$$
$$= (-1)^r[(-1)^r p_r] = (-1)^{2r} p_r = p_r$$

Hence, the required value is $1 + p_1 + p_2 + \ldots + p_n$

Example 13 *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $x^n + nax - b = 0$, show that*

$$(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\ldots(\alpha_1 - \alpha_n) = n\,(\alpha_1^{n-1} + a)$$

Solution

$$f(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

$$\log f(x) = \log(x-\alpha_1) + \log(x-\alpha_2) + \ldots + \log(x-\alpha_n)$$

$$f'(x) = \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \ldots + \frac{f(x)}{x-\alpha_n} \text{ or } nx^{n-1} + na = [(x-\alpha_2)(x-\alpha_3)\ldots$$

$(x-\alpha_n)]$ + [other $(n-1)$ factors of the above type each of which will contain $x-\alpha_1$ as a factor].

Putting $x = \alpha_1$ in both sides, we get

$$n(\alpha_1^{n-1} + a) = \{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)\}$$

$\because$ All other factors vanish because of $(x-\alpha_1)$ in them.

Example 14 *Show that all the roots of the equation*

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$$

can be obtained when they are in AP.

Solution The roots of the given equation being in AP. may be taken to be $a, a+d,$

$$a + 2d \ldots a + (n-1)d.$$

We have to find values of a and d

$$\Sigma\alpha = na + \{1 + 2 + 3 + 4 + \ldots + (n-1)\}d = -p_1$$

or
$$na + \frac{(n-1)n}{2}d = -p_1 \qquad \ldots\text{(i)}$$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

i.e.,
$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta) = p_1^2 - 2p_2$$

The above relation holds good for any number of roots.

or
$$a^2 + (a+d)^2 + (a+2d)^2 + \ldots + \{a + (n-1)d\}^2 = p_1^2 - 2p_2$$

or
$$na^2 - 2ad\,\{1 + 2 + 3 + \ldots + (n-1)\} + d^2\,\{1^2 + 2^2 + \ldots + (n-1)^2\} = p_1^2 - 2p_2$$

$$\Sigma n = \frac{n(n+1)}{2} \text{ and } \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

Putting $n = n-1$ in above formulae,

$$na^2 + 2ad\,\frac{n(n-1)}{2} + \frac{d^2n(n-1)(2n-1)}{6} = p_1^2 - 2p_2 \qquad \ldots\text{(ii)}$$

Now, if we square (i) and subtract it from n times (ii) a will be eliminated and we can find d^2. Flowing found d we can find a from Eq. (i). When a and d are known, all the roots are known.

Example 15 *If $f(x)=0$ is a cubic equation whose roots are α, β, γ and α is the harmonic mean of the roots of $f'(x)=0$. Show that $\alpha^2=\beta\gamma$.*

Solution Let α, β, γ be roots of

$$x^3 - px^2 + qx - r = 0$$

$\therefore$ $\Sigma\alpha = p, \Sigma\alpha\beta = q, \alpha\beta\gamma = r$ and $f'(x) = 3x^2 - 2px + q = 0$

If its roots be a, b, then $a+b = 2p/3$ and $ab = q/3$

Now, α being the HM of a, b.

$\therefore a, \alpha, b$ are in HP.

$$\therefore \quad \alpha = \frac{2ab}{a+b} = 2\left(\frac{q}{3}\right) \div \frac{2p}{3} = \frac{q}{p} = \frac{\Sigma\alpha\beta}{\Sigma\alpha}$$

$$\therefore \quad \alpha(\alpha+\beta+\gamma) = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$\therefore \quad \alpha^2 = \beta\gamma$$

Example 16 *Solve for x, y, z the equations*

$$x + y + z = 5, \quad yz + zx + xy = 7 \text{ and}$$
$$xyz = 3.$$

Solution The given equations are symmetric with respect to x, y, z.

The equation whose roots are x, y, z is

$$t^3 - (x+y+z)t^2 + (yz+zx+xy)t - xyz = 0$$

or $$t^3 - 5t^2 + 7t - 3 = 0 \qquad \ldots\text{(i)}$$

By trial, $t=1$ satisfies it.

Dividing LHS of Eq. (i) by $t-1$

The depressed equation is $t^2 - 4t + 3 = 0$ or $(t-1)(t-3) = 0 \;\therefore\; t = 1, 3$

values of x, y, z are 1, 1, 3.

Concept To transform an equation into another whose roots are the reciprocals of the roots of the given equation.

Let $$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

be the given equation. If x be a root of the given equation and y that of the transformed equation, then $y = \frac{1}{x}$ or $x = \frac{1}{y}$

Hence, the transformed equation is obtained by putting $x = \frac{1}{y}$ in $f(x) = 0$ and is therefore $f\left(\frac{1}{y}\right) = 0$.

i.e., $$a_0\frac{1}{y^n} + a_1\frac{1}{y^{n-1}} + a_2\frac{1}{y^{n-2}} + \ldots + a_{n-1}\frac{1}{y} + a_n = 0$$

or $$a_ny^n + a_{n-1}y^{n-1} + \ldots + a_2y^2 + a_1y + a_0 = 0$$

Rule If the equation given be complete (if not, it may be made to take that form), then the above transformation is effected, if we take the last coefficient to be the Ist, last but one to be the IInd and so on.

Concept All those equations which remain unchanged when x is replaced by $1/x$ are called reciprocal equation. These are of two types

(i) those in which the coefficient of terms, equidistant from the begining and the end, are equal and of the same sign. *e.g.*,

$$x^4 + 6x^3 + 8x^2 + 6x + 1 = 0$$

(ii) those in which these coefficients are equal but of opposite sign *e.g.*,

$$x^5 - 4x^4 + 7x^3 - 7x^2 + 4x - 1 = 0$$

Thus, we conclude that if α is a root of the reciprocal equation, then $1/\alpha$ must be its root. Hence, the root of a reciprocal equation occur in pairs of α, $1/\alpha$, β, $1/\beta$, γ, $1/\gamma$ and so on.

In case, the equation be of odd degree then it will be seen that one of its roots must be either +1 or −1. In case, the equation be of even degree and of IInd type, then it will be seen that $x^2 - 1$ will always be its factor *e.g.*,

$$6x^6 + 5x^5 - 44x^4 + 44x^2 - 5x - 6 = 0$$
$$6(x^6 - 1) + 5x(x^4 - 1) - 44x^2(x^2 - 1) = 0$$
$$(x^2 - 1)[6(x^4 + x^2 + 1) + 5x(x^2 + 1) - 44x^2] = 0$$
$$(x^2 - 1)(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$$

Above shows that $(x^2 - 1)$ is a factor of the given equation. The equation $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$, is of even degree and of the first type *i.e.*, the coefficient of terms equidistant from the beginning and the end are equal and is called in the standard form to which all the reciprocal equations can be reduced.

Concept To transform an equation into another, whose roots are the roots of the given equation with sign changed.

If y be the root of the transformed equation, then $y = -x$ or $x = -y$. Hence, transformed equation is obtained by putting $x = -y$ in $f(x) = 0$,

$\therefore f(-y) = 0$, which takes the form

$$a_0 y^n - a_1 y^{n-1} + a_2 y^{n-2} - \ldots + (-1)^n a_n = 0$$

Rule If the given equation be complete (if not, it may be made to take that form), then the transformation is effected by changing the sign of IInd, IVth, VIth, *i.e.*, all even terms, *i.e.*, by changing the sign of every alternate term beginning from the IInd.

Concept To transform a given equation into another, whose roots are the roots of the given equation multiplied by a given number m.

If y be a root of the transformed equation, then $y = mx$ or $x = y/m$. Hence, the transformed equation is obtained by putting $x = y/m$ in $f(x) = 0$ and $\therefore f(y/m) = 0$

i.e.,
$$a_0 \frac{y^n}{m^n} + a_1 \frac{y^{n-1}}{m^{n-1}} + a_2 \frac{y^{n-2}}{m^{n-2}} + \ldots + a_n = 0$$

or
$$a_0 y^n + a_1 m y^{n-1} + a_2 m^2 y^{n-2} + \ldots + m^n a_n = 0$$

Rule If the given equation be complete (if not, it may be made complete), then the above transformation is effected by multiplying the successive terms beginning from the IInd by m, m^2, $m^3, \ldots, m^n$ respectively.

Note 1. The above transformation is very useful when we are dealing with equations with fractional coefficients. We can get rid of fractional coefficients by multiplying the roots of the given equation by the LCM of the denominators of the fractional coefficients. Similarly, if the coefficient of leading term be not unity but k and we want to make it unity, then it can be done so by multiplying the roots of the given equation by k.

2. If we have to divide the roots of the given equation by m, we say that we have to multiply its roots by $1/m$.

Concept To transform a given equation into another, whose roots are the roots of the given equation diminished (or increased) by a constant h.

Let $$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0 \qquad \dots(i)$$

If y be a root of the transformed equation, then $y = x - h$ or $x = y + h$. Hence, the transformed equation is obtained by putting $x = y + h$ in $f(x) = 0$ and is therefore,

$$f(y + h) = 0$$

or $$a_0(y + h)^n + a_1(y + h)^{n-1} + a_2(y + h)^{n-2} + \dots + a_n = 0$$

The simplification of the above equation will be difficult and let us suppose that this equation, when simplified and arranged in descending powers of y takes the form.

$$A_0y^n + A_1y^{n-1} + A_2y^{n-2} + \dots + A_{n-1}y + A_n = 0 \qquad \dots(ii)$$

The problem is to find $A_0, A_1, A_2, \dots, A_n$

Now, $$y = x - h$$

$\therefore$ $$A_0(x - h)^n + A_1(x - h)^{n-1} + A_2(x - h)^{n-2} + \dots + A_{n-1}(x - h) + A_n = 0$$

or $$[A_0(x - h)^{n-1} + A_1(x - h)^{n-2} + A_2(x - h)^{n-3} + \dots + A_{n-1}] \times (x - h) + A_n = 0 \qquad \dots(iii)$$

The LHS of the above is identical with LHS of line (i) and hence, if $f(x)$ be divided by $(x - h)$, then the remainder is the value of A_n and the quotient is $A_0(x - h)^{n-1} + A_1(x - h)^{n-2} + \dots + A_{n-1}$ and the quotient when again divided by $(x - h)$ leaves the remainder A_{n-1}. If we continue the above process, then we shall find $A_n, A_{n-1}, \dots, A_2, A_1$ and the last quotient A_0 is clearly equal to a_0.

Rule In order to find the successive coefficients of the transformed equation, we have to divide the given complete equation by $(x - h)$, the quotient again by $x - h$ and so on. The successive remainders that are left in the above procedure of division are the required coefficients, the first coefficient being the same as that of the given equation.

Example 1 *Solve the equation* $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

Solution Dividing throughout by x^2 it can be written as

$$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0$$

Put $$x + \frac{1}{x} = y, \ \therefore\ x^2 + \frac{1}{x^2} = y^2 - 2$$

$\therefore$ $$(y^2 - 2) - 10y + 26 = 0$$

$$y^2 - 10y + 24 = 0$$

$\Rightarrow$ $$(y - 6)(y - 4) = 0$$

$\therefore$ $$y = 6, 4 \quad \text{or} \quad x + \frac{1}{x} = 6$$

$\therefore$ $$x^2 - 6x + 1 = 0,\ x = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$$

and $$x + \frac{1}{x} = 4$$

$\therefore$ $$x^2 - 4x + 1 = 0$$

$$x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

Example 2 *Form equation, whose roots are the roots of the equation $x^5 - x^2 + x - 3 = 0$ with their sign changed.*

Solution The given equation being not complete, may be written as

$$x^5 + 0x^4 + 0x^3 - x^2 + x - 3 = 0$$

and the transformed equation is

$$x^5 - 0x^4 + 0x^3 + x^2 + x + 3 = 0$$

$$\therefore \qquad x^5 + x^2 + x + 3 = 0$$

Example 3 *If the reciprocal of every root of $x^3 + x^2 + ax + b = 0$ is also a root. Prove that $a = b = 1$ or $a = b = -1$. In each case find the roots.*

Solution Let $f(x) = x^3 + x^2 + ax + b = 0$ ...(i)

If α is a root, then $\frac{1}{\alpha}$ is also a root.

$\therefore$ It is a reciprocal equation.

It remains unchanged by the transformation

$$y = \frac{1}{x}$$

On putting $x = \frac{1}{y}$, we get

$$f\left(\frac{1}{y}\right) = \frac{1}{y^3} + \frac{1}{y^2} + a \cdot \frac{1}{y} + b = 0$$

or $$1 + y + ay^2 + by^3 = 0$$

or $$by^3 + ay^2 + y + 1 = 0$$

Writing it in x,

$$\therefore \qquad bx^3 + ax^2 + x + 1 = 0 \qquad \text{...(ii)}$$

Eqs. (i) and (ii) represent the same equation, *i.e.*, they have same roots.

On comparing coefficients of Eqs. (i) and (ii), we get

$$\frac{1}{b} = \frac{1}{a} = \frac{a}{1} = \frac{b}{1} \Rightarrow a^2 = 1 \text{ and } a = b$$

$$\therefore \qquad a = b = \pm 1$$

(i) If $a = b = 1$, Eq. (i) becomes

$$x^3 + x^2 + x + 1 = 0$$

or n $$x^2(x + 1) + (x + 1) = 0$$

$$(x^2 + 1)(x + 1) = 0$$

$$x = -1, \pm i$$

(ii) If $a = b = -1$, Eq. (i) becomes

$$x^3 + x^2 - x - 1 = 0$$

$$\Rightarrow \qquad x^2(x + 1) - 1(x + 1) = 0$$

$$\Rightarrow \qquad 1(x + 1)(x^2 - 1) = 0$$

$$\Rightarrow \qquad (x + 1)(x + 1)(x - 1) = 0$$

$\therefore$ Roots are $-1, -1, 1$

Example 4 *If α, β and γ are roots of the equation $x^3 - 6x^2 + 12x - 8 = 0$, find an equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$. Hence, find roots of the given equation.*

Solution Roots of equation

$$x^3 - 6x^2 + 12x - 8 = 0 \qquad \ldots(i)$$

and α, β, γ. Diminishing its roots by 2

2	1	−6	−12	−8
		2	−8	8
	1	−4	4	0
		2	−4	
	1	−2	0	
		2		
	1	0		
	1			

$$\therefore \qquad x^3 = 0 \qquad \ldots(ii)$$

Roots of Eq. (ii) are 0, 0, 0

i.e., $\alpha - 2 = \beta - 2 = \gamma - 2 = 0$

$\therefore$ $\alpha = \beta = \gamma = 2$

Roots of Eq. (i) are 2, 2, 2.

Example 5 *Find the equation whose roots exceed by 2, the roots of the equation $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$.. Hence, solve the equation.*

Solution Equation is

$$4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0 \qquad \ldots(i)$$

Dividing $4x^4 + 32x^3 + 83x^2 + 76x + 21$ successively by $x + 2$

−2	4	32	83	76	21
		−8	−48	−70	−12
	4	24	35	6	9
		−8	−32	−6	
	4	16	3	0	
		−8	−16		
	4	8	−13		
		−8			
	4	0			
	4				

Transformed equation is

$$4x^4 - 13x^2 + 9 = 0 \qquad \ldots(ii)$$

i.e., $(4x^2 - 9)(x^2 - 1) = 0$

$\therefore$ $x = \pm 3/2, \pm 1$

Roots of Eq. (ii) are the roots of Eq. (i) increased by 2

Roots of Eq. (i) are the roots of Eq. (ii) decreased by 2

Roots of Eq. (i) are $3/2 - 2, -3/2 - 2, 1 - 2, -1 - 2$.

i.e., $1/2, -7/2, -1, -3$

Example 6 *The difference between two roots of the equation* $2x^3 + x^2 - 7x - 6 = 0$, *is* 3. *Solve it by diminishing the roots by* 3.

Solution Let $\alpha, \alpha + 3, \beta$ be the roots of given equation

$$2x^3 + x^2 - 7x - 6 = 0 \quad \ldots(i)$$

Diminishing its roots by 3

3	2	1	–7	–6
		6	21	42
	2	7	14	36
		6	39	
	2	13	53	
		6		
	2	19		
	2			

Transformed equation is

$$2x^3 + 19x^2 + 53x + 36 = 0 \quad \ldots(ii)$$

Roots of this equation are $\alpha - 3, \alpha, \beta - 3$.

Thus, Eqs. (i) and (ii) have a common root α, finding the HCF of the LHS of Eqs. (i) and (ii)

2	1	–7	–6	2	19	53	36	1
			× 3	2	1	–7	–6	
6	3	–21	–18	6	18	60	42	
6	20	14			3	10	7	3
–1	–17	–35	–18		3	3		
	17	35	18		7	7	7	
			× 3			1	1	1
	51	105	54			1	1	
	51	170	119			×		
	–65	–65	–65					

HCF = $x + 1$, equating it to zero, $x = -1$

$\therefore$ –1 is a root of given equation.

Dividing Eq. (i) by $x + 1$

–1	2	1	–7	–6
		–2	1	6
	2	–1	–6	0

Depressed equation is $2x^2 - x - 6 = 0$

$$\therefore \quad x = \frac{1 \pm \sqrt{1 + 48}}{4}$$

$$= \frac{1 \pm 7}{4} = 2, \frac{-3}{2}$$

Hence, the roots of the given equation are –1, 2, –3/2.

Example 7 *If α, β, γ are the roots of $8x^3 - 4x^2 + 6x - 1 = 0$, find the equations whose roots are*

$$\alpha + 1/2, \beta + 1/2 \text{ and } \gamma + 1/2.$$

Solution

$$8x^3 - 4x^2 + 6x - 1 = 0 \qquad \ldots(i)$$

Roots of required equation are the roots of given equation increased by 1/2 (or decreased by $-1/2$) Diminishing the roots of Eq. (i) by $-1/2$

$$\begin{array}{r|rrrr}
-\frac{1}{2} & 8 & -4 & 6 & -1 \\
 & & -4 & 4 & -5 \\
\hline
 & 8 & -8 & 10 & -6 \\
 & & -4 & 6 & \\
\hline
 & 8 & -12 & 16 & \\
 & & -4 & & \\
\hline
 & 8 & -16 & & \\
 & 8 & & &
\end{array}$$

Hence, required equation is

$$8x^3 - 16x^2 + 16x - 6 = 0$$

or

$$4x^3 - 8x^2 + 8x - 3 = 0$$

Symmetric Functions of the Roots

A symmetric function of the roots of an equation is a function in which all the roots are involved alike, so that the expressions remains unaltered when two of the roots are interchanged.

e.g., The expression $\alpha\beta + \beta\gamma + \gamma\alpha$ is a symmetric function of the roots α, β, γ of a cubic. For the sake of brevity, we generally denote such symmetric functions by attaching Σ to one of its terms.

Thus, $\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha$

Rule To find the number of terms in a symmetric function.

If n = the total number of roots of the equation.

r = the number of roots occuring in the symmetric function.

k = the number of roots having the same index (degree), then the total number of terms in the symmetric function.

$$= \frac{n!}{(n-r)!\,k!}$$

For Example

1. If α, β and γ are the roots of a cubic equation, then the total number of terms in

$$\Sigma\alpha^2\beta \text{ is } \frac{3!}{(3-3)!\,0!} = 6$$

$$\Sigma\frac{\alpha\beta}{\gamma^2} = \Sigma\alpha\beta\gamma^{-2} \text{ is } \frac{3!}{(3-2)!\,2!} = 3$$

2. If α, β, γ and δ are the roots of a biquadratic equation, then the total number of terms in

$$\Sigma\alpha^2\beta\gamma \text{ is } \frac{4!}{(4-1)!\,2!} = 2$$

$$\Sigma\frac{\alpha}{\beta} = \Sigma\alpha\beta^{-1} \text{ is } \frac{4!}{(4-2)!\,0!} = 12$$

Example 1 *Calculate the values of the following symmetric functions for the cubic $x^3 + px^2 + qx + r = 0$, whose roots are α, β, γ*

(i) $\Sigma\alpha^2\beta$ (ii) $\Sigma\alpha^2$ (iii) $\Sigma\alpha^3$ (iv) $\Sigma\alpha^2\beta\gamma$

(v) $\Sigma\alpha^2\beta^2$ (vi) $\Sigma\alpha^2\beta$ (vii) $\Sigma\alpha^4$ (viii) $\Sigma\alpha^3\beta^2$

Solution (i) $\Sigma\alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta$ [Number of terms 6]

Now, $\Sigma\alpha\Sigma\alpha\beta = (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \lambda\alpha)$ [Number of terms 9]

$$= \Sigma\alpha^2\beta + 3\alpha\beta\gamma$$

$$\Sigma\alpha^2\beta = \Sigma\alpha\,\Sigma\alpha\beta - 3\alpha\beta\gamma = (-p)(q) - 3(-r) = 3r - pq$$

(ii) $\Sigma\alpha^2 = \alpha^2 + \beta^2 + \gamma^2$

Now, $(\Sigma\alpha)^2 = (\alpha + \beta + \gamma)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$ [Number of terms 9]

$\therefore$ $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma(\alpha\beta) = p^2 - 2q$

Note Above relation is true for equations of all degrees and will be used frequently.

(iii) $\Sigma\alpha^3 = \alpha^3 + \beta^3 + \gamma^3$

Now, $\Sigma\alpha\,\Sigma\alpha^2 = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2)$

$$= \Sigma\alpha^3 + \Sigma\alpha^2\beta$$

$\therefore$ $\Sigma\alpha^3 = \Sigma\alpha\,\Sigma\alpha^2 - \Sigma\alpha^2\beta$

The values of $\Sigma\alpha^2$ and $\Sigma\alpha^2\beta$ have already been calculated in relations (i) and (ii)

$\therefore$ $\Sigma\alpha^3 = (-p)(p^2 - 2q) - (3r - pq) = 3pq - p^3 - 3r$

(iv) $\Sigma\alpha^2\beta\gamma = \alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta = \alpha\beta\gamma\cdot\Sigma\alpha$

Hence, $\Sigma\alpha^2\beta\gamma = \alpha\beta\gamma\,\Sigma\alpha = pr$

(v) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

Now, $(\Sigma\alpha\beta)^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \Sigma\alpha^2\beta^2 + 2\Sigma\alpha^2\beta\gamma$

$$\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha^2\beta\gamma = q^2 - 2pr$$

(vi) $\Sigma a^3\beta = \alpha^3\beta + \alpha^3\gamma + \beta^3\alpha + \beta^3\gamma + \gamma^3\alpha + \gamma^3\beta$

Now, $\Sigma\alpha^2\,\Sigma\alpha\beta = (\alpha^2 + \beta^2 + \gamma^2)(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$= \Sigma\alpha^3\beta + \Sigma\alpha^2\beta\gamma$$

$\therefore$ $\Sigma\alpha^3\beta = \Sigma\alpha^2\Sigma\alpha\beta - \alpha\beta\gamma\cdot\Sigma\alpha$

$$= q(p^2 - 2q) - pr$$

$$= p^2q - 2q^2 - pr$$

(vii) $\Sigma\alpha^4 = (\Sigma\alpha^2)^2 - 2(\Sigma\alpha^2\beta^2)$

$= (p^2 - 2q)^2 - 2(q^2 - 2pr)$ [by relations (ii) and (v)]

$= p^4 - 4p^2q + 2q^2 + 4pr$

(viii) $\Sigma\alpha^3\beta^2$

Now, $\Sigma\alpha\,\Sigma\alpha^2\beta^2 = \Sigma\alpha^3\beta^2 + \Sigma\alpha^2\beta^2\gamma$

$\therefore$ $\Sigma\alpha^3\beta^2 = \Sigma\alpha\,\Sigma\alpha^2\beta^2 - \alpha\beta\gamma\,\Sigma\alpha\beta$

$$= -p[q^2 - 2pr] - (-r)q$$

$$= -pq^2 + 2p^2r + qr$$

Example 2 *Find the equation, whose roots are the squares of the roots of $x^3 + qx + r = 0$.*

Solution Given equation is

$$x^3 + qx + r = 0 \quad \ldots(i)$$

If y be a root of the transformed equation, then

$$y = x^2 \quad \ldots(ii)$$

Now, we have to eliminate x from Eqs. (i) and (ii)

[**Method** Collect terms with odd powers of x on one side and with even power of x on the other side. Square and put $x^2 = y$]

From Eq. (i) $\quad x^3 + qx = -r$

Squaring, we get $\quad x^6 + 2qx^4 + q^2x^2 = r^2$

Putting $\quad x^2 = y$

$$y^3 + 2qy^2 + q^2y - r^2 = 0$$

which is required equation.

Example 3 *Find the equation, whose roots are the squares of the roots of*

$$x^3 + px^2 + qx + r = 0$$

Solution

$$x^3 + px^2 + qx + r = 0 \quad \ldots(i)$$

If y be a root of the transformed equation, then $y = x^2$ ...(ii)

we have, now to eliminate x from Eqs. (i) and (ii)

From Eq. (i), $x^3 + qx = -(px^2 + r)$

On squaring both sides

$$x^6 + 2qx^4 + q^2x^2 = p^2x^4 + 2prx^2 + r^2$$

Putting $x^2 = y$, we get

$$y^3 + (2q - p^2)y^2 + (q^2 - 2pr)y - r^2 = 0$$

which is required equation.

Example 4 *Find the equation whose roots are the squares of the roots of*

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

and if α, β, γ and δ are the roots, find the values of

(i) $\Sigma\alpha^2$ (ii) $\Sigma\alpha^2\beta^2$ (iii) $\Sigma\alpha^2\beta^2\gamma^2$

Solution (a) $x^4 + ax^3 + bx^2 + cx + d = 0$ ...(i)

If y be a root of the transformed equation, then

$$y = x^2 \quad \ldots(ii)$$

We have to now eliminate x between Eqs. (i) and (ii).

From Eq. (i) $\quad x^4 + bx^2 + d = -(ax^3 + cx)$

On squaring both sides, we get

$$x^8 + b^2x^4 + d^2 + 2bx^6 + 2bdx^2 + 2dx^4 = a^2x^6 + c^2x^2 + 2acx^4$$

On putting $x^2 = y$, we get

$$y^4 + b^2y^2 + d^2 + 2by^3 + 2bdy + 2dy^2 = a^2y^3 + c^2y + 2acy^2$$

$$\Rightarrow \quad y^4 + (2b - a^2)y^3 + (b^2 + 2d - 2ac)y^2 + (2bd - c^2)y - d^2 = 0 \quad \ldots(iii)$$

which is required equation.

(b) If α, β, γ and δ are the roots of Eq. (i), then the roots of Eq. (iii) are $\alpha^2, \beta^2, \gamma^2, \delta^2$

(i) $\Sigma\alpha^2 = -(2b - a^2) = a^2 - 2b$

(ii) $\Sigma\alpha^2\beta^2 = b^2 + 2d - 2ac$

(iii) $\Sigma\alpha^2\beta^2\gamma^2 = -(2bd - c^2) = c^2 - 2bd$

Example 5 *Form an equation whose roots are* $\beta^2+\gamma^2, \gamma^2+\alpha^2, \alpha^2+\beta^2$, *where* α, β, γ *are roots of* $z^3+pz+q=0$

Solution $\because \alpha, \beta, \gamma$ are the roots of

$$z^3+pz+q=0 \quad \ldots(\text{i})$$

$$\therefore \quad \left.\begin{array}{r}\alpha+\beta+\gamma=0\\ \alpha\beta\gamma=-q\end{array}\right\} \quad \ldots(\text{ii})$$

If the transformed equation is in terms of y, then

$$y=\beta^2+\gamma^2=(\beta+\gamma)^2-2\beta\gamma$$

$$=(-\alpha)^2-\frac{2\alpha\beta\gamma}{\alpha}=\alpha^2+\frac{2q}{\alpha} \quad [\because \text{Eq. (i)}]$$

$$\therefore \quad y=z^2+\frac{2q}{z} \text{ or } z^3-zy+2q=0 \quad \ldots(\text{iii})$$

Subtracting Eq. (iii) from Eq. (i)

$$(p+y)z-q=0, \therefore z=\frac{q}{p+y}$$

Putting this value of z in Eq. (i), we get

$$\frac{q^3}{(p+y)^3}+p\cdot\frac{q}{p+y}+q=0$$

$$q^2+p(p+y)^2+(p+y)^3=0$$

or $$y^3+4py^2+5p^2y+(2p^3+q^2)=0$$

which is the required equation.

Concept To find the sum of the integral power of the roots of an equation.

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0 \quad \ldots(\text{A})$$

If $\alpha_1, \alpha_2, \alpha_3, \ldots$, are the roots of the polynomial equation, then let us represent $s_r=\Sigma\alpha_1^r$

i.e., $$s_1=\Sigma\alpha_1=\alpha_1+\alpha_2+\alpha_3+\ldots$$

$$s_2=\Sigma\alpha_1^2=\alpha_1^2+\alpha_2^2+\alpha_3^3+\ldots$$

$$s_3=\Sigma\alpha_1^3=\alpha_1^3+\alpha_2^3+\alpha_3^3+\ldots, \text{ and so on}$$

The following results will help us to find out the value of $s_1, s_2, s_3, \ldots, s_r$.

$$a_0s_1+a_1=0 \quad \ldots(\text{i})$$

$$a_0s_2+a_1s_1+2a_2=0 \quad \ldots(\text{ii})$$

$$a_0s_3+a_1s_2+a_2s_1+3a_3=0 \quad \ldots(\text{iii})$$

$$a_0s_r+a_1s_{r-1}+a_2s_{r-2}+\ldots+a_{r-1}s+ra_r=0 \quad \ldots(\text{iv})$$

From result (i), we can find the value of s_1 and by putting its value in result (ii) we can find s_2. Proceeding in the same way we can find the value of s_r, provided $r<n$. If however $r\geq n$, then multiply Eq. (A) by x^{r-n}

$$a_0x^r+a_1x^{r-1}+a_2x^{r-2}+\ldots+a_{n-1}x^{r-n+1}+a_n^{r-n}=0 \quad \ldots(\text{v})$$

Putting $x=\alpha_1, \alpha_2, \ldots, \alpha_n$ in succession,

$$a_0\alpha_1^r+a_1\alpha_1^{r-1}+a_2\alpha_1^{r-2}+\ldots+a_{n-1}\alpha_1^{r-n+1}+a_n\alpha^{r-n}=0$$

$$a_0\alpha_2^r+a_1\alpha_2^{r-1}+a_2\alpha_2^{r-2}+\ldots+a_{n-1}\alpha_2^{r-n+1}+a_n\alpha_2^{r-n}=0$$

$$\dots\ \dots\ \dots$$
$$\dots\ \dots\ \dots$$
$$\dots\ \dots\ \dots$$
$$a_0\alpha_n^r + a_1\alpha_n^{r-1} + a_2\alpha_n^{r-2} + \dots + a_{n-1}\alpha_n^{r-n+1} + a_n\alpha_n^{r-n} = 0$$

Adding, we have

$$a_0s_r + a_1s_{r-1} + a_2s_{r-2} + \dots + a_{n-1}s_{r-n+1} + a_ns_{r-n} = 0$$

Putting $r = n, n+1, n+2$ etc, we get

$$a_0s_n + a_1s_{n-1} + a_2s_{n-2} + \dots + a_{n-1}s_1 + na_n = 0$$
$$[\because s_0 = \alpha_1^0 + \alpha_2^0 + \dots + \alpha_n^0 = n]$$
$$a_0s_{n+1} + a_1s_n + a_2s_{n-1} + \dots + a_{n-1}s_2 + a_ns_1 = 0$$
$$a_0s_{n+2} + a_1s_{n+1} + a_2s_n + \dots + a_{n-1}s_3 + a_ns_2 = 0$$

and so on. These results gives values of

$$s_n, s_{n+1}, s_{n+2}, \dots.$$

Example 1 *Find the sum of the cubes of the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$.*

Solution Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be the roots of the equation.

Here, $a_0 = 1, a_1 = a, a_2 = b, a_3 = c, a_4 = d$

$\therefore$ $a_0s_1 + a_1 = 0 \Rightarrow s_1 + a = 0$

$\therefore$ $s_1 = -a$

$\Rightarrow$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -a$

Now, $a_0s_2 + a_1s_1 + 2a_2 = 0$

$\Rightarrow$ $s_2 + a(-a) + 2b = 0$

$\therefore$ $s_2 = a^2 - 2b$

$\Rightarrow$ $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = a^2 - 2b$

Now, $a_0s_3 + a_1s_2 + a_2s_1 + 3a_3 = 0$

$\Rightarrow$ $s_3 + a(a^2 - 2b) + b(-a) + 3c = 0$

$\therefore$ $s_3 = -a^3 + 3ab - 3c$

$\Rightarrow$ $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_4^3 = 3ab - a^3 - 3c$

Example 2 *If α, β and γ be the roots of the equation $x^3 + px + q = 0$, find value of Σx^5.*

Solution $a_0 = 1, a_1 = 0, a_2 = p, a_3 = q$

$\therefore$ $a_0s_1 + a_1 = 0 \Rightarrow s_1 = 0$

$$a_0s_2 + a_1s_1 + 2a_2 = 0 \Rightarrow s_2 = -2p$$
$$a_0s_3 + a_1s_2 + a_2s_1 + 3a_3 = 0 \Rightarrow s_3 = -3q$$

Multiplying both sides of given equation by x^2

$$x^5 + px^3 + qx^2 = 0$$

Putting $x = \alpha, \beta, \gamma$ in succession and adding

$$s_5 + ps_3 + qs_2 = 0$$

or $s_5 + p(-3q) + q(-2p) = 0$

$\therefore$ $s_5 = \Sigma x^5 = 5pq$

Example 3 *If α, β, γ be the roots of the equation $x^3 - 7x + 7 = 0$. Find $\alpha^{-4} + \beta^{-4} + \gamma^{-4}$.*

Solution Roots of the equation $x^3 - 7x + 7 = 0$ are α, β, γ. Changing x to $1/x$ and multiplying by x^3, we get

$$7x^3 - 7x^2 + 1 = 0 \quad \ldots(i)$$

Its roots being the reciprocals of the roots of given equation are $1/\alpha, 1/\beta, 1/\gamma$.

Let us denote them by α', β', γ'.

We have to find

$$\alpha^{-4} + \beta^{-4} + \gamma^{-4} = (\alpha')^4 + (\beta')^4 + (\gamma')^4 = s'_4$$

Here, $a_0 = 7, a_1 = -7, a_2 = 0, a_3 = 1$

$\therefore$ $a_0 s'_1 + a_1 = 0 \Rightarrow 7s'_1 - 7 = 0$

$\therefore$ $s'_1 = 1$

$$a_0 s'_2 + a_1 s'_1 + 2a_2 = 0 \Rightarrow 7s - 7 = 0$$

$\therefore$ $s'_2 = 1$

$$a_0 s'_3 + a_1 s'_2 + a_2 s'_1 + 3a_3 = 0$$

$\Rightarrow$ $7s'_3 - 7 + 3 = 0$

$\therefore$ $s'_3 = 4/7$

Multiplying Eq. (i) by x, we get

$$7x^4 - 7x^3 + x = 0$$

Putting $x = \alpha', \beta', \gamma'$ in succession and adding

$$7s'_4 - 7s'_3 + s'_1 = 0 \Rightarrow 7s'_4 - 4 + 1 = 0$$

Hence, $\alpha^{-4} + \beta^{-4} + \gamma^{-4} = 3/7$

$\therefore$ $s'_4 = 3/5$

Example 4 *If $\alpha + \beta + \gamma = 6$; $\alpha^2 + \beta^2 + \gamma^2 = 14$ and $\alpha^3 + \beta^3 + \gamma^3 = 36$. Prove that*

$$\alpha^4 + \beta^4 + \gamma^4 = 98.$$

Solution Let α, β, γ be the roots of the equation

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0 \; (a_0 \neq 0) \quad \ldots(i)$$

We are given that $s_1 = 6, s_2 = 14$ and $s_3 = 36$

$\therefore$ $a_0 s_1 + a_1 = 0 \Rightarrow 6a_0 + a_1 = 0$

$\therefore$ $a_1 = -6a_0$

$\therefore$ $a_0 s_2 + a_1 s_1 + 2a_2 = 0$

$\Rightarrow$ $14a_0 - 6a_0(6) + 2a_2 = 0$

$\therefore$ $a_2 = 11a_0$

Now, $a_0 s_3 + a_1 s_2 + a_2 s_1 + 3a_3 = 0$

$\Rightarrow$ $36a_0 - 6a_0(14) + 11a_0(6) + 3a_3 = 0$

$\therefore$ $a_3 = -6a_0$

Putting these values of a_1, a_2, a_3 in Eq. (i) and dividing by a_0, we get

$$x^3 - 6x^2 + 11x - 6 = 0 \quad \ldots(ii)$$

Multiplying both sides of Eq. (ii) by x, we get

$$x^4 - 6x^3 + 11x^2 - 6x = 0$$

Putting $x = \alpha, \beta$ and γ in succession and adding

$$s_4 - 6s_3 + 11s_2 - 6s_1 = 0$$

$$s_4 - 6(36) + 11(14) - 6(6) = 0$$

$\therefore$ $$s_4 = 216 - 154 + 36 = 98$$

Hence, $$\alpha^4 + \beta^4 + \gamma^4 = 98$$

Example 5 *Find $x^5 + y^5 + z^5$, it being given that*

$$x + y + z = 1; x^2 + y^2 + z^2 = 2;$$
$$x^3 + y^3 + z^3 = 3.$$

Solution Let x, y, z be the roots of the equation

$$a_0 t^3 + a_1 t^2 + a_2 t + a_3 = 0 \quad (a_0 \neq 0) \qquad \text{...(i)}$$

We are given that $s_1 = 1$, $s_2 = 2$, $s_3 = 3$

$$a_0 s_1 + a_1 = 0 \Rightarrow a_0 + a_1 = 0$$

$\therefore$ $$a_1 = -a_0$$

$$a_0 s_2 + a_1 s_1 + 2a_2 = 0 \Rightarrow 2a_0 - a_0 + 2a_2 = 0$$

$\therefore$ $$a_2 = -\frac{a_0}{2}$$

$$a_0 s_3 + a_1 s_2 + a_2 s_1 + 3a_3 = 0$$

$\Rightarrow$ $$3a_0 - a_0(2) - \frac{a_0}{2} \cdot 1 + 3a_3 = 0 \quad \Rightarrow \quad 3a_3 + \frac{a_0}{2} = 0$$

$\therefore$ $$a_3 = -\frac{a_0}{6}$$

Putting these values of a_1, a_2, a_3 in Eq. (i) and dividing by a_0, we get

$$t^3 - t^2 - \frac{1}{2}t - \frac{1}{6} = 0$$

or $$6t^3 - 6t^2 - 3t - 1 = 0 \qquad \text{...(ii)}$$

Multiplying both sides of Eq. (ii) by t

$$6t^4 - 6t^3 - 3t^2 - t = 0$$

Putting $t = x, y, z$ in succession and adding

$$6s_4 - 6s_3 - 3s_2 - s_1 = 0$$

or $$6s_4 - 6(3) - 3(2) - 1 = 0$$

$\therefore$ $$s_4 = \frac{25}{6}$$

Multiplying both sides of Eq. (ii) by t^2

$$6t^5 - 6t^4 - 3t^3 - t^2 = 0$$

Putting $t = x, y, z$ in succession and adding

$$6s_5 - 6s_4 - 3s_3 - s_2 = 0$$

$$6s_5 - 6\left(\frac{25}{6}\right) - 3(3) - 2 = 0$$

$\therefore$ $$s_5 = 6$$

Hence, $$x^5 + y^5 + z^5 = 6$$

Example 6 *If α, β and γ are the roots of $x^3 + px + q = 0$. Prove that*

(i) $\dfrac{\alpha^5+\beta^5+\gamma^5}{5} = \dfrac{\alpha^3+\beta^3+\gamma^3}{3} \cdot \dfrac{\alpha^2+\beta^2+\gamma^2}{2}$

(ii) $3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5) = 5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4)$

Solution Here, $a_0 = 1, a_1 = 0, a_2 = p, a_3 = q$

$$a_0s_1 + a_1 = 0 \Rightarrow s_1 = 0$$
$$a_0s_2 + a_1s_1 + 2a_2 = 0 \Rightarrow s_2 = -2p$$
$$a_0s_3 + a_1s_2 + a_2s_1 + 3a_3 = 0$$
$$\Rightarrow \qquad s_3 = -3q$$

(i) Multiplying both sides of given equation by x^2.

$$x^5 + px^3 + qx^2 = 0$$

Putting $x = \alpha, \beta, \gamma$ in succession and adding

$$s_5 + ps_3 + qs_2 = 0$$
or
$$s_5 - 3pq - 2pq = 0$$
$$\therefore \qquad s_5 = 5pq \qquad \ldots(\text{i})$$

Now,
$$\frac{s_5}{5} = pq;$$
$$\frac{s_3}{3} \cdot \frac{s_2}{2} = (-q)(-p) = pq$$
$$\therefore \qquad \frac{s_5}{5} = \frac{s_3}{3} \cdot \frac{s_2}{2}$$

Hence,
$$\frac{\alpha^5+\beta^5+\gamma^5}{5} = \frac{\alpha^3+\beta^3+\gamma^3}{3} \cdot \frac{\alpha^2+\beta^2+\gamma^2}{2}$$

(ii) Multiplying both sides of the given equation by x,

$$x^4 + px^2 + qx = 0$$

Putting $x = \alpha, \beta, \gamma$ in succession and adding

$$s_4 + ps_2 + qs_1 = 0 \quad \text{or} \quad s_4 - 2p^2 = 0$$
$$\therefore \qquad s_4 = 2p^2$$

Also,
$$s_5 = 5pq \qquad [\text{from Eq. (i)}]$$
$$3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5) = 3s_2s_5$$
$$= 3(-2p)(5pq) = -30p^2q$$
$$5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4) = 5s_3s_4$$
$$= 5(-3q)(2p^2) = -30p^2q$$

Hence,
$$3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5)$$
$$= 5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4)$$

Example 7 *If $a+b+c=0$, show that $a^5+b^5+c^5 = -5abc(bc+ca+ab)$*

Solution Let a, b, c be the roots of the equation

$$a_0x^3 + a_2x + a_3 = 0 \qquad \ldots(\text{i})$$

[IInd term is missing $\because$ $a+b+c=0$]

We are given that $s_1 = 0$;

$$\therefore \quad a_1 = 0$$

$$a_0 s_2 + a_1 s_1 + 2a_2 = 0 \Rightarrow s_2 = \frac{2a_2}{a_0}$$

$$a_0 s_3 + a_1 s_2 + a_2 s_1 + 3a_3 = 0$$

$$\Rightarrow \quad s_3 = \frac{-3a_3}{a_0}$$

Multiplying both sides of the Eq. (i) by x^2,

$$a_0 x^5 + a_2 x^3 + a_3 x^2 = 0$$

Putting $x = a, b, c$ in succession and adding

$$a_0 s_5 + a_2 s_3 + a_3 s_2 = 0$$

or
$$a_0 s_5 - \frac{3a_2 a_3}{a_0} - \frac{2a_2 a_3}{a_0} = 0$$

$$\therefore \quad s_5 = \frac{5a_2 a_3}{a_0^2} \quad \text{or} \quad a^5 + b^5 + c^5 = \frac{5a_2 a_3}{a_0^2}$$

Also, $bc + ca + ab = \frac{a_2}{a_0}$ and $abc = -\frac{a_3}{a_0}$

$$\therefore \quad -5abc(bc + ca + ab) = -5\left(-\frac{a_3}{a_0}\right)\left(\frac{a_2}{a_0}\right) = \frac{5a_2 a_3}{a_0^2}$$

Hence, $a^5 + b^5 + c^5 = -5abc\,(bc + ca + ab)$

Example 8 *Show that sum of products of the first integers taken three at a time is*

$$n^2(n+1)^2(n-2)(n-1)/48.$$

Solution The equation whose roots are first n integers is

$$f(x) = (x-1)(x-2)(x-3)\ldots(x-n) \qquad \text{[say]}$$

$$= x^n + p_1 x^{n-1} + p_2 x^{n-2} + p_3 x^{n-3} + \ldots = 0$$

We have to find the symmetric function of the type.

$\Sigma\alpha\beta\gamma = -p_3$. Here, $\alpha, \beta, \gamma, \ldots$ means 1, 2, 3 ... respectively. *i.e.*, first n integers.

Now, in usual notation

$$s_1 = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

$$s_2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$s_3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Now,
$$s_1 + p_1 = 0;$$

$$\therefore \quad p_1 = -s_1 = \frac{-n(n+1)}{2}$$

$$\because \quad a_0 = 1$$

$$s_2 + p_1 s_1 + 2p_2 = 0$$

or
$$\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} + 2p_2 = 0$$

$$p_2 = \frac{1}{2}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]$$

$$s_3 + p_1s_2 + p_2s_1 + 3p_3 = 0$$

or
$$\frac{n^2(n+1)^2}{4} - \frac{n(n+1)}{2} \cdot \frac{n(n+1)(2n+1)}{6}$$
$$+ \frac{n(n+1)}{4} \times \left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right] + 3p_3 = 0$$

or
$$\frac{n^2(n+1)^2}{4}\left[1 + \frac{n(n+1)}{4}\right] - \frac{n^2(n+1)^2(2n+1)}{12} \times \left(1 + \frac{1}{2}\right) + 3p_3 = 0$$

or
$$\frac{n^2(n+1)^2}{4}\left[\frac{4+n^2+n}{4} - \frac{(2n+1)}{3} \cdot \frac{3}{2}\right] + 3P_3 = 0$$

or
$$\frac{n^2(n+1)^2}{4}\left[\frac{4+n^2+n-4n-2}{4}\right] + 3p_3 = 0$$

$\therefore$
$$-p_3 = \frac{n^2(n+1)^2}{48}(n^2 - 3n + 2)$$

or
$$\Sigma\alpha\beta\gamma = -p_3 = \frac{n^2(n+1)^2(n-1)(n-2)}{48}$$

Example 9 *If $\alpha, \beta, \gamma, \ldots$ be the roots of the equation $f(x) = 0$. Prove that $\frac{f'(x)}{f(x)} = \Sigma S_{m-1}x^{-m}$ where S_m is sum of the nth powers of the roots.*

Solution $f(x) = (x-\alpha)(x-\beta)(x-\gamma)\ldots n$ factors. Taking log on both sides,

$$\log f(x) = \log(x-\alpha) + \log(x-\beta) + \log(x-\gamma) + \ldots$$

Differentiating w.r.t. x, we get

$$\frac{1}{f(x)}f'(x) = \frac{1}{x-\alpha} + \frac{1}{x-\beta} + \frac{1}{x-\gamma} + \ldots \qquad \ldots(i)$$

Now,
$$\frac{1}{x-\alpha} = \frac{1}{x}\left(1 - \frac{\alpha}{x}\right)^{-1}$$
$$= \frac{1}{x}\left(1 + \frac{\alpha}{x} + \frac{\alpha^2}{x^2} + \frac{\alpha^3}{x^3} + \ldots\right) = \frac{1}{x} + \frac{\alpha}{x^2} + \frac{\alpha^2}{x^3} + \frac{\alpha^3}{x^4} + \ldots$$

$\therefore$
$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{\alpha}{x^2} + \frac{\alpha^2}{x^3} + \frac{\alpha^3}{x^4} + \ldots + \frac{1}{x} + \frac{\beta}{x^2} + \frac{\beta^2}{x^3} + \frac{\beta^3}{x^4} + \ldots \text{ for } n \text{ roots}$$
$$= n\frac{1}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \frac{s_3}{x^4} + \ldots$$

Now,
$$S_0 = 1 + 1 + 1 + \ldots = n$$
$$\frac{f'(x)}{f(x)} = s_0x^{-1} + s_1x^{-2} + s_2x^{-3} + s_3x^{-4} + \ldots$$
$$= \Sigma S_{m-1}x^{-m}$$

Hence proved.

Concept Let $f(x) = a_0x^n + a_1x^{n-1} + \dots, a_{n-1}\ x + a_n$ polynomial of degree $n \geq 1$ with integral coefficients and $a_n \neq 0$, then every integral zero of $f(x)$ is a divisor of the constant terms a_n. But the converse is not always true. *i.e.*, every integral divisor of the constant term need not be a zero of the polynomial with integral coefficients.

For example let $f(x) = x^2 + 6$.

Here, 3 divides 6. The constant term of the polynomial $f(x)$ with integral coefficients.

But $$f(3) = 3^2 + 6 = 15 \neq 0$$

$\therefore$ 3 is not a zero of the polynomial $f(x)$.

Hence, the converse may or may not be true.

Concept Upper and lower bounds for the real roots of an equation.

Two real numbers M and m $[m \leq M]$ are called the upper and the lower bounds respectively of the real roots of the real equation $f(x) = 0$, if the real roots of $f(x) = 0$ (if any) lie between m and M.

Method of grouping for finding an upper bound of the real roots.

(1) Combine each $-$ve term with a + ve term. (the power of positive term must be greater than $-$ve term). No $-$ve term is to be left alone.

(2) Choose a value of x say M which makes every group +ve or zero.

Then, M is called an upper bound of the real roots of the real equation $f(x) = 0$.

Method of grouping for finding a lower bound of the real roots.

Let $f(-x) = g(x)$, if m is the upper bound of the real roots of $g(x) = 0$, then $-m$ is the lower bound of real roots of $f(x) = 0$.

Example 1 *Find all the integral roots of* $5x^3 - 11x^2 + 12x - 2 = 0$.

Solution $f(x) = 5x^3 - 11x^2 + 12x - 2 = 0$

Here, constant term $= -2$

Divisors of constant term are $\pm 1, \pm 2$.

So, possible values for integral roots are $\pm 1, \pm 2$

Now,
$$\begin{aligned} f(1) &= 5 - 11 + 12 - 2 \neq 0 \\ f(-1) &= -5 - 11 - 12 - 2 \neq 0 \\ f(2) &= 40 - 44 + 24 - 2 \neq 0 \\ f(-2) &= -40 - 44 - 24 - 2 \neq 0 \end{aligned}$$

Hence, there is no integral root of this equation.

Example 2 *Find the integral roots, if any of the equation* $x^4 + x^3 - 2x^2 + 4x - 24 = 0$. *Also solve the equation.*

Solution Let $f(x) = x^4 + x^3 - 2x^2 + 4x - 24 = 0$

$$\begin{aligned} f(x) &= x^4 - 2x^2 + x^3 - 24 + 4x \\ &= x^2(x^2 - 2) + (x^3 - 24) + 4x \end{aligned}$$

Clearly, $x = 3$ makes every groups on RHS +ve or zero.

M (an upper bound) of real roots of $f(x) = 0$ is 3.

$$\begin{aligned} f(-x) &= 4x^4 - 4x^3 - 8x^2 - 16x - 96 \\ &= x^4 - 4x^3 + x^4 - 8x^2 + x^4 - 16x + x^4 - 96 \\ &= x^3(x - 4) + x^2(x^2 - 8) + x(x^3 - 16) + x^4 - 96 \end{aligned}$$

Clearly, $x = 4$ makes RHS +ve or zero.

m (lower bound) of $f(x) = 0$ is 4.

Divisors of -24 between -4 and 3 are $-4, -3, -2, -1, 1, 2, 3$, so 0 is not a root.

Now, $$f(1) = 20 \neq 0$$

$\therefore$ 1 is not a root.

Divisors decreased by 1 are $-5, -4, -3, -2, 1, 2, 3$ does not divide $f(1) = -20$

$\therefore$ -2 rejected

$f(-1) = -30 \neq 0$. So, -1, is not a root.

Divisors increased by 1 are $-3, -2, -1, 3, 4$.

4 does not divide 30

$\therefore$ 3 rejected

Divisors left $-4, -3, 2$

Hence, integrals roots are $-3, 2$.

Concept Continuation and variation of signs.

Continuation (or Permanence) or of sign

In any polynomial $f(x)$, whose terms are arranged in order, when a + ve sign follows a + ve sign and a − ve sign follows a − ve sign, a continuation of sign is said to be occur.

Variation or Change of sign

In any polynomial $f(x)$, whose terms are arranged in order, when a + ve sign follows a − ve sign or a − ve sign follows a + ve sign, a variation or a change of sign is said to occur.

Some Important Observations

1. If an equation of degree n is complete, then it has $(n + 1)$ terms. In this case (the number of continuation in sign) + (the number of variation in sign) = n.

   e.g. In the polynomial
   $$2x^5 + 7x^4 - 5x^3 - 4x^2 + x + 5$$
   Number of continuation in signs = 3

   Number of variations in signs = 2

   So, $$n = 5$$

2. If the equation is incomplete, then (the number of continuation in sign) + (number of changes in sign) = total number of non-zero terms in the polynomial − 1.

   e.g., Consider the polynomial $x^4 - 3x^2 + 5$

   The signs are + − +

   Number of continuation in sign = 0

   Number of variation in sign = 2

   Number of non-zero terms = 3

   $\therefore$ $$0 + 2 = 3 - 1$$

3. If $f(x) = 0$ is a complete equation, then a continuation in sign $f(x)$ becomes a variation in sign in $f(-x)$ and *vice-versa*. But if the equation is not complete, the result may or may not be true.

   e.g., Let $f(x) = x^3 - 3x^2 + 4x - 5$ be a complete polynomial.

   The sign are + − + −

   There are 3 changes of signs and no continuation in sign.

   Again $f(-x) = -x^3 - 3x^2 - 4x - 5$. There are 3 continuation of sign.

[Ist, IInd, IIIrd, IVth] and no changes of sign.

Let $$f(x) = x^4 + x^2 + 5$$

Signs are + + +

There are 2 continuations of sign and no change of sign.

So, result need not be true, if the polynomial is incomplete.

4. If between two like signs (either both + ve or both – ve), we introduce any number of + ve or – ve signs, then the total number of resulting variations will be an even number.
5. If between two unlike signs (one + ve and other is – ve), we introduce any number of + ve or – ve signs, then the total number of resulting variations will be an odd number.

Descarte's Rule of Signs

The polynomial equations $f(x) = 0$ with real coefficient cannot have more.

1. + ve roots than the number of changes of sign in $f(x)$.
2. – ve roots than the number of changes of signs in $f(x)$.

Concept Location of Zeros

Consider the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \dots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

For every large value of $|x|$, the quantity in the bracket is very near to 1.

$\therefore$ $f(x)$ behaves like $a_n x^n$. Hence, for every large values of $|x|$, the graph of $y = f(x)$ is similar to the graph of $y = a_n x^n$.

It moves up (or down) as $x \to \pm \infty$

Thus, for $a_n > 0$, we have

(i) $a_n x^n \longrightarrow \infty$ as $x \longrightarrow \pm \infty$

(ii) $a_n x^n \longrightarrow \infty$ as $x \longrightarrow \infty$, if n is odd.

(iii) $a_n x^n \longrightarrow -\infty$ as $x \longrightarrow \infty$, if n is odd.

Same is true for $f(x)$.

Theorem *Let $f(x)$ be a real polynomial of degree $n\ (\geq 1)$ and a, b be two real numbers such that $a < b$.*

(i) *If $f(a)$ and $f(b)$ are of opposite signs, then the polynomial $f(x)$ has atleast one and always an odd number of real zeros in (a, b).*

(ii) *If $f(a)$ and $f(b)$ are of the same sign, then the polynomial $f(x)$ has either one real zero or an even number of real zeros in (a, b).*

Corollary 1 *Every equation of an odd degree (having + ve leading coefficients) has atleast one real root of a sign opposite to that of its last term.*

Proof Let $f(x) = 0$, where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1 x + a_0$$

$$= a_n x^n \left[1 + \frac{a_{n-1}}{a_n x} + \dots\dots + \frac{a_0}{a_n x^n}\right]; \quad a_n > 0;$$

n being odd. We have, $f(+\infty) = \infty; f(0) = a_0; f(-\infty) = -\infty$

If $a_0 > 0$, then there must be one root of $f(x) = 0$ in $(-\infty, 0)$ *i.e.*, a real - ve root.

If $a_0 < 0$, then there must be a root between 0 and ∞ *i.e.*, a real positive root.

Corollary 2 *Every equation of an even degree whose constant term is - ve has atleast 2 real roots, one + ve and other - ve (leading coefficient being + ve).*

Proof Let $f(x) = 0$, where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$
$$= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \dots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

$\because$ n is even and $a_n > 0$

$$f(+\infty) = +\infty;\ f(0) = a_0;\ f(-\infty) = -\infty$$

$\therefore$ $\quad a_0 < 0$

$\therefore$ Atleast one root lies between $-\infty$ and 0 and another between 0 and $+\infty$

There must be atleast one + ve and atleast one -ve root.

Rolle's Theorem

Between two consecutive real roots of a polynomial equation $f(x) = 0$ with real coefficients there lies atleast one and always an odd number of real roots of the equation $f'(x) = 0$.

Corollary 1 *Between two consecutive roots of $f'(x) = 0$ there lie almost one real root of $f(x) = 0$*

Corollary 2 *Between any two consecutive real roots a and b of $f'(x) = 0$ there lies,*

(a) *no real root of $f(x) = 0$, if $f(a)f(b) > 0$*

(b) *a unique real root of $f(x) = 0$, iff $f(a)f(b) < 0$*

Method to determine the number of distinct real roots and to locate them.

Let $f(x)$ be a given real polynomial equation.

1. Solve $f'(x) = 0$ for real roots.
2. Arrange the real roots of $f'(x) = 0$ in ascending order say $\alpha_1, \alpha_2, \dots, \alpha_m$.
3. Determine signs of $f(x)$ at $-\infty, \alpha_1, \alpha_2, \dots, \alpha_n, \infty$.
4. The number of changes of sign in the above sequence of signs determine the number of real roots of $f(x) = 0$.

Note It follows from above that if $f(x) = 0$ has n real roots, then $f(x) = 0$, can have at most $m + 1$ real roots.

$\because$ If deg $f(x) = n$, then equation $f(x) = 0$ has atleast $n - (m + 1) = n - m - 1$ non-real roots. Also, $f'(x) = 0$ has $(n - 1) - m = n - m - 1$ non-real roots.

Hence, the number of non real roots of $f(x) = 0$ cannot be less than the number of non-real roots of $f'(x) = 0$.

Example 1 *Consider the polynomial*

$$\underset{+}{}2x^5 \underset{+}{-} 7x^4 \underset{-}{-} 5x^3 \underset{-}{-} 4x^2 \underset{+}{+} x \underset{+}{+} 5$$

There are two changes or variation of sign.

[IInd, IIIrd place + –; IVth, Vth place – +]

Ambiguity When any term of a polynomial $f(x)$ has double sign $\pm$ or $\mp$, an ambiguity is said to be occur.

Example 2 *Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ be a real polynomial of degree n and $a_0 \neq 0$. Let r and s denote the number of variations in signs in $f(x)$ and $f(-x)$ respectively. Show that $n - r - s$ is even.*

Solution Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

is a real polynomial of degree n.

$\therefore$ $\quad f(0) = a_0 \neq 0$ (given)

$\therefore$ 0 is not a root of the equation $f(x)=0$.

$\therefore$ Every real root of $f(x)=0$ is either +ve or –ve.

Let p be the number of + ve roots of $f(x)=0$ and q be the number of –ve roots

$\because f(x)$ is a real polynomial.

$\therefore$ Non-real roots of $f(x)=0$ occur in conjugate pairs

$\because$ Number of non-real roots of $f(x)=0$ is even say $2k$ where k is a +ve integer.

$\therefore$ $$p+q+2k=n \quad \ldots\text{(i)}$$

[$\because$ degree of $f(x)=n \therefore f(x)=0$ has n roots]

By Descarte's rule of sign p which is the number of +ve roots of $f(x)=0$ is $\le r$, the numbers of variations in sign $f(x)$ and falls short of it by an even number.

Let $p=r-2t$ where t is an integer ...(ii)

Similarly, $q=s-2l$ ...(iii)

From Eqs. (i), (ii) and (iii), we have

$$r-2t+s-2l+2k=n$$

$$\Rightarrow \quad n-r-s=2k-2t-2l$$

$$=2[k-t-l]=2k_1$$

where $k_1=k-t-l=$ is an integer

Here, $n-r-s$ is even.

Example 3 *If $a>0$, prove that $x^3+ax+b=0$ has two complex roots.*

Solution **Case I** $b>0$

Let $f(x)=x^3+ax+b$

Signs are +, +, +.

Since, there is no change of sign in $f(x)$.

$\therefore f(x)=0$ has no + ve root.

Again, $f(-x)=-x^3-ax+b$

Signs are –, –, –.

Since, there is one change of sign in $f(-x)$

$\therefore$ $f(x)=0$ has one – ve root

$f(x)=0$ has 2 complex roots

Case II $b<0$

$$f(x)=x^3+ax+b$$

Signs are +, +, –.

$\because$ There is one change of sign in $f(x)$

$\therefore$ $f(x)=0$ has one + ve root

Again, $f(-x)=-x^3-ax+b$

Signs are –, –, –.

Since, there is no change of sign in $f(-x)$

$\therefore$ There is no – ve root of $f(x)=0$

$\therefore$ $f(x)=0$ has 2 complex roots

Case III $b=0$

$$x^3 + ax + b = 0 \text{ becomes } ax + x^3 = 0$$

$\Rightarrow$ $$x(x^2 + a) = 0$$

$\Rightarrow$ $$x = 0 \quad \text{or} \quad x^2 + a = 0$$

$\therefore$ $$x^2 + a = 0 \text{ gives } x = \pm \sqrt{a}\, i \qquad [\because a > 0]$$

$$f(x) = 0 \text{ has 2 complex roots}$$

Example 4 *If $f(x)=0$ is a complete equation and has all its roots real, then the number of + ve roots equals the number of variation in signs and the number of – ve roots equals the number of continuation in sign of $f(x)$.*

Solution Let p be number of variation in signs and

q be number of continuation in sign of $f(x)$.

Let P be number of + ve roots and

N be number of – ve roots of $f(x) = 0$

Let $f(x)=0$ be the equation of degree n

$\because$ $f(x)=0$ is a complete equation

Now, $$p+q=n \qquad \text{...(i)}$$

and equation does not have zero as a root.

$\because$ $f(x) = 0$ has all its roots real

So, $$P+N=n \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$p+q=P+N \qquad \text{...(iii)}$$

We shall prove that $P=p$ and $N=q$

Let it possible $P \neq p$. Either $P>p$ or $P<p$

But $P>p$, then, number of + ve roots of

$[f(-x)=0] >$ Number of variation in sign of $f(x)$.

Which is contrary to Descarte's Rule.

If $P<p$, then from Eq. (iii) $N>q$.

$\Rightarrow$ Number of – ve roots of $[f(x)=0] >$ Number of continuations of sign of $f(x)$.

$\Rightarrow$ Number of – ve roots of $[f(x)=0] >$ Number of variations in sign of $f(-x)$.

Which is contrary to Descarte's Rule.

$\therefore$ Our supposition is wrong.

Hence $P=p$

Now, from Eq. (iii), $p+q=P+N \Rightarrow q=N$

Example 5 *Separate the real roots of the equation.* $3x^4 + 4x^3 - 6x^2 - 12x + 2 = 0$.

Solution
$$f(x) = 3x^4 + 4x^3 - 6x^2 - 12x + 2$$

$$f'(x) = 12x^3 + 12x^2 - 12x - 12 = 12(x^3 + x^2 - x - 1)$$

$$= 12(x+1)(x^2-1) = 12(x-1)(x+1)^2$$

$\therefore$ Roots of $f'(x) = 0$ are given by $12(x-1)(x+1)^2 = 0$

$\Rightarrow$ $-1, 1$ are the only roots of $f'(x) = 0$.

x	$-\infty$	-1	1	∞
$f(x)$	$+\infty$	7	-9	$+\infty$
	$(+)$	$(+)$	$(-)$	$(+)$

Note that for large $|x|$, $f(x)$ behaves as $3x^4$.

$\therefore$ $f(x) = 0$ has a unique real root in each of the intervals $(-1, 1)$ and $(1, \infty)$.

It also shows that $f(x) = 0$ has 2 real roots.

Example 6 *Find the interval in which k should lie so that the roots of equation*

$$2x^3 - 9x^2 + 12x - k = 0 \text{ are real.}$$

Solution $f(x) = 2x^3 - 9x^2 + 12x - k$

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

Roots of $f'(x) = 0$ are given by

$$6(x-1)(x-2) = 0$$

$\Rightarrow$ $x = 1, 2$ which are real roots of $f'(x) = 0$

x	$-\infty$	1	2	$+\infty$
$f(x)$	$-\infty$	$5-k$	$4-k$	$+\infty$
	$(-)$	$(+)$	$(-)$	$(+)$

For large values of $|x|$, $f(x)$ behaves like $2x^3$. Since, all roots of $f(x) = 0$.

$\therefore$ $5 - k > 0$ and $4 - k < 0$ *i.e.*, $k < 5$ and $4 < k$

i.e., $4 < k < 5$ $\therefore$ $k \in (4, 5)$

Example 7 *Show that $x^5 - 5ax + 4b = 0$ has 3 real roots or only one real root, according as $a^5 >$ or $< b^4$, a and b are both + ve.*

Solution $f(x) = x^5 - 5ax + 4b$

$$f'(x) = 5x^4 - 5a = 5(x^4 - a) = 5(x^2 - \sqrt{a})(x^2 + \sqrt{a})$$

$\therefore$ The real roots of $f'(x) = 0$ are given by

$$x^2 - \sqrt{a} = 0$$

$\Rightarrow$ $x = \pm a^{1/4}$

$\therefore$ Only real roots of $f(x) = 0$ are $-a^{1/4}$ and $a^{1/4}$

x	$-\infty$	$-a^{1/4}$	$a^{1/4}$	$+\infty$
$f(x)$	$-\infty$	$4a^{5/4} + 4b$	$-4a^{5/4} + 4b$	$+\infty$
	$(-)$	$(+)$	$(-)$	$+$

For large $|x|$, $f(x)$ behaves as x^5.

(i) If $-4a^{5/4} + 4b < 0$ *i.e.*, $a^{5/4} > b$ or $a^5 > b^4$

then $f(x) = 0$ has 3 real roots one each in

$$(-\infty, -a^{1/4}); (-a^{1/4}, a^{1/4}) \text{ and } (a^{1/4}, +\infty)$$

(ii) If $-4a^{5/4} + 4b > 0$ *i.e.*, $a^5 < b^4$, then $f(x) = 0$

has only one real root in $(-\infty, -a^{1/4})$.

Additional Solved Examples

Additional Solved Examples

Example 1. *Solve* $x(y+z)=44; y(z+x)=50; z(x+y)=54.$

Solution On adding the given equations

$$2(xy+yz+zx)=148$$

$$\Rightarrow \quad xy+yz+zx=74 \quad \text{...(i)}$$

On subtracting given equations one by one from 1, we get

$$yz=30,\ zx=24,\ xy=20$$

On multiplying the three, given above together.

$$x^2y^2z^2=30\times 24\times 20 \Rightarrow xyz=\pm 120 \quad \text{...(ii)}$$

On dividing Eq. (ii) by the values of xy, yz and zx respectively, we get

$$x=\pm 4, y=\pm 5, z=\pm 6$$

Example 2. *Solve the system*

$$x(x+y+z)=a^2; y(x+y+z)=b^2;$$

$$z(x+y+z)=c^2.$$

Solution On adding the given equations termwise, we get

$$(x+y+z)^2=a^2+b^2+c^2$$

$$\Rightarrow \quad x+y+z=\pm\sqrt{a^2+b^2+c^2}$$

Consequently,

$$x=\frac{a^2}{\pm\sqrt{a^2+b^2+c^2}}, y=\frac{b^2}{\pm\sqrt{a^2+b^2+c^2}}$$

$$z=\frac{c^2}{\pm\sqrt{a^2+b^2+c^2}}$$

Example 3. *Solve the system*

$$x(x+y+z)=a-yz, y=(x+y+z)=b-xz$$

$$z(x+y+z)=c-xy.$$

Solution Given system can be rewritten as $(x+z)(x+y)=a;$

$$(y+z)(y+x)=b; (z+x)(z+y)=c$$

Multiplying these equations and extracting a square root from both members of the obtained equality, we have

$$(x+z)(x+y)(y+z)=\pm\sqrt{abc}$$

Hence,

$$y+z=\pm\frac{\sqrt{abc}}{a};\ x+z=\pm\frac{\sqrt{abc}}{b};$$

$$x+y=\pm\frac{\sqrt{abc}}{c}$$

On adding these equalities termwise, we get

$$x+y+z=\pm\frac{\sqrt{abc}}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \quad \therefore \quad x=\pm\frac{\sqrt{abc}}{2}\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)$$

$$y=\pm\frac{\sqrt{abc}}{2}\left(\frac{1}{a}+\frac{1}{c}-\frac{1}{b}\right);$$

$$z=\pm\frac{\sqrt{abc}}{2}\left(\frac{1}{b}+\frac{1}{a}-\frac{1}{c}\right)$$

Example 4. *Solve the system* $x+y+z=a; x+y+v=b; x+z+v=c; y+z+v=d.$

Solution Adding all the given equations, we get $x+y+z+v=\dfrac{a+b+c+d}{3}$ consequently,

$$v=(x+y+z+v)-(x+y+z)$$
$$=\frac{a+b+c+d}{3}-a=\frac{b+c+d-2a}{3}$$

On likewise, we get

$$z=\frac{a+c+d-2b}{3}, y=\frac{a+b+d-2c}{3},$$
$$x=\frac{a+b+c-2d}{3}$$

Example 5. *Solve the system* $ay+bx=c; cx+az=b$ *and* $bz+cy=a$.

Solution Dividing the first equation by ab, the second by ac and third by bc (assuming $abc\neq 0$), we get

$$\frac{y}{b}+\frac{x}{a}=\frac{c}{ab}; \frac{x}{a}+\frac{z}{c}=\frac{b}{ac}; \frac{z}{c}+\frac{y}{b}=\frac{a}{bc}$$

Adding all these equations termwise, we find

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{1}{2}\left(\frac{c}{ab}+\frac{b}{ac}+\frac{a}{bc}\right)$$

Hence, $$\frac{z}{c}=\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)-\left(\frac{x}{a}+\frac{y}{b}\right)=\frac{1}{2}\left(\frac{c}{ab}+\frac{b}{ac}+\frac{a}{bc}\right)-\frac{c}{ab}$$

$\Rightarrow$ $$\frac{z}{c}=\frac{a^2+b^2-c^2}{2abc}$$

$\Rightarrow$ $$z=\frac{a^2+b^2-c^2}{2ab}$$

Similarly, $$y=\frac{a^2+c^2-b^2}{2ac}, x=\frac{b^2+c^2-a^2}{2bc}$$

Example 6. *Solve the system* $cy+bz=2dyz; ax+cx=2d'zx; bx+ay=2d''xy$.

Solution First of all we have an obvious solution $x=y=z=0$. Let us now look for non-zero solutions *i. e.*, for such in which x, y, z are not equal to zero. Dividing the first of the given equations by yz, the second by zx and third by xy.

$$\frac{c}{z}+\frac{b}{y}=2d; \frac{a}{x}+\frac{c}{z}=2d'; \frac{b}{y}+\frac{a}{x}=2d''$$

Hence, $$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=d+d'+d''$$

$\therefore$
$$\frac{a}{x}=d'+d''-d;\ \frac{b}{y}=d+d''-d';$$
$$\frac{c}{z}=d+d'-d''$$

Finally,
$$x=\frac{a}{d'+d''-d};\ y=\frac{b}{d+d''-d'};$$
$$z=\frac{c}{d+d'-d''}$$

Example 7. *Solve the system*
$$\frac{xy}{ay+bx}=c;\ \frac{xz}{az+cx}=b;\ \frac{yz}{bz+cy}=a.$$

Solution Rewrite the system in the following way
$$\frac{ay+bx}{xy}=\frac{1}{c};\ \frac{az+cx}{xz}=\frac{1}{b};\ \frac{bz+cy}{yz}=\frac{1}{a}$$

Hence,
$$\frac{a}{x}+\frac{b}{y}=\frac{1}{c};\ \frac{a}{x}+\frac{c}{z}=\frac{1}{b};\ \frac{b}{y}+\frac{c}{z}=\frac{1}{a}.$$

Proceeding in the same way as in example 6
$$x=\frac{2a^2bc}{ac+ab-bc};\ y=\frac{2ab^2c}{bc+ab-ac};$$
$$z=\frac{2abc^2}{bc+ac-ab}$$

Example 8. *Solve the system* $y+z-x=\dfrac{xyz}{a^2}$;
$$z+x-y=\frac{xyz}{b^2};\ x+y-z=\frac{xyz}{c^2}.$$

Solution Obvious solution is $x=y=z=0$.

Dividing both members of each equation of our system by xyz, we get
$$\frac{1}{xz}+\frac{1}{xy}-\frac{1}{yz}=\frac{1}{a^2};\ \frac{1}{xy}+\frac{1}{yz}-\frac{1}{xz}=\frac{1}{b^2};$$
$$\frac{1}{yz}+\frac{1}{xz}-\frac{1}{xy}=\frac{1}{c^2}$$

On adding pairwise, we find
$$\frac{2}{xy}=\frac{1}{a^2}+\frac{1}{b^2};\ \frac{2}{yz}=\frac{1}{b^2}+\frac{1}{c^2};\ \frac{2}{xz}=\frac{1}{a^2}+\frac{1}{c^2}$$

Consequently,
$$xy=\frac{2a^2b^2}{a^2+b^2};\ yz=\frac{2b^2c^2}{b^2+c^2},$$
$$xz=\frac{2a^2c^2}{a^2+c^2}$$

On multiplying the equalities, we get
$$x^2y^2z^2=\frac{8a^4b^4c^4}{(a^2+b^2)(b^2+c^2)(a^2+c^2)}$$

$$\Rightarrow \qquad xyz = \pm \frac{2\sqrt{2}\,a^2b^2c^2}{\sqrt{(a^2+b^2)(b^2+c^2)(a^2+c^2)}}$$

Using the equality
$$xy = \frac{2a^2b^2}{a^2+b^2}$$

We find for z two values which differ in the sign. By the obtained value of z we find corresponding values of y and x from the equalities. Thus, we get 2 sets of values for x, y and z satisfying our equation.

Example 9. *Solve the system*

$$z + ay + a^2x + a^3 = 0$$
$$z + by + b^2x + b^3 = 0$$
$$z + cy + c^2x + c^3 = 0.$$

Solution The given equations show that the polynomial $\alpha^3 + x\alpha^2 + y\alpha + z$ vanishes at three different values of α namely at $\alpha = a, \alpha = b$ and at $\alpha = c$ (assuming that a, b, c are not equal to one another).

Set up a difference

$$\alpha^3 + x\alpha^2 + y\alpha + z - (\alpha - a)(\alpha - b)(\alpha - c)$$

This difference also becomes zero at α equal to a, b, c.

Expanding this expression in powers of α, we get

$$(x + a + b + c)\alpha^2 + (y - ab - ac - bc)\alpha + z + abc$$

This second degree trinomial vanishes at three different values of α and therefore it equals zero identically and consequently, all its coefficients are equal to zero. *i.e.*,

$$x + a + b + c = 0; y - ab - ac - bc = 0;$$
$$z + abc = 0$$

Hence,
$$x = -(a + b + c)$$
$$y = ab + ac + bc$$
$$z = -abc$$

is the solution of our system.

Example 10. *Solve the system*

$$x_1 + x_2 + x_3 + \ldots + x_n = 1$$
$$x_1 + x_3 + \ldots + x_n = 2$$
$$x_1 + x_2 + x_4 + \ldots + x_n = 3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_1 + x_2 + \ldots + x_{n-1} = n$$

Solution Let $x_1 + x_2 + x_3 + \ldots + x_n = s = 1$

Then $s - x_2 = 2$;

$$s - x_3 = 3, \ldots, s - x_{n-1} = n - 1,$$
$$s - x_n = n$$

Consequently, (since $s = 1$)

$$x_2 = -1, x_3 = -2, \ldots, x_n = -(n-1)$$

Hence,

$$x_2 + x_3 + \ldots + x_n = -[1 + 2 + \ldots + (n-1)] = -\frac{n(n-1)}{2}$$

Finally,
$$x_1 = 1 - (x_2 + x_3 + \ldots + x_n) = 1 + \frac{n(n-1)}{2}$$

Similarly, we can find other values.

Example 11. *Solve the system*

$$x \sin a + y \sin 2a + z \sin 3a = \sin 4a$$
$$x \sin b + y \sin 2b + z \sin 3b = \sin 4b$$
$$x \sin c + y \sin 2c + z \sin 3c = \sin 4c$$

Solution We have, $\sin 2a = 2\sin a \cos a$

$$\sin 3a = \sin a(4\cos^2 a - 1)$$
$$\sin 4a = 4\sin a(2\cos^3 a - \cos a)$$

$\therefore$ The first equation of our system is rewritten in the following way

$$x + 2y\cos a + z(4\cos^2 a - 1) = 4(2\cos^3 a - \cos a)$$

The remaining two are similar. Expand this equation in powers of $\cos a$, we have

$$8\cos^3 a - 4z\cos^2 a - (2y + 4)\cos a + z - x = 0$$

Putting $\cos a = t$ and dividing both members by 8, we get

$$t^3 - \frac{z}{2}t^2 - \frac{y+2}{4}t + \frac{z-x}{8} = 0$$

Our system of equations is equivalent to the statement that the equations has three roots : $t = \cos a$; $t = \cos b$ and $t = \cos c$, which follows :

$$\frac{z}{2} = \cos a + \cos b + \cos c$$
$$\frac{y+2}{4} = -(\cos a \cos b + \cos a \cos c + \cos b \cos c)$$
$$\frac{x-z}{8} = \cos a \cos b \cos c$$

$\therefore$ The solution of our system will be

$$x = 2(\cos a + \cos b + \cos c) + 8(\cos a \cos b \cos c)$$
$$y = -2 - 4(\cos a \cos b + \cos a \cos c + \cos b \cos c)$$
$$z = 2(\cos a + \cos b + \cos c)$$

Example 12. *Solve the system*

$$x^2 = a + (y - z)^2$$
$$y^2 = b + (x - z)^2$$
$$z^2 = c + (x - y)^2$$

Solution Reduce the system to the following form

$$(x + y - z)(x + z - y) = a$$
$$(y + z - x)(y + x - z) = b$$
$$(x + z - y)(z + y - x) = c$$

Multiplying and taking a square root, we get

$$(x + y - z)(x + z - y)(y + z - x) = \pm\sqrt{abc}$$

Further, $\quad y + z - x = \pm\sqrt{\frac{bc}{a}}\ x + z - y = \pm\sqrt{\frac{ac}{b}}\quad x + y - z = \pm\sqrt{\frac{ab}{c}}$

Consequently, $\quad x = \pm\frac{1}{2}\left(\sqrt{\frac{ac}{b}} + \sqrt{\frac{ab}{c}}\right),\ y = \pm\frac{1}{2}\left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ab}{c}}\right), z = \pm\frac{1}{2}\left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ac}{b}}\right)$

Example 13. *Solve the system*

$$x^2 + y^2 + xy = c^2$$
$$z^2 + x^2 + xz = b^2$$
$$y^2 + z^2 + yz = a^2$$

Solution Subtracting the equations term-by-term, we have

$$(x - y)(x + y + z) = b^2 - a^2$$
$$(x - z)(x + y + z) = c^2 - a^2$$

Put $x + y + z = t$, then

$$(x - y)t = b^2 - a^2; (x - z)t = c^2 - a^2$$

On adding these two equations termwise, we have

$$[3x - (x + y + z)]t = b^2 + c^2 - 2a^2$$

Hence,
$$x = \frac{t^2 + b^2 + c^2 - 2a^2}{3t}$$
$$y = \frac{t^2 + a^2 + c^2 - 2b^2}{3t}$$
$$z = \frac{t^2 + a^2 + b^2 - 2c^2}{3t}$$

Substituting these values of x, y, z in one of the equations, we find

$$t^4 - (a^2 + b^2 + c^2)t^2 + a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2 = 0$$

Hence,
$$t^2 = \frac{(a^2 + b^2 + c^2) \pm \sqrt{\begin{array}{c}3(a + b + c)(-a + b + c)\\(a - b + c)(a + b - c)\end{array}}}{2}$$

Knowing t, we obtain the values of x, y, z.

Example 14. *Solve*

$$x^2 + y^2 + z^2 = 14$$
$$xy + yz + zx = 11$$
$$x + y + 2z = 9$$

Solution We have,

$$x^2 + y^2 + z^2 = 14 \quad \text{...(i)}$$
$$xy + yz + zx = 11 \quad \text{...(ii)}$$
$$x + y + 2z = 9 \quad \text{...(iii)}$$

Adding double of Eq. (ii) in Eq. (i), we get

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 36$$

or
$$(x + y + z)^2 = 36$$

or
$$x + y + z = \pm 6 \quad \text{...(iv)}$$

Subtracting Eq. (iv) from Eq. (iii)

$$z = 9 \pm 6 = 3, 15$$
$$z = 3, x + y = 3 \quad \text{[from Eq. (iii)]}$$
$$z = 15, x + y = -21 \quad \text{[from Eq. (iii)]}$$

Also from Eq. (ii),
$$xy + z(x + y) = 11$$
$$xy + 3 \times 3 = 11$$

or $$xy = 2$$
when $$(x + y) = 3, xy = 2$$
$\therefore$ $$x = 1, 2; y = 2, 1$$
From Eq. (ii), when $x + y = -21$
$$xy + 15(-21) = 11$$
$$xy = 326$$
$\therefore$ $$x = \frac{-21 \pm \sqrt{-863}}{2}; \; y = \frac{-21 \pm \sqrt{-863}}{2}$$
Hence, $$x = 1, 2, \frac{-21 \pm \sqrt{-863}}{2}$$
$$y = 2, 1, \frac{-21 \pm \sqrt{-863}}{2}$$
$$z = 3, 3, 15$$

Example 15. *Solve the equations*
$$x + y + z = 14$$
$$x^2 + y^2 + z^2 = 91$$
$$y^2 = zx.$$

Solution We have
$$x + y + z = 14 \quad \ldots\text{(i)}$$
$$x^2 + y^2 + z^2 = 91 \quad \ldots\text{(ii)}$$
$$y^2 = zx \quad \ldots\text{(iii)}$$
Squaring the Eq. (i), we get
$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 196$$
Putting the values of $x^2 + y^2 + z^2$ from Eq. (ii) and of zx from Eq. (iii), we get
$$91 + 2xy + 2yz + 2y^2 = 196$$
$$2xy + 2yz + 2y^2 = 105$$
$$2y(x + y + z) = 105$$
$$2y(14) = 105 \quad \text{or} \quad y = \frac{105}{28} = \frac{15}{4}$$
Hence, $x + z = 14 - \dfrac{15}{4}$ [from Eq. (i)]
or $$x + z = \frac{41}{4}$$
Also, $$zx = \left(\frac{15}{4}\right)^2$$ [from Eq. (ii)]
or $$zx = \frac{225}{16}$$
Hence, x and z are the roots of the equation.
$$t^2 - (x + z)t + (xz) = 0 \quad \text{or} \quad t^2 - \frac{41}{4}t + \frac{225}{16} = 0$$
or $$16t^2 - 164t + 225 = 0 \Rightarrow t = \frac{164 \pm \sqrt{(164)^2 - 4 \times 16 \times 225}}{32}$$

$$\Rightarrow \qquad l = \frac{41 \pm \sqrt{41^2 - 900}}{8} = \frac{41 \pm \sqrt{(41+30)(41-30)}}{8}$$

$$= \frac{41 \pm \sqrt{71 \times 11}}{8} = \frac{41 \pm \sqrt{781}}{8}$$

Hence,
$$x = \frac{41 \pm \sqrt{781}}{8}$$
$$y = \pm \frac{15}{4}$$
$$z = \frac{41 \pm \sqrt{781}}{8}$$

Example 16. *Solve the equations*
$$x + y + z = 9$$
$$x^2 + y^2 + z^2 = 29$$
$$x^3 + y^3 + z^3 = 99$$

Solution We have
$$x + y + z = 9 \quad \ldots\text{(i)}$$
$$x^2 + y^2 + z^2 = 29 \quad \ldots\text{(ii)}$$
$$x^3 + y^3 + z^3 = 99 \quad \ldots\text{(iii)}$$

On squaring both sides of Eq. (i),
$$x^2 + y^2 + z^2 + 2(xy + yz + zx) = 81$$

or $\quad 2(xy + yz + zx) = 52 \quad$ [from Eq. (ii)]

or $\quad xy + yz + zx = 26$

Also, $\quad x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$\therefore \quad 99 - 3xyz = 9(29 - 26)$

or $\quad 3xyz = 72$ or $xyz = 24 \quad \ldots$(iv)

Now, $\quad xy + yz + zx = 26$

or $\quad \dfrac{24}{z} + z(x + y) = 26 \quad$ [from Eq. (iv)]

or $\quad \dfrac{24}{z} + z(9 - z) = 26 \quad$ [from Eq. (i)]

or $\quad z^3 - 9z^2 + 26z - 24 = 0$

or $\quad z = 2, 3, 4$

Since, the equation is symmetrical.

Hence,
$$x = 2, 3, 4$$
$$y = 3, 4, 2$$
$$z = 4, 2, 3.$$

Example 17. *Solve the equations*
$$4x(x + 2y + 4z) = 21$$
$$y(x + 2y + 4z) = \frac{21}{2}$$
$$z(x + 2y + 4z) = 21$$

Solution
$$4x(x + 2y + 4z) = 21 \quad \ldots\text{(i)}$$
$$y(x + 2y + 4z) = \frac{21}{2} \quad \ldots\text{(ii)}$$

$$z(x+2y+4z)=21 \quad \ldots(iii)$$

From Eqs. (i) and (ii) $\frac{4x}{y}=2$ or $y=2x$

From Eqs. (i) and (iii) $\frac{4x}{z}=1$ or $z=4x$

Substituting the values of y and z in Eq. (i), we get

$$4x(x+4x+16x)=21$$

or $$84x^2=21 \text{ or } x^2=\frac{21}{84}$$

$$x=\pm 1/2$$

$$y=\pm 1$$

$$z=\pm 2$$

Example 18. *Determine the values x_1, x_2, x_3, x_4, x_5 satisfying $x_5+x_2=yx_1; x_1+x_3=yx_2; x_2+x_4=yx_3;$ $x_3+x_5=yx_4; x_4+x_1=yx_5$ where y is a given parameter.*

Solution

$$x_5+x_2=yx_1 \quad \ldots(i)$$

$$x_1+x_3=yx_2 \quad \ldots(ii)$$

$$x_2+x_4=yx_3 \quad \ldots(iii)$$

$$x_3+x_5=yx_4 \quad \ldots(iv)$$

$$x_4+x_1=yx_5 \quad \ldots(v)$$

Express x_5 and x_3 from Eqs. (i) and (ii)

$$x_5=yx_1-x_2 \quad \ldots(vi)$$

$$x_3=yx_2-x_1 \quad \ldots(vii)$$

Substitute Eq. (vi) into Eq. (v), we get

$$x_4=(y^2-1)x_1-yx_2 \quad \ldots(viii)$$

After substituting Eqs. (viii) and (vii) into Eq. (iii) and ordering the equation

$$(y^2+y-1)(x_1-x_2)=0 \quad \ldots(ix)$$

Substituting Eqs. (vi), (vii) and (viii) into Eq. (iv), we get

$$(y^2+y-1)((y-1)x_1-x_2)=0 \quad \ldots(x)$$

If $$y^2+y-1=0,$$

i.e., $y=\frac{-1\pm\sqrt{5}}{2}$, then Eqs. (ix) and (x) is satisfied for arbitrary x_1 and x_2 ; since all our previous transformations are reversible, these values uniquely determine x_3, x_4, x_5. If
$y^2+y-1\neq 0,$

Then Eqs. (ix) and (x) gives $x_1-x_2=0$

$$(y-1)x_1-x_2=0 \quad \ldots(xi)$$

Implying $$(y-2)x_1=0 \quad \ldots(xii)$$

For $y=2$ the value of $x_1=x_2$ can be arbitrary, if $x_1=x_2=c$, then $x_3=x_4=x_5=c$
Finally, if $y\neq 2$, then $x_1=0$ from Eq. (xii) and $x_2=0$ from Eq. (xi) implying that all unknowns in the original system are equal to zero.

Example 19. *Solve*

$$\log_2 x + \log_4 y + \log_4 z = 2$$
$$\log_3 y + \log_9 z + \log_9 x = 2$$
$$\log_4 z + \log_{16} x + \log_{16} y = 2$$

Solution

$$\log_2 x + \log_4 y + \log_4 z = 2$$

$\Rightarrow$ $$(\log_4 x)\log_2 4 + \log_4 y + \log_4 z = 2$$

$\Rightarrow$ $$x^2yz = 16 \qquad [\because \log_2 4 = 2]$$

Similarly, the remaining equations reduce to

$$y^2zx = 81 \text{ and } z^2xy = 256$$

Solving $x^2yz = 16; xy^2z = 81$ and $xyz^2 = 256$,

we get

$$x = 2/3; y = 27/8; z = 32/3$$

Example 20. *If* $a + b + c = 1; a^2 + b^2 + c^2 = 9$ *and* $a^3 + b^3 + c^3 = 1$, *find* $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Solution Since,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(bc + ca + ab)$$

$\therefore$ $$bc + ca + ab = [1^2 - 9]/2 = -4 \qquad \text{...(i)}$$

Also, $$a^3 + b^3 + c^3 - 3abc$$
$$= (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab) = 1(9 - (-4))$$
$$a^3 + b^3 + c^3 - 3abc = 13 \qquad \text{...(ii)}$$

So that $$abc = (1 - 13)/3 = -4 \qquad \text{...(iii)}$$

Now, $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{(bc + ca + ab)}{abc} = (-4)/(-4) = 1$$

Example 21. *Solve the following system of equations for real* x, y, z

$$x + y - z = 4; x^2 - y^2 + z^2 = -4; xyz = 6.$$

Solution Given equations can be written as

$$x - z = 4 - y; x^2 + z^2 = y^2 - 4; xz = 6/y$$

By using the identity $(x - z)^2 + 2xz = x^2 + z^2$, we eliminate x and z from the above equations, so as to get

$$(4 - y)^2 + \frac{12}{y} = y^2 - 4$$

$\Rightarrow$ $$y(4 - y)^2 + 12 = y(y^2 - 4)$$

$\Rightarrow$ $$2y^2 - 5y - 3 = 0$$

$\Rightarrow$ $$y = -1/2 \text{ or } 3$$

When $y = -1/2$; $x^2 + z^2 = -15/4$ which is not possible for any real values of x and z.

When $y = 3, x - z = 1, xz = 2$, so that x and $-z$ are the roots of $t^2 - t - 2 = 0$, giving $t = 2$ or -1 *i.e.*, either

$$x = 2, z = 1$$

or $$x = -1, z = -2$$

Example 22. *Solve the equations*

$$x + y + z = ab$$
$$x^{-1} + y^{-1} + z^{-1} = a^{-1}b$$
$$xyz = a^3.$$

Solution

$$x + y + z = ab \quad \text{...(i)}$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{b}{a} \quad \text{...(ii)}$$
$$xyz = a^3 \quad \text{...(iii)}$$

By Eqs. (ii) and (iii), we have

$$xy + yz + zx = a^2b \quad \text{...(iv)}$$

Now, by Eqs. (i), (iii) and (iv), it is clear that x, y, z are roots of

$$t^3 - abt^2 + a^2bt - a^3 = 0 \quad \text{...(v)}$$

We see that the above equations vanishes for $t = a$, *i.e.*, $(t - a)$ is a factor of Eq. (v).
So, by remainder theorem, Eq. (v) is

$$t^2(t - a) + at(t - a) + a^2(t - a) - abt(t - a) = 0$$

or $$(t - a)(t^2 + at + a^2 - abt) = 0$$

or $$(t - a)\{t^2 + t(a - ab) + a^2\} = 0$$

i.e., either $t = a$,

or $$t = \frac{-(a - ab) \pm \sqrt{[(a - ab)^2 - 4a^2]}}{2}$$

i.e., x, y, z are

$$a, \frac{1}{2}a\,[b - 1 + \sqrt{b^2 - 2b - 3}],$$
$$\frac{1}{2}a\,[b - 1 - \sqrt{b^2 - 2b - 3}] \text{ respectively.}$$

Example 23. *Solve the equations*

$$x^3 + y^3 + z^3 = 495; x + y + z = 15; xyz = 105.$$

Solution

$$x^3 + y^3 + z^3 = 495 \quad \text{...(i)}$$
$$x + y + z = 15 \quad \text{...(ii)}$$
$$xyz = 105 \quad \text{...(iii)}$$

From Eqs. (i) and (iii)

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

or $$495 - 315 = 15[x^2 + y^2 + z^2 - xy - yz - zx]$$

or $$\frac{180}{15} = [(x + y + z)^2 - 3(xy + yz + zx)]$$

$$12 = [225 - 3(xy + yz + zx)]$$

or $$xy + yz + zx = 71$$

$$xy + yz + zx = 71 \quad \text{...(iv)}$$

From Eq. (iv), $$x(y + z) + yz = 71$$

or $$x(15 - x) + \frac{105}{x} = 71 \quad \text{[using Eq. (ii)]}$$

$$x^2(15-x)+105-71x=0$$
$$x^3-15x^2+71x-105=0$$

This equation is satisfied when $x = 3, 5, 7$

$\therefore$ By symmetry $y = 5, 7, 3$

$z = 7, 3, 5$

Example 24. *Eliminate x, y, z, u from the equations*

$$x = by + cz + du\,;\, y = cz + du + ax$$
$$z = du + ax + by\,;\, u = ax + by + cz.$$

Solution Now,

$$x + ax = ax + by + cz + du$$
$$y + by = ax + by + cz + du$$
$$z + cz = ax + by + cz + du$$

and
$$u + du = ax + by + cz + du$$

If
$$ax + by + cz + du = k \qquad \text{(say)}$$

Then,
$$x + ax = k$$

$\Rightarrow$
$$x = \frac{k}{1+a}$$

Similarly,
$$y = \frac{k}{1+b};$$
$$z = \frac{k}{1+c} \text{ and } u = \frac{k}{1+d}$$

Substituting these values in

$ax + by + cz + du = k$, we get

$$\frac{ak}{1+a} + \frac{bk}{1+b} + \frac{ck}{1+c} + \frac{dk}{1+d} = k$$

or
$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1$$

Example 25. *Eliminate x, y, z from the equations*

$$x + y + z = 0\,;\, x^2 + y^2 + z^2 = a^2,$$
$$x^3 + y^3 + z^3 = b^3 \quad \textit{and} \quad x^5 + y^5 + z^5 = c^5.$$

Solution Given,

$$x + y + z = 0 \qquad \text{...(i)}$$
$$x^2 + y^2 + z^2 = a^2 \qquad \text{...(ii)}$$
$$x^3 + y^3 + z^3 = b^3 \qquad \text{...(iii)}$$
$$x^5 + y^5 + z^5 = c^5 \qquad \text{...(iv)}$$

From Eq. (i) we can conclude

$$x^3 + y^3 + z^3 = 3xyz \Rightarrow b^3 = 3xyz, \qquad \text{[using Eq. (iii)]}$$

Squaring first equation

$$x^2 + y^2 + z^2 + 2(xy + yz + zx) = 0$$

or $$a^2 + 2(xy + yz + zx) = 0$$

$\therefore$ $$xy + yz + zx = -\frac{a^2}{2} \qquad \ldots\text{(ii)}$$

Now, multiplying Eqs. (ii) and (iii), we get

$$(x^2 + y^2 + z^2)(x^3 + y^3 + z^3) = a^2b^3$$

$$x^5 + y^5 + z^5 + x^2y^3 + x^2z^3 + y^2z^3 + y^2x^3 + z^2x^3 + z^2x^3 = a^2b^3$$

or $$c^5 + x^2y^2(x + y) + y^2z^2(y + z) + z^2x^2(z + x) = a^2b^3$$

or $$c^5 + x^2y^2(-z) + y^2z^2(-x) + z^2x^2(-y) = a^2b^3 \qquad [\because x + y + z = 0]$$

or $$c^5 - x^2y^2z - xy^2z^2 - yx^2z^2 = a^2b^3$$

or $$c^5 - xyz(xy + yz + xy) = a^2b^3$$

From Eqs. (i) and (ii) it becomes

$$c^5 - \frac{b^3}{3}\left(-\frac{a^2}{2}\right) = a^2b^3$$

or $$6c^5 + a^2b^2 = 6a^2b^3 \quad \text{or} \quad 6c^5 = 5a^2b^3$$

Example 26. *Eliminate x, y from the equations*

$$x + y = a; \quad x^2 + y^2 = b^2; \quad x^4 + y^4 = c^4.$$

Solution $\because$ $$x^2 + y^2 = (x + y)^2 - 2xy$$

$\therefore$ $$b^2 = a^2 - 2xy \quad \therefore 2xy = a^2 - b^2$$

Now, $$x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2$$

or $$c^4 = b^4 - 2\left(\frac{a^2 - b^2}{2}\right)^2$$

$\Rightarrow$ $$2c^4 = 2b^4 - (a^4 + b^4 - 2a^2b^2)$$

or $$b^4 - a^4 - 2c^4 + 2a^2b^2 = 0$$

Example 27. *Eliminate x, y, z from the equations*

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = a; \frac{x}{z} + \frac{y}{x} + \frac{z}{y} = b; \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right) = c.$$

Solution Multiplying the given first two equations

$$ab = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right) = 3 + \sum\frac{x^2}{yz} + \sum\frac{xy}{z^2} \qquad \ldots\text{(i)}$$

Also, $$c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$$

$\Rightarrow$ $$c = \left(\frac{x}{z} + \frac{z}{y} + \frac{y^2}{z^2} + \frac{y}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right) \Rightarrow c = 1$$

$$c = 1 + \frac{x^2}{yz} + \frac{z^2}{xy} + \frac{zx}{y^2} + \frac{y^2}{zx} + \frac{xy}{z^2} + \frac{yz}{x^2} + 1$$

$$\Rightarrow \qquad c = 2 + \Sigma \frac{x^2}{yz} + \Sigma \frac{xy}{z^2} \qquad \text{...(ii)}$$

Hence, from Eqs. (i) and (ii)

$$ab - 3 = c - 2$$
$$ab = c + 1$$

Example 28. *Eliminate x, y, z from the equations*

$$\frac{x^2(y+z)}{a^3} = \frac{y^2(z+x)}{b^3} = \frac{z^2(x+y)}{c^3} = \frac{xyz}{abc} = 1.$$

Solution Given equations can be written as

$$x^2(y+z) = a^3 \qquad \text{...(i)}$$
$$y^2(z+x) = b^3 \qquad \text{...(ii)}$$
$$z^2(x+y) = c^3 \qquad \text{...(iii)}$$
$$xyz = abc \qquad \text{...(iv)}$$

From Eqs. (i), (ii) and (iii)

$$x^2y^2z^2(y+z)(z+x)(x+y) = a^3 \times b^3 \times c^3$$
$$a^2b^2c^2(y+z)(z+x)(x+y) = a^3b^3c^3$$

Using Eq. (iv),

$\therefore$

$$abc = (y+z)(z+x)(x+y)$$
$$= (yz + yx + z^2 + zx)(x+y)$$
$$= xyz + x^2y + xz^2 + x^2z + y^2z + y^2x + yz^2 + xyz$$
$$= 2xyz + x^2(y+z) + y^2(z+x) + z^2(x+y) = 2abc + a^3 + b^3 + c^3$$

Hence,

$$a^3 + b^3 + c^3 + abc = 0$$

Example 29. *Eliminate x, y, z from the equations*

$$(x+y-z)(x-y+z) = ayz;$$
$$(y+z-x)(y-z+x) = bzx;$$
$$(z+x-y)(z-x+y) = cxy.$$

Solution Given equations are

$$(x+y-z)(x-y+z) = ayz \qquad \text{...(i)}$$
$$(y+z-x)(y-z+x) = bzx \qquad \text{...(ii)}$$
$$(z+x-y)(z-x+y) = cxy \qquad \text{...(iii)}$$

Multiplying Eqs. (i), (ii), (iii), we get

$$(x+y-z)^2(x-y+z)^2(y+z-x)^2 = abcx^2y^2z^2$$

or

$$(-x^3 - y^3 - z^3 + y^2z + yz^2 + z^2x + zx^2 + x^2y + xy^2 - 2xyz)^2 = abcx^2y^2z^2 \qquad \text{...(iv)}$$

On dividing both sides of Eq. (iv) by $x^2y^2z^2$

or

$$abc = \left(-\frac{x^2}{yz} - \frac{y^2}{zx} - \frac{z^2}{xy} + \frac{y}{x} + \frac{z}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{z} - 2\right) \qquad \text{...(v)}$$

But Eq. (i) may be written as

$$a = \frac{x^2 - y^2 - z^2 + 2yz}{yz} = \frac{x^2}{yz} - \frac{y}{z} - \frac{z}{y} + 2$$

or $$a-2=\frac{x^2}{yz}-\frac{y}{z}-\frac{z}{y}$$

Similarly, $$b-2=\frac{y^2}{zx}-\frac{z}{x}-\frac{x}{z}$$

and $$c-2=\frac{z^2}{xy}-\frac{x}{y}-\frac{y}{x}$$

Now, from Eq. (v)

$$abc=[(2-a)+(2-b)+(2-c)-2]^2$$

$$abc=(4-a-b-c)^2$$

Example 30. *Show that*

$$(a+b+c)^3-4(b+c)(c+a)(a+b)+5abc=0$$

is the eliminant of $ax^2+by^2+cz^2$

$$=ax+by+cz=yz+zx+xy=0.$$

Solution Given equations are

$$ax^2+by^2+cz^2=0 \quad \text{...(i)}$$

$$ax+by+cz=0 \quad \text{...(ii)}$$

$$yz+zx+xy=0 \quad \text{...(iii)}$$

Multiplying Eq. (ii) by $(x+y+z)$, we have

$$(ax+by+cz)(x+y+z)=0$$

or $$ax^2+by^2+cz^2+xy(a+b)+yz(b+c)+zx(c+a)=0$$

But $$ax^2+by^2+cz^2=0$$

$\therefore$ $$xy(a+b)+yz(b+c)+zx(c+a)=0$$

Also, $$xy+yz+zx=0$$

Hence, $$\frac{xy}{b-a}=\frac{yz}{c-b}=\frac{zx}{a-c}=\frac{1}{k} \quad \text{(say)}$$

Dividing each ratio by xyz

$$\frac{1}{z(b-a)}=\frac{1}{x(c-b)}=\frac{1}{y(a-c)}=\frac{1}{k}$$

$\therefore$ $$x=\frac{k}{(c-b)},\ y=\frac{k}{(a-c)},\ z=\frac{k}{(b-a)}$$

Substituting these values in Eq. (ii)

$$a\frac{k}{c-b}+b\frac{k}{a-c}+c\frac{k}{b-a}=0$$

$$a(b-a)(a-c)+b(c-b)(b-a)+c(c-b)(a-c)=0$$

or $$a^3+b^3+c^3-(a+b)(b+c)(c+a)+5abc=0$$

or $$a^3+b^3+c^3+3(a+b)(b+c)(c+a)-4(a+b)(b+c)(c+a)+5abc=0$$

or $$(a+b+c)^3-4(a+b)(b+c)(c+a)+5abc=0$$

Example 31. *Eliminate a, b, c from the system*

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c};a^2+b^2+c^2=1;$$

$$a+b+c=1.$$

Solution Put
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{1}{\lambda}$$

Then, $$a=x\lambda;b=y\lambda;c=z\lambda \qquad \text{...(i)}$$

Now, $$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc \qquad \text{...(ii)}$$

$\because$ $$a+b+c=1 \text{ and } a^2+b^2+c^2=1 \qquad \text{(given)}$$

On putting these values in Eq. (ii), we get

$$ab+bc+ac=0$$

Taking into the consideration equalities Eq. (i), we get

$$xy+xz+yz=0$$

Example 32. *Eliminate x, y, z from the system*

$$y^2+z^2-2ayz=0;$$

$$z^2+x^2-2bxz=0;$$

$$x^2+y^2-2cxy=0.$$

Solution We have,
$$\frac{y}{z}+\frac{z}{y}=2a;\frac{z}{x}+\frac{x}{z}=2b;$$

and
$$\frac{x}{y}+\frac{y}{x}=2c$$

Squaring these equalities and adding them, we get

$$\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}+6=4a^2+4b^2+4c^2$$

On multiplying these equalities, we get

$$\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}+2=8abc$$

Consequently the result of eliminating x, y, z from the given system is

$$a^2+b^2+c^2-2abc=1$$

Example 33. *Eliminate x, y between the equations*

$$x^2-y^2=px-qy;4xy=qx+py;x^2+y^2=1.$$

Solution Given equations are

$$x^2-y^2=px-qy \qquad \text{...(i)}$$

$$4xy=qx+py \qquad \text{...(ii)}$$

$$x^2+y^2=1 \qquad \text{...(iii)}$$

Multiplying the Eq. (i) by x and Eq. (ii) by y, we get

$$x^3-xy^2=px^2-qxy \qquad \text{...(iv)}$$

and
$$4xy^2=py^2+qxy \qquad \text{...(v)}$$

From Eqs. (iv) and (v)

$$x^3 + 3xy^2 = p(x^2 + y^2)$$

Hence, by Eq. (iii)

$$p = x^3 + 3xy^2 ; q = 3x^2y + y^3$$

$$p + q = (x + y)^3 ; p - q = (x - y)^3$$

$$\therefore \quad (p + q)^{\frac{2}{3}} + (p - q)^{\frac{2}{3}} = (x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$$

$$\therefore \quad (p + q)^{2/3} + (p - q)^{2/3} = 2$$

Example 34. *Eliminate x, y, z between the equations*

$$y^2 + z^2 = ayz ; z^2 + x^2 = bzx; x^2 + y^2 = cxy.$$

Solution We have,

$$\frac{y}{z} + \frac{z}{y} = a ; \frac{z}{x} + \frac{x}{z} = b ; \frac{x}{y} + \frac{y}{x} = c$$

Multiplying together these three equations

$$2 + \frac{y^2}{z^2} + \frac{z^2}{y^2} + \frac{z^2}{x^2} + \frac{x^2}{z^2} + \frac{x^2}{y^2} + \frac{y^2}{x^2} = abc$$

$$2 + (a^2 - 2) + (b^2 - 2) + (c^2 - 2) = abc$$

$$\therefore \quad a^2 + b^2 + c^2 - 4 = abc$$

Example 35. *Eliminate x, y, z between the equations*

$$\frac{y}{z} - \frac{z}{y} = a ; \frac{z}{x} - \frac{x}{z} = b ; \frac{x}{y} - \frac{y}{x} = c.$$

Solution We have,

$$a + b + c = \frac{x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2)}{xyz}$$

$$= \frac{(y - z)(z - x)(x - y)}{xyz}$$

If we change the sign of x, the sign of b and c are changed, while the sign of a remains unaltered.

Hence,

$$a - b - c = \frac{(y - z)(z + x)(x + y)}{xyz}$$

$$b - c - a = \frac{(y + z)(z - x)(x - y)}{xyz}$$

$$c - a - b = \frac{(y + z)(z + x)(x - y)}{xyz}$$

$$\therefore (a + b + c)(b + c - a)(c + a - b)(a + b + c)$$

$$= \frac{-(y^2 - z^2)^2(z^2 - x^2)^2(x^2 - y^2)^2}{x^4y^4z^4}$$

$$= -\left(\frac{y}{z} - \frac{z}{y}\right)^2\left(\frac{z}{x} - \frac{x}{z}\right)^2\left(\frac{x}{y} - \frac{y}{x}\right)^2$$

$$= -a^2b^2c^2$$

$$\therefore \quad 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 + a^2b^2c^2 = 0$$

Example 36. *If α, β, γ be the roots of the cubic $ax^3 + 3bx^2 + 3cx + d = 0$. Prove that the equation in y whose roots are $\frac{\beta\gamma - \alpha^2}{\beta + \gamma - 2\alpha}$; $\frac{\gamma\alpha - \beta^2}{\gamma + \alpha - 2\beta}$; $\frac{\alpha\beta - \gamma^2}{\alpha + \beta - 2\gamma}$; is obtained by the transformation $axy + b(x + y) + c = 0$. Hence, form the equation with above roots.*

Solution $\because \alpha, \beta, \gamma$ are the roots of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0 \quad \text{...(i)}$$

$$\therefore \quad \alpha + \beta + \gamma = -\frac{3b}{a}; \alpha\beta + \beta\gamma + \gamma\alpha = \frac{3c}{a};$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Now,
$$y = \frac{\beta\gamma - \alpha^2}{\beta + \gamma - 2\alpha} = \frac{\frac{\alpha\beta\gamma}{\alpha} - \alpha^2}{(\alpha + \beta + \gamma) - 3\alpha} = \frac{-\frac{d}{a\alpha} - \alpha^2}{-\frac{3b}{a} - 3\alpha}$$

$$= \frac{d + a\alpha^3}{3\alpha(b + a\alpha)} = \frac{d + ax^3}{3x(b + ax)}$$

$$\therefore \quad 3xy(b + ax) = d + ax^3$$

or
$$ax^3 - 3ayx^2 - 3byx + d = 0 \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$3(b + ay)x^2 + 3(c + by)x = 0$$

$$(b + ay)x + c + by = 0 \quad [\because x \neq 0]$$

$$axy + b(x + y) + c = 0$$

Which is the required transformation.

Now,
$$(ay + b)x = -(by + c)$$

$$\therefore \quad x = -\frac{by + c}{ay + b}$$

Putting this value of x in Eq. (i), we get

$$-a\left(\frac{by + c}{ay + b}\right)^3 + 3b\left(\frac{by + c}{ay + b}\right)^2 - 3c\left(\frac{by + c}{ay + b}\right) + d = 0$$

or
$$a(by + c)^3 - 3b(by + c)^2(ay + b) + 3c(by + c)(ay + b)^2 - d(ay + b)^3 = 0$$

Which is the required equation.

Example 37. *If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x + 1 = 0$. Form an equation whose roots are $\frac{1}{\beta^3} + \frac{1}{\gamma^3} - \frac{1}{\alpha^3}$; $\frac{1}{\alpha^3} + \frac{1}{\gamma^3} - \frac{1}{\beta^3}$; $\frac{1}{\alpha^3} + \frac{1}{\beta^3} - \frac{1}{\gamma^3}$.*

Solution Roots of the equation

$$x^3 + 2x^2 + 3x + 1 = 0 \quad \text{...(i)}$$

are α, β, γ. Let us form an equation, whose roots are $\alpha^3, \beta^3, \gamma^3$. If y is a root of the transformed equation, then

$$y = x^3 \quad \text{...(ii)}$$

To eliminate x between Eqs. (i) and (ii)

Eq. (i) can be written as $x^3 + 1 = -(2x^2 + 3x)$

On cubing both sides, we get

$$x^9 + 3x^6 + 3x^3 + 1 = -[8x^6 + 27x^3 + 18x^3(2x^2 + 3x)]$$

$$x^9 + 3x^6 + 3x^3 + 1 = -[8x^6 + 27x^3 + 18x^3(-x^3 - 1)]$$

Putting $x^3 = y$ in this equation

$$y^3 + 3x^2 + 3y + 1 = -8y^2 - 27y + 18y^2 + 18y$$

or
$$y^3 - 7y^2 + 12y + 1 = 0 \quad \text{...(iii)}$$

Its roots are $\alpha^3, \beta^3, \gamma^3$

Changing y to $1/y$, Eq. (iii) becomes

$$\frac{1}{y^3} - \frac{7}{y^2} + \frac{12}{y} + 1 = 0$$

or
$$y^3 + 12y^2 - 7y + 1 = 0 \quad \text{...(iv)}$$

Its roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$.

Let us denote them by a, b, c.

$$\therefore \quad a + b + c = -12$$

We have to form an equations whose roots are

$$\frac{1}{\beta^3} + \frac{1}{\gamma^3} - \frac{1}{\alpha^3}; \frac{1}{\gamma^3} + \frac{1}{\alpha^3} - \frac{1}{\beta^3}; \frac{1}{\alpha^3} + \frac{1}{\beta^3} - \frac{1}{\gamma^3}$$

i.e., whose roots are $b + c - a, c + a - b, a + b - c$ where a, b, c are the roots of Eq. (iv). If the new equation is in terms of z, then

$$z = b + c - a = (a + b + c) - 2a = -12 - 2y$$

$$\therefore \quad y = -\frac{12 + z}{2}$$

Putting this value of y in Eq. (iv), we have

$$-\frac{(12 + z)^3}{8} + 12 \cdot \frac{(12 + z)^2}{4} + 7 \cdot \frac{(12 + z)}{2} + 11 = 0$$

$$\frac{(12 + z)^3}{8} - 12\frac{(12 + z)^2}{4} - 7\frac{(12 + z)}{2} - 11 = 0$$

$$(12 + z)^3 - 24(12 + z)^2 - 28(12 + z) - 88 = 0$$

or
$$(12 + z)^3 - 24(12 + z)^2 - 28(12 + z) - 88 = 0$$

or
$$z^3 + 12z^2 - 172z - 2152 = 0$$

Which is required equation.

Example 38. *If x_1, x_2, x_3 are the roots of $x^3 - x^2 + 4 = 0$, form the equation whose roots are $x_1 + x_2^2 + x_3^2$; $x_2 + x_3^2 + x_1^2$; $x_3 + x_1^2 + x_2^2$.*

Solution $\because$ x_1, x_2, x_3 are the roots of equation

$$x^3 - x^2 + 4 = 0 \quad \text{...(i)}$$

$$\therefore \quad x_1 + x_2 + x_3 = 1; x_1x_2 + x_2x_3 + x_3x_1 = 0;$$
$$x_1x_2x_3 = -4$$

If the transformed equation is in terms of y, then

$$y = x_1 + x_2^2 + x_3^2 = x_1 + (x_2 + x_3)^2 - 2x_2x_3$$

$$= x_1 + (1 - x_1)^2 - \frac{2x_1x_2x_3}{x_1} = x_1 + (1 - x_1)^2 + \frac{8}{x_1}$$

$\therefore$ $$y = x + (1 - x)^2 + \frac{8}{x} = x^2 - x + 1 + \frac{8}{x}$$

or $$x^3 - x^2 + x - xy + 8 = 0 \qquad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$xy - x - 4 = 0 \Rightarrow x = \frac{4}{y - 1}$$

Putting this value of x in Eq. (i), we get

$$\frac{64}{(y-1)^3} - \frac{16}{(y-1)^2} + 4 = 0$$

$\Rightarrow$ $$(y - 1)^3 - 4(y - 1) + 16 = 0$$

or $$y^3 - 3y^2 - y + 19 = 0$$

Example 39. *If α, β, γ are the roots of the cubic $x^3 + px^2 + qx + r = 0$, find the value of* $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$ in terms of p, q, r.

Solution Roots of the equation

$$x^3 + px^2 + qx + r = 0 \qquad \text{...(i)}$$

are α, β, γ.

Let us first form an equation whose roots are

$$\beta + \gamma, \gamma + \alpha, \alpha + \beta.$$

Let y be a root of the transformed equation, then

$$y = \beta + \gamma = \alpha + \beta + \gamma - \alpha$$

$$= - p - \alpha \qquad (\because \alpha + \beta + \gamma = - p)$$

$$= - p - x$$

$\therefore$ $$x = - (y + p)$$

Putting this value of x in Eq. (i), we have

$$-(y + p)^3 + p(y + p)^2 - q(y + p) + r = 0$$

or $$y^3 + 2py^2 + (p^2 - q)y + (pq - r) = 0 \qquad \text{...(ii)}$$

Its roots are $\beta + \gamma, \gamma + \alpha, \alpha + \beta$.

$\therefore$ $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$

$$= \text{product of the roots of Eq. (ii)} = - (pq - r) = r - pq$$

Example 40. *If α is a root of equation $x^4 + ax^3 - 6x^2 - ax + 1 = 0$*, then show that $\frac{1 + \alpha}{1 - \alpha}$ is also a root. Hence, show that the other two roots are

$$\frac{-1}{\alpha}, \frac{\alpha - 1}{\alpha + 1}.$$

Solution Since, α is a root.

$\therefore$ $$\alpha^4 + a\alpha^3 - 6\alpha^2 - a\alpha + 1 = 0 \qquad \text{...(i)}$$

Putting $x = \frac{1+\alpha}{1-\alpha}$, we get

$$\frac{[(1+\alpha)^4 + (1-\alpha)^4] + a[1+\alpha][1-\alpha]}{[1+\alpha]^2 - [1-\alpha]^2 - 6[(1+\alpha)^2(1-\alpha)^2]} = 0$$

On simplification it reduces to Eq. (i). Hence, $\frac{1+\alpha}{1-\alpha}$ is also a root. Again if we replace x by $-1/x$, we get the same equation.

$\therefore$ If α is a root, then $-1/\alpha$ is also a root and if $-\frac{1}{\alpha}$ is a root, then $\frac{1+(-1/\alpha)}{1-(-1/\alpha)}$ *i.e.*, $\frac{\alpha-1}{\alpha+1}$ is also a root.

Example 41. *If α, β, γ be the roots of $x^3 + 2x^2 - 3x - 1 = 0$.*

Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3}$.

Solution Roots of the equation,

$$x^3 + 2x^2 - 3x - 1 = 0 \quad \text{...(i)}$$

are α, β, γ.

Let us first form an equation whose roots are $\alpha^3, \beta^3, \gamma^3$.

If y is a root of the transformed equation, then

$$y = x^3 \quad \text{...(ii)}$$

To eliminate x between Eqs. (i) and (ii), Eq. (i) can be written as

$$x^3 - 1 = -(2x^2 - 3x)$$

On cubing both sides of above equations

$\Rightarrow$ $x^9 - 3x^6 + 3x^3 - 1 = -[8x^6 - 27x^3 - 18x^3(2x^2 - 3x)]$

$\Rightarrow$ $x^9 - 3x^6 + 3x^3 - 1 = -[8x^6 - 27x^3 - 18x^3(1 - x^3)]$

$\Rightarrow$ $x^9 - 3x^6 + 3x^3 - 1 = -8x^6 + 27x^3 + 18x^3 - 18x^6$

$\Rightarrow$ $x^9 + 23x^6 - 42x^3 - 1 = 0$

Putting $x^3 = y$ in this equation, we get

$$y^3 + 23y^2 - 42y - 1 = 0 \quad \text{...(iii)}$$

Its roots are $\alpha^3, \beta^3, \gamma^3$.

Changing y to $\frac{1}{y}$, Eq. (iii) becomes

$$\frac{1}{y^3} + \frac{23}{y^2} - \frac{42}{y} - 1 = 0$$

$$y^3 + 42y^2 - 23y - 1 = 0 \quad \text{...(iv)}$$

Its roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$.

$\therefore$ $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3}$ = sum of roots of Eq. (iv) = -42

Example 42. *Find the value of k for which the roots of the* $kx^3 + 2x^2 - 3x + 1 = 0$ *are in HP.*

Solution Let α, β, γ be the roots of

$$kx^3 + 2x^2 - 3x + 1 = 0 \quad \text{...(i)}$$

which are in HP.

Transform the given equation whose roots are in AP. replace x by $\frac{1}{x}$.

$$x^3 - 3x^2 + 2x + k = 0 \quad \text{...(ii)}$$

Let the roots of Eq. (ii) be

$$a - d, a, a + d$$

Sum of roots, $3a = 3$

$\Rightarrow$ $a = 1$

Since, $a = 1$ is one of the root of Eq. (ii)

$\therefore$ It will satisfy the equation.

$\therefore$ We get, $1 - 3 + 2 + k = 0$

$\Rightarrow$ $k = 0$

Which is a contradiction as the given equation is a degree three polynomial.

$\therefore$ There is no value of k for which the roots are in HP.

Example 43. *If* α, β, γ *be the roots of the cubic equation* $x^3 + 3x + 2 = 0$, *find the equation whose roots are* $(\alpha - \beta)(\alpha - \gamma), (\beta - \gamma)(\beta - \alpha), (\gamma - \alpha)(\gamma - \beta)$. *Hence, show that the above cubic has two imaginary roots.*

Solution Let $z = (\alpha - \beta)(\alpha - \gamma) = \alpha^2 - \alpha\beta - \alpha\gamma + \beta\gamma$

$$= \alpha^2 - \Sigma\alpha\beta + \frac{2\alpha\beta\gamma}{\alpha}$$

or $\alpha z = \alpha^3 - 3\alpha + 2(-2)$ $\quad (\because \Sigma\alpha\beta = 3; \alpha\beta\gamma = -2)$ ...(i)

Also, $\alpha^3 + 3\alpha + 2 = 0$, we get

$$\alpha^2 - 3\alpha = -6\alpha - 2$$

On putting this value in Eq. (i)

$$\alpha z = -6\alpha - 6 \text{ or } \alpha(z + 6) = -6$$

$$\therefore \quad \alpha = -\frac{6}{z + 6}$$

But α is a root of $x^3 + 3x + 2 = 0$

$$\therefore \quad \left(-\frac{6}{z+6}\right)^3 + 3\left(-\frac{6}{z+6}\right) + 2 = 0$$

$$(z + 6)^3 - 9(z + 6)^2 - 108 = 0$$

$$z^3 + 9z^2 - 216 = 0$$

Let z_1, z_2, z_3 be the roots of above equation, then

$$z_1 z_2 z_3 = (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)(\beta - \alpha)(\gamma - \alpha)$$

$$(\gamma - \beta) = 216$$

or $$-(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = 216$$

$$(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = 216$$

Hence, one of the factors in RHS must be -ve say $(\alpha - \beta)^2$ is -ve. *i.e.*, $\alpha - \beta$ = pure imaginary, showing that α and β are conjugate complex.
Hence, the given equation has two imaginary roots.

Example 44. *Find the remainder when $(x + 1)^n$ is divided by $(x - 1)^3$.*

Solution Let $x - 1 = y$ (so that $x = y + 1$)

We find that
$$(x + 1)^n = (y + 2)^n$$
$$= 2^n + ny2^{n-1} + \frac{n(n-1)}{2}y^2 \cdot 2^{n-2} + \dots$$

The remainder when RHS is divided by y^3
$$= 2^n + ny \cdot 2^{n-1} + \frac{n(n-1)}{2}y^2 2^{n-2}$$
$$= 2^n + n(x-1)2^{n-1} + \frac{n(n-1)}{2}(x-1)^2 \cdot 2^{n-2}$$
$$= n(n-1) \cdot 2^{n-3}x^2 + x[-n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1}] + n(n-1) \cdot 2^{n-3} - n \cdot 2^{n-1} + 2^n$$
$$= n(n-1) \cdot 2^{n-3}x^2 - 2^{n-2}x[n^2 - 3n] + 2^{n-3}[n^2 - 5n + 8]$$

Example 45. *Show that $(x - 1)^2$ is a factor of $x^n - nx + n - 1$.*

Solution Let $f(x) = x^n - nx + n - 1$

Since, $f(x) = 0$

$\therefore$ $f(x)$ is divisible by $x - 1$ [by factor theorem]

so, $f(x) = (x^n - 1) - n(x - 1)$
$$= (x-1)\{x^{n-1} + x^{n-2} + \dots + 1 - n\} = (x-1)g(x)$$

where $g(x) = x^{n-1} + x^{n-2} + \dots + 1 - n$

Since, $g(1) = 0$

$\therefore$ $x - 1$ is a factor of $g(x)$. [by factor theorem]

$\therefore$ $f(x)$ is divisible by $(x - 1)^2$

Example 46. *Show that $(x - 1)^2$ is a factor of $x^{n+1} - x^n - x + 1$.*

Solution
$$x^{n+1} - x^n - x + 1 = (x^n - 1)(x - 1)$$
$$= (x-1)^2(x^{n-1} + x^{n-2} + \dots + 1)$$

It is clear that
$(x - 1)^2$ is a factor of $x^{n+1} - x^n - x + 1$

Example 47. *$f(x)$ is a polynomial of degree atleast two with integer coefficients. Show that which it is divided by $(x - a)(x - b)$ where $a \neq b$, the remainder is* $x\left[\dfrac{f(a)-f(b)}{a-b}\right] + \dfrac{af(b)-bf(a)}{a-b}$.

Solution Since, the divisor is a polynomial of degree 2 the remainder will be of the form $Ax + b$.
By division algorithm, we have
$$f(x) = q(x)(x-a)(x-b) + Ax + B \qquad \dots(i)$$
where $q(x)$ is the quotient.

Substituting $x = a, b$ successively in Eq. (i), we get $f(a) = Aa + B$; $f(b) = Ab + B$.
Solving for A and B, we have

$$A = [f(a) - f(b)] / (a - b)$$
$$B = [af(b) - bf(a)] / (a - b)$$

Example 48. *Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer M such that $p(n)p(n+1) = p(M)$.*

Solution Let the zeros of $p(x)$ be α, β so that $p(x) = (x - \alpha)(x - \beta)$.

Then, $$p(n) = (n - \alpha)(n - \beta),$$
$$p(n+1) = (n + 1 - \alpha)(n + 1 - \beta)$$

We have to show that $p(n)p(n+1)$ can be written as $(t - \alpha)(t - \beta)$ for some integer t (which will depend upon n)

$$\begin{aligned} p(n)p(n+1) &= (n - \alpha)(n - \beta)(n + 1 - \alpha)(n + 1 - \beta) \\ &= \{(n - \alpha)(n + 1 - \beta)\}\{(n - \beta)(n + 1 - \alpha)\} \\ &= \{n(n+1) - n(\alpha + \beta) - \alpha + \alpha\beta\} \times \{n(n+1) - n(\alpha + \beta) - \beta + \alpha\beta\} \\ &= \{n(n+1) + na + b - \alpha\}\{n(n+1) + na + b - \beta\} \\ &= (t - \alpha)(t - \beta); t = n(n+1) + an + b = p(t) \end{aligned}$$

Thus, $p(n)p(n+1)$ can be written as $p(M)$ for

$$M = n(n+1) + an + b$$

Example 49. *A polynomial $f(x)$ with rational coefficients leaves remainder 15, when divided by $x - 3$ and remainder $2x + 1$, when divided by $(x - 1)^2$. Find the remainder when $f(x)$ is divided by $(x - 3)(x - 1)^2$.*

Solution Let quotient be $q(x)$ and remainder be $r(x)$ when $f(x)$ is divided by $(x - 3)(x - 1)^2$.

Now, as divisor is a polynomial of degree 3 the remainder must be a polynomial of degree at most 2 *i.e.*, it must be of the form $ax^2 + bx + c$; a, b, c are some rational numbers.

$$\begin{aligned} ax^2 + bx + c &= a[(x - 1) + 1]^2 + b[(x - 1) + 1] + c \\ &= a(x - 1)^2 + (2a + b)(x - 1) + a + b + c \end{aligned}$$

By division algorithm,

$$f(x) \equiv q(x)(x - 3)(x - 1)^2 + a(x - 1)^2 + (2a + b)(x - 1) + a + b + c \quad \text{...(i)}$$

Now, according to given condition $f(x)$ leaves a remainder 15 when divided by $x - 3$

$\therefore$ $$f(3) = 15$$

Now, putting $x = 3$ in Eq. (i), we have

$$15 = 9a + 3b + c \quad \text{...(ii)}$$

Also, from Eq. (i), we find that remainder when $f(x)$ is divided by $(x - 1)^2$ is $(2a + b)(x - 1) + (a + b + c)$.

Since, this is given to be $2x + 1$, we have

$$(2a + b)(x - 1) + (a + b + c) = 2x + 1$$

Putting $x = 1$, we get

$$a + b + c = 3 \quad \text{...(iii)}$$

Putting $x = 0$ throughout, we get

$$-a + c = 1 \quad \text{...(iv)}$$

From Eqs. (ii), (iii) and (iv), we get

$$a = 2, b = -2, c = 3$$

$\therefore$ $$\text{Remainder} = ax^2 + bx + c$$
$$= 2x^2 - 2x + 3$$

Example 50. *For every pair p, q of + ve integers, whose HCF is 1. Show that*

$$(x^p - 1)(x^q - 1) \text{ divides } (x^{pq} - 1)(x - 1)$$

Solution Since, $y - 1$ divide $y^{n} - 1$ for every + ve integers n.

$\therefore$ By writing $x^{pq} - 1$ as $(x^p)^q - 1$ we find that $x^p - 1$ divides $x^{pq} - 1$. Similarly, $x^q - 1$ also divides $x^{pq} - 1$, where p and q are prime to each other

$\therefore$ GCD of $x^p - 1$ and $x^q - 1$ is $x - 1$.

Consequently, $$x^p - 1 = (x - 1) f(x)$$

and $$x^q - 1 = (x - 1) g(x)$$

where $f(x), g(x)$ have no common factor.

Now, $f(x), g(x), (x - 1)$ have no factor in common and each of them divides $x^{pq} - 1$

$\therefore$ $(x - 1) f(x) g(x)$ divides $x^{pq} - 1$.

Consequently, $(x - 1)^2 f(x) g(x)$ divides $(x^{pq} - 1)(x - 1)$.

i.e., $(x^p - 1)(x^q - 1)$ divides $(x^{pq} - 1)(x - 1)$.

Example 51. *Prove that the polynomial*

$$x^{9999} + x^{8888} + x^{7777} + \ldots\ldots + x^{1111} + 1$$

is divisible by $x + 1$.

Solution Let

$$M = x^{9999} + x^{8888} + x^{7777} + \ldots + x^{1111} + 1$$

and $$N = x^9 + x^8 + x^7 + \ldots + x^1 + 1$$

$$M - N = x^9 (x^{9990} - 1) + x^8 (x^{8880} - 1) + x^7 (x^{7770} - 1) + \ldots + x (x^{1110} - 1)$$
$$= x^9 [(x^{10})^{999} - 1] + x^8 [(x^{10})^{888} - 1] + x^7 [(x^{10})^{777} - 1] + \ldots + x [(x^{10})^{111} - 1] \quad \ldots \text{(i)}$$

Now, $(x^{10})^n$ is divisible by $x^{10} - 1, \forall n \geq 1$

RHS of Eq. (i) is divisible by $x^{10} - 1$

$\therefore$ $M - N$ is divisible by $x^{10} - 1$

and hence divisible by $x^9 + x^8 + \ldots + 1$.

Example 52. *If n is an odd + ve integers not divisible by 3. Show that $xy(x + y)(x^2 + xy + y^2)$ is a factor of $(x + y)^n - x^n - y^n$.*

Solution We have, $xy (x + y)(x^2 + y^2 + xy)$

$$= xy (x + y)(x - \omega y)(x - \omega^2 y)$$

[ω, ω^2 are non-real cube roots of unity]

It is enough to show that $(x + y)^n - x^n - y^n$ vanishes for $x = 0; y = 0$.

Now, $$x = -y; x = \omega y \text{ and } x = \omega^2 y$$

The polynomial obviously vanishes for $x = \omega y$

$$(\omega y + y)^n - (\omega y)^n - y^n = y^n [(\omega + 1)^n - \omega^n - 1]$$
$$= y^n [(-\omega^2)^n - \omega^n - 1] = -y^n [(\omega + 1)^n - \omega^n - 1]$$
$$= y^n [(-\omega^2)^n - \omega^n - 1] = -y^n [\omega^{2n} + \omega^n + 1] \qquad (\because n \text{ is odd})$$

Let $n = 3p + 1$, then $\omega^n = \omega^{3p+1}$

and $$\omega^{2n} = \omega^{6p+2} = \omega^2$$

$\therefore$ $$\text{Above expression} = -y^n [\omega^2 + \omega + 1] = 0$$

If $n = 3p + 2$, then $\omega^n = \omega^2, \omega^{2n} = \omega$ and the above expression is zero. We can similarly prove that the given polynomial vanishes for $x = \omega^2 y$. If n is an odd $+ve$ integer not divisible by 3, then $(x + y)^n - x^n - y^n$ is divisible by $xy(x^2 + xy + y^2)$.

Example 53. *If $g(x)$ and $h(x)$ are polynomials with real coefficients and $f(x) = g(x^3) + x\,h(x^3)$ is divisible by $x^2 + x + 1$. Show that $g(x)$ and $h(x)$ are both divisible by $x - 1$.*

Solution If $f(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$. We can write

$$f(x) = q(x)(x^2 + x + 1)$$

Let ω be a non-real cube root of unity. Then,

$$f(\omega) = q(\omega)(\omega^2 + \omega + 1) = 0$$
$$f(\omega^2) = q(\omega^2)(\omega + \omega^2 + 1) = 0$$

$\therefore$ $$g(1) + \omega h(1) = 0 \text{ and } g(1) + \omega^2 h(1) = 0$$

So, $$g(1) = h(1) = 0$$

Both $g(x)$ and $h(x)$ are divisible by $x - 1$.

Example 54. *Let z be a root of $x^5 - 1 = 0$ with $z \neq 1$. Find the value of $z^{15} + z^{16} + z^{17} + \ldots + z^{50}$*

Solution $z^{15} + z^{16} + \ldots + z^{50}$

$$= z^{15}(z^1 + \ldots + z^{35}) = 1\left(\frac{z^{36} - 1}{z - 1}\right) = 1\left(\frac{z - 1}{z - 1}\right) = 1$$

Example 55. *Let $f(x)$ be a polynomial leaving the remainder A when divided by $(x - a)$ and remainder B when divided by $x - b$ $(a \neq b)$. Find remainder left by this polynomial when divided by $(x - a)(x - b)$.*

Solution Since, the products $(x - a)(x - b)$ is a second degree trinomial when divided by it, the polynomial $f(x)$ will necessary leave a remainder which is a first degree polynomial in $x, a\alpha + \beta$.

Thus, there exists the following identity.

$$f(x) = (x - a)(x - b)Q(x) + \alpha x + \beta$$

It only remains to determine $\alpha + \beta$. Putting in this identity first $x = a$ and then $x = b$, we get

$$f(a) = \alpha a + \beta ; f(b) = \alpha\beta + \beta$$

Remainder from dividing $f(x)$ by $x - a$ is equal to $f(a)$, so $f(a) = A ; f(b) = B$

Thus, for determining α and β, we get the following system of two equations in two unknowns

$$\alpha a + \beta = A ; \alpha\beta + \beta = B$$

Hence, $$\alpha = \frac{1}{a - b}(A - B) \text{ and } \beta = \frac{aB - bA}{a - b}$$

Example 56. *Find out at what values of p and q the binomial* $x^4 + 1$ *is divisible by* $x^2 + px + q$?

Solution Let us suppose

$$x^4 + 1 = (x^2 + px + q)(x^2 + p'x + q')$$
$$= x^4 + (p + p')x^3 + (q + q' + pp')x^2 (pq' + qp')x + qq'$$

For determining p, q, p' and q' we have four equations

$$p + p' = 0 \quad \ldots(\text{i})$$
$$pp' + q + q' = 0 \quad \ldots(\text{ii})$$
$$pq' + qp' = 0 \quad \ldots(\text{iii})$$
$$qq' = 1 \quad \ldots(\text{iv})$$

From Eqs. (i) and (iii), we find $p' = -p(q' - q) = 0$

Assume

Case I $p = 0, p' = 0, q + q' = 0, qq' = 1, q^2 = -1$

$$q = \pm i ; q' = \pm i$$

The corresponding factorization has the form

$$x^4 + 1 = (x^2 + i)(x^2 - i)$$

Case II $q' = q, q^2 = 1, q = \pm 1$

Suppose first $q' = q = 1$, then $pp' = -2, p + p' = 0, p^2 = 2, p = \pm\sqrt{2}, p' = \pm\sqrt{2}$

The corresponding factorization is

$$x^4 + 1 = (x^2 - \sqrt{2x} + 1)(x^2 + \sqrt{2x} + 1)$$

Assume, then

$$q = q' = -1 ; p + p' = 0 ; pp' = 2 ; p = \pm\sqrt{2}\, i$$
$$p' = \pm\sqrt{2}\, i$$

Factorization will be

$$x^4 + 1 = (x^2 + \sqrt{2x}\, i - 1)(x^2 - \sqrt{2x}\, i - 1)$$

Example 57. *Prove that*

(i) *the polynomial* $x(x^{n-1} - na^{n-1}) + a^n(n-1)$ *is divisible by* $(x - a)^2$.

(ii) *the polynomial* $(1 - x^n)(1 + x) - 2nx^n(1 - x) - n^2x^n(1 - x)^2$ *is divisible by* $(1 - x)^3$.

Solution (i) Rewrite the polynomial as

$$x^n - a^n - nxa^{n-1} + na^n = (x^n - a^n) - na^{n-1}(x - a)$$
$$= (x - a)(x^{n-1} + ax^{n-2} + \ldots + a^{n-1} - na^{n-1})$$

At $x = a$, the second factor of the last product vanishes and consequently is divisible by $x - a$.

$\therefore$ Given polynomial is divisible by $(x - a)^2$

(ii) Let us denote polynomial by P_n and set up the difference $P_n - P_{n-1}$. Transforming this difference, we easily prove that it is divisible by $(1 - x)^3$.

Since, it is true for any + ve integer n we obtain a number of equalities

$$P_n - P_{n-1} = (1 - x)^3 \phi_n(x)$$

$$P_{n-1} - P_{n-2} = (1-x)^3 \phi_{n-1}(x)$$

....

....

....

....

$$P_3 - P_2 = (1-x)^3 \phi_2(x)$$

$$P_2 - P_1 = (1-x)^3 \phi_1(x)$$

Where $\phi_1(x)$ are polynomials with respect to x.

Hence, $$P_n - P_1 = (1-x)^3 \phi(x)$$

But since $$P_1 = (1-x)^3$$

It follows that P_n is divisible by $(1-x)^3$ and our proposition is proved.

Example 58. *Find out whether the polynomial*

$$x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$$

(a, b, c, d are + ve integers) is divisible by $x^3 + x^2 + x + 1$.

Solution Put $$f(x) = x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$$

$$x^3 + x^2 + x + 1 = (x+1)(x^2+1) = (x+1)(x+i)(x-i)$$

It only remains to show that

$$f(-1) = f(i) = f(-i) = 0$$

Try yourself

Example 59. *Show that the expression.*

$(x+y+z)^m - x^m - y^m - z^m$ *(m is odd) is divisible by* $(x+y+z)^3 - x^3 - y^3 - z^3$.

Solution It is known that

$$(x+y+z)^3 - x^3 - y^3 - z^3 = 3(x+y)(x+z)(y+z)$$

Let us prove that $(x+y+z)^m - x^m - y^m - z^m$ is divisible by $x + y$. Considering our polynomial rearranged in powers of x. We put in it $x = -y$.

We have, $$(-y+y+z)^m - (-y)^m - y^m - z^m = 0 \qquad [\because m \text{ is odd}]$$

Consequently, our polynomial is divisible by $(x + y)$. Likewise we make sure that it is divisible by $(x+z)$ and $(y+z)$.

Example 60. *Prove that the polynomial* $(\cos\phi + x\sin\phi)^n - \cos n\phi - x\sin n\phi$ *is divisible by* $x^2 + 1$.

Solution Put

$$f(x) = (\cos\phi + x\sin\phi)^n - \cos n\phi - x\sin\phi$$

But $$x^2 + 1 = (x+i)(x-i)$$

$$f(i) = (\cos\phi + i\sin\phi)^n - (\cos n\phi + i\sin n\phi) = 0$$

Likewise we make sure that $f(-i) = 0$

Our supposition is proved.

Example 61. *Find out at what n the polynomial* $1 + x^2 + x^4 + \ldots + x^{2n-2}$ *is divisible by the polynomial* $1 + x + x^2 + x^{n-1}$.

Solution
$$1 + x^2 + x^4 + \ldots + x^{2n-2} = \frac{x^{2n} - 1}{x^2 - 1}$$
$$1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

It is required to find out at what n
$$\left(\frac{x^{2n} - 1}{x^2 - 1}\right) \Bigg/ \left(\frac{x^n - 1}{x - 1}\right) \text{ will be a polynomial in } x.$$

we find,
$$\left(\frac{x^{2n} - 1}{x^2 - 1}\right) \Bigg/ \left(\frac{x^n - 1}{x - 1}\right) = \frac{x^n + 1}{x + 1}$$

For $x^n + 1$ to be divisible by $x + 1$, it is necessary and sufficient that $(-1)^n + 1 = 0$ *i.e.*, n is odd.

Thus, $1 + x^2 + \ldots + x^{2n-2}$ is divisible by $1 + x + x^2 + \ldots + x^{n-1}$, if n is odd.

Example 62. *Find the condition necessary and sufficient for* $x^3 + y^3 + z^3 + kxyz$ *to be divisible by* $x + y + z$.

Solution The condition necessary and sufficient for a polynomial $f(x)$ to be divisible by $x - a$, consists in that $f(a) = 0$

Let
$$f(x) = x^3 + kxyz + y^3 + z^3$$

For this polynomial to be divisible by $x + y + z$, it is necessary and sufficient that
$$f(-y - z) = 0$$

However,
$$f(-y - z) = -(y + z)^3 - kyz(y + z) + y^3 + z^3$$
$$= -(k + 3)yz(y + z)$$

Simplifying, we get $k = -3$

Thus, for $x^3 + y^3 + z^3 + kxyz$ to be divisible by $x + y + z$ it is necessary and sufficient that $k = -3$.

Example 63. *Solve the system of equation for real* x *and* y
$$5x\left(1 + \frac{1}{x^2 + y^2}\right) = 12\ ; 5y\left(1 - \frac{1}{x^2 + y^2}\right) = 4.$$

Solution $$(5x)^2 + (5y)^2 = \frac{12^2}{\left(1 + \frac{1}{x^2 + y^2}\right)^2} + \frac{4^2}{\left(1 - \frac{1}{x^2 + y^2}\right)^2}$$

Put $x^2 + y^2 = \frac{1}{t}$, we have
$$\frac{25}{t} = \frac{144}{(1 + t)^2} + \frac{16}{(1 - t)^2}$$

So that
$$25(1 - t^2)^2 = 144t(1 - t)^2 + 16t(1 + t)^2$$

or
$$25t^4 - 160t^3 - 206t^2 - 160t + 25 = 0$$

Dividing by t^2, we get

$$25t^2 - 160t + 206 - \frac{160}{t} + \frac{25}{t^2} = 0$$

i.e.,

$$25\left(t^2 + \frac{1}{t^2}\right) - 160\left(t + \frac{1}{t}\right) + 206 = 0$$

Let $t + \frac{1}{t} = u$, we have

$$25(u^2 - 2) - 160u + 206 = 0$$

i.e.,

$$25u^2 - 160u + 156 = 0$$

$\therefore$

$$u = \frac{160 \pm \sqrt{(160^2 - 4 \cdot 25 \cdot 156)}}{50} = \frac{160 \pm 100}{50} = \frac{6}{5}, \frac{26}{5}$$

When $u = \frac{6}{5}$, $t + \frac{1}{t} = \frac{6}{5}$ which does not give any real values of t.

If $u = \frac{26}{5}$, $t + \frac{1}{t} = \frac{26}{5}$, which gives $t = \frac{1}{5}$ or 5

$\therefore$ $t = 5$ or $\frac{1}{5}$ so that $x^2 + y^2 = \frac{1}{5}$ or 5

When $x^2 + y^2 = \frac{1}{5}$, we get

$$5x(1 + 5) = 12, 5y(1 - 5) = 4,$$

so that $x = \frac{2}{5}, y = -\frac{1}{5}$

When $x^2 + y^2 = 5$, we get

$$5x\left(1 + \frac{1}{5}\right) = 12 \; ; 5y\left(1 - \frac{1}{5}\right) = 4,$$

so that $x = 2, y = -1$

Thus, $x = \frac{2}{5}, y = -\frac{1}{5}$ and $x = 2, y = -1$

Example 64. *Prove that the equation*

$$5x^2 + 5y^2 - 8xy - 2x - 4y + 5 = 0$$

is not satisfied by any pair of real numbers x and y.

Solution Rewriting the given equation as a quadratic in y

$$5y^2 - y(8x + 4) + 5x^2 - 2x + 5 = 0 \quad \text{...(i)}$$

Solving Eq. (i), we have

$$y = \frac{(8x + 4) \pm \sqrt{(8x + 4)^2 - 20(5x^2 - 2x + 5)}}{10}$$

$$= \frac{(8x + 4) \pm 2\sqrt{(-9x^2 + 26x - 21)}}{10} \quad \text{...(ii)}$$

Now, the expression under the radical sign in Eq. (ii) can never be +ve and therefore y cannot take a real value for any real value of x. In fact

$$-9x^2 + 26x - 21 = -9\left[\left(x - \frac{13}{9}\right)^2 - \frac{20}{81}\right]$$

Which is -ve, whatever real value x may have.

Given equation is not satisfied by any pair of real numbers x and y.

Example 65. *Prove that if the coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers, then the roots of the equation cannot be rational numbers.*

Solution Suppose $\frac{p}{q}$ is a root of quadratic equation $ax^2 + bx + c = 0$. Then,

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0, \text{ i.e., } ap^2 + bpq + cq^2 = 0.$$

If p and q are both odd, then all the terms of the expression $ap^2 + bpq + cq^2$ are odd and so the expression cannot be equal to zero. If one of p and q is odd and other is even, then two of the terms are even. One term is odd.

$\therefore$ The expression cannot be equal to zero.

Also, p and q cannot be both even as p and q are prime to each other.

Hence proved.

Example 66. *Given that $x^4 + px^3 + qx^2 + rx + s = 0$ has four real +ve roots. Prove that*

(i) $pr - 16s \geq 0$ (ii) $q^2 - 36s \geq 0$ *with equality in each case holds if and only if four roots are equal.*

Solution Let the roots of the equation.

$$x^4 + px^3 + qx^2 + rx + s = 0 \text{ be } \alpha, \beta, \gamma, \delta \text{ so that}$$

$$\alpha > 0, \beta > 0, \gamma > 0, \delta > 0$$

Now,
$$\Sigma\alpha = -p$$
$$\Sigma\alpha\beta = q$$
$$\Sigma\alpha\beta\gamma = -r$$
$$\alpha\beta\gamma\delta = s$$

(i) $pr = \Sigma\alpha \, \Sigma\alpha\beta\gamma$

By the inequality of the means

$$\frac{1}{4}\Sigma\alpha \geq (\alpha\beta\gamma\delta)^{1/4} \quad \text{...(i)}$$

$$\frac{1}{4}\Sigma\alpha\beta\gamma \geq (\alpha\beta\gamma\delta)^{3/4} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1}{16}\Sigma\alpha \, \Sigma\alpha\beta\gamma \geq \alpha\beta\gamma\delta$$

i.e.,
$$pr - 16s \geq 0 \quad \text{...(iii)}$$

Equality hold in Eq. (iii) $\Leftrightarrow$ Equalities hold in both Eqs. (i) and (ii) $\Leftrightarrow \alpha, \beta, \gamma, \delta$ are all equal and $\alpha\beta\gamma$, $\alpha\beta\delta, \alpha\gamma\delta, \beta\gamma\delta$ are all equal $\Leftrightarrow \alpha = \beta = \gamma = \delta$.

(ii) $q^2 = (\Sigma\alpha\beta)^2 \geq [6(\alpha\beta \cdot \alpha\gamma \cdot \alpha\delta \cdot \beta\gamma \cdot \beta\delta . \gamma\delta)^{1/6}]^2 = 36\alpha\beta\gamma\delta = 36s$

$\therefore$
$$q^2 \geq 36s$$

Equality holds if and only if $\alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta$ are all equal, *i.e.*, if and only if $\alpha = \beta = \gamma = \delta$.

Example 67. *Let a, b, c, d be four real numbers, not all equal to zero. Prove that the zeros of the polynomial $f(x) = x^6 + ax^3 + bx^2 + cx + d$ cannot be all real.*

Solution Let us suppose that all the 6 roots of the given equation are real.

Let us denote the roots by $\alpha, \beta, \gamma, \delta, \lambda$ and μ $\Sigma\alpha = 0, \Sigma\alpha\beta = 0$

$\therefore$
$$\Sigma\alpha^2 = (\Sigma\alpha^2) - 2\Sigma\alpha\beta = 0$$

$\because \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \lambda^2 + \mu^2 = 0$ and $\alpha, \beta \ldots$ are all real.

$\therefore \quad \alpha = \beta = \gamma = \delta = \lambda = \mu = 0$

Consequently we must have $a = b = c = d = 0$ which is impossible, since we are given a, b, c, d are all not zero.

Hence, roots of given equation cannot be all real.

Example 68. *Prove that if $a_1, a_2, \ldots, a_n$ are all distinct, then the polynomial $(x - a_1)^2 (x - a_2)^2 \ldots (x - a_n)^2 + 1$ can never be written as the product of two polynomials with integer coefficients.*

Solution Suppose that there exists polynomial $f(x) \cdot g(x)$, with integer coefficients such that

$$f(x) g(x) = (x - a_1)^2 (x - a_2)^2 \ldots (x - a_n)^2 + 1 \quad \ldots \text{(i)}$$

$\because$ RHS is always +ve.

$\therefore$ $f(x)$ can never vanish

So, its sign never changes.

Similarly, $g(x)$ can never vanish and its sign never changes

$\because$ $f(x), g(x)$ are always + ve, so $f(x)$ and $g(x)$ are both always +ve,

Substituting $x = a_1, a_2, \ldots, a_n$ in Eq. (i), we get

$$f(a_1) g(a_1) = 1, f(a_2) g(a_2) = 1, \ldots, f(a_n) g(a_n) = 1$$

$\because$ $f(a_1), \ldots, f(a_n)$ are all +ve integers. It follows that

$$f(a_1) = f(a_2) = \ldots = f(a_n) = 1$$

Similarly, $\quad g(a_1) = g(a_2) = \ldots = g(a_n) = 1$

$\because$ $f(x) - 1, g(x) - 1$ vanish when $\quad x = a_1, a_2, \ldots, a_n$

$\therefore \quad f(x) - 1 = p(x)(x - a_1)(x - a_2) \ldots (x - a_n)$

By Factor theorem

$$g(x) - 1 = q(x)(x - a_1)(x - a_2) \ldots (x - a_n)$$

$p(x), q(x)$ are polynomials with integer coefficients.

$\because$ $f(x) g(x)$ is of degree $2n$, $p(x)$ and $q(x)$ must be both constants. Suppose $p(x) = a, q(x) = b$.

Then, $\quad f(x) = a(x - a_1)(x - a_2) \ldots (x - a_n) + 1$

$\quad g(x) = b(x - a_1)(x - a_2) \ldots (x - a_n) + 1$

Substituting $f(x)$ and $g(x)$ in Eq. (i), we get

$$ab = 1, a + b = 0$$

$\because$ There do not exist any real numbers a, b satisfying these conditions (these conditions imply $a^2 = -1, b^2 = -1$).

$\therefore$ There is a contradiction and given polynomial cannot be expressed as the product of two polynomials with integer coefficients.

Example 69. *If $p(x)$ is a polynomial with integer coefficients and a, b, c are three distinct integers, then show that it is impossible to have $p(a) = b$, $p(b) = c$, $p(c) = a$.*

Solution Suppose it is possible that $p(a) = b, p(b) = c$ and $p(c) = a$.

$\because \quad p(x) - p(b)$ vanishes, when $x = b$

$\therefore \quad (x - b)$ must be a factor of $p(x) - p(b)$.

$a - b$ must divide $p(a) - p(b)$ i.e., $b - c$

Similarly, $b - c$ must divide $a - b$.

$\therefore \quad b - c = \pm (a - b)$

$\because$ a, b, c are all distinct,

So, we cannot have

$$b - c = -(a - b)$$

Consequently, $\quad b - c = a - b$

This means that $b = \frac{1}{2}(a + c)$ so that b must lie between a and c. Similarly, we conclude that, c lies between a and b. It is obvious that both the conclusion cannot be simultaneous true.

Hence, our supposition must be wrong.

It cannot be possible that $p(a) = b, p(b) = c$ and $p(c) = a$.

Example 70. *Let $f(x)$ be a polynomial with integer coefficients and suppose that for five distinct integers a_1, a_2, a_3, a_4, a_5, one has $f(a_1) = f(a_2) f(a_3) = f(a_4) = f(a_5) = 2$. Show that there does not exist an integer b such that $f(b) = 9$.*

Solution Suppose there exist an integer b such that $f(b) = 9$

$\because \quad f(x) - f(b) = 0$, when $x = b$

$\therefore \quad x - b$ must divide $f(x) - f(b)$

$a_1 - b$ must divide $f(a_1) - f(b)$

We are given that $f(a_1) - f(b) = -7$

$$\therefore \quad (a_1 - b) | (-7)$$

$\because$ Only divisors of -7 are $-7, -1, +1, +7$

$\therefore$ $a_1 - b$ must have one of these values.

Similarly, $a_2 - b, a_3 - b, a_4 - b, a_5 - b$ must take one of the values $\pm 1, \pm 7$.

At least some two of the five numbers $a_i - b, i = 1, 2, \ldots, 5$ must be equal.

This is not possible $\because a_i$ are all distinct.

$f(x) \neq 9$ for any integer $x = b$.

Example 71. *Solve in R the equation*

$$2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + \ldots + 2x + 3 = 0.$$

Solution $2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + \ldots + 2x + 3$

$$= (2x + 3)(x^{98} + x^{96} + x^{95} + \ldots + 1)$$

$$= (2x + 3)\left(\frac{x^{98} - 1}{x - 1}\right)$$

The equation $x^{98} - 1 = 0$ has only two real roots *i.e.*, ± 1.

$\therefore$ The given equation has only 2 real roots *i.e.*, $-\frac{3}{2}$ and -1.

Example 72. *If x, y, z are three real numbers such that $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$. Show that each of x, y, z lies in the closed interval $\left[\frac{2}{3}, 2\right]$ i.e., $\frac{2}{3} \le x \le 2, \frac{2}{3} \le y \le 2, \frac{2}{3} \le z \le 2$. Can x attain extreme value $\frac{2}{3}$ or 2?*

Solution Rewriting the given equation in the form

$$y + z = 4 - x, \; y^2 + z^2 = 6 - x^2, \text{ we get}$$

$$yz = \frac{1}{2}[(y + z)^2 - (y^2 + z^2)] = x^2 - 4x + 5$$

$\therefore$ y, z must be the roots of equation.

$$t^2 - (4-x)t + x^2 - 4x + 5 = 0 \quad \ldots(\text{i})$$

$\because$ y, z are real.

Discriminant of Eq. (i) must be non-negative, *i.e.*,

$$(4-x)^2 - 4(x^2 - 4x + 5) \geq 0$$

$$\Leftrightarrow \quad 3x^2 - 8x + 4 \leq 0$$

$$\Leftrightarrow \quad (3x-2)(x-2) \leq 0$$

$$\Leftrightarrow \quad \frac{2}{3} \leq x \leq 2$$

Given equations are symmetrical in x, y, z.

$\therefore$ We must have $\frac{2}{3} \leq y \leq 2$, $\frac{2}{3} \leq z \leq 2$

when $x = \frac{2}{3}$, Eq. (i) becomes $t^2 - \frac{10}{3}t + \frac{25}{9} = 0$

So that $t = \frac{5}{3}, \frac{5}{3}$. We have solution $\left(\frac{2}{3}, \frac{5}{3}, \frac{5}{3}\right)$. Consequently x take the value $\frac{2}{3}$.

Similarly, we find that when $x = 2$, Eq. (i) reduces to $t^2 - 2t + 1 = 0$ so that $t = 1, 1$.

We have, solution (2, 1, 1) and we conclude that x can take the extreme value 2 as well.

Example 73. *Determine $x, y, z \in R$ such that*

$$2x^2 + y^2 + 2z^2 - 8x + 2y - 2xy + 2xz - 16z + 35 = 0$$

Solution

$$2x^2 + y^2 + 2z^2 - 8x + 2y - 2xy + 2xz - 16z + 35 = 0$$

$$\Rightarrow \quad (x-y)^2 + (x+z)^2 + z^2 - 16z - 8x + 2y + 35 = 0$$

$$\Rightarrow \quad (x-y-1)^2 + (x+z-3)^2 + z^2 - 10z + 25 = 0$$

$$\Rightarrow \quad (x-y-1)^2 + (x+z-3)^2 + (z-5)^2 = 0$$

Thus, $\quad x - y = 1, x + z = 3, z = 5$

Hence, $\quad x = -2, y = -3$

Solution is $\quad x = -2,\ y = -3$ and $z = 5$

Example 74. *Find all real x, y that satisfying $x^3 + y^3 = 7$ and $x^2 + y^2 + x + y + xy = 4$.*

Solution Let $x + y = \alpha, xy = \beta$ and hence $x^2 + y^2 = \alpha^2 - 2\beta$

$$(x^3 + y^3) = (x+y)(x^2 - xy + y^2)$$

$$\Rightarrow \quad \alpha(\alpha^2 - 3\beta) = 7$$

$$\Rightarrow \quad \alpha^3 - 3\alpha\beta = 7 \quad \ldots(\text{i})$$

and $\quad x^2 + y^2 + x + y + xy = 4$

$$\Rightarrow \quad \alpha^2 - 2\beta + \alpha + \beta = 4$$

$$\Rightarrow \quad \alpha^2 - \beta + \alpha = 4$$

$$\Rightarrow \quad \beta = \alpha^2 + \alpha - 4 \quad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we have

$$\alpha^3 - 3\alpha(\alpha^2 + \alpha - 4) = 7$$

$$f(\alpha) = 2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = 0$$

$\Rightarrow$ $$f(1) = 2 + 3 - 12 + 7 = 0$$

Hence, $(\alpha - 1)$ is a factor.

So, $$f(\alpha) = 2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = 0$$

$$= (\alpha - 1)(2\alpha^2 + 5\alpha - 7) = 0$$

$$= (\alpha - 1)(\alpha - 1)(2\alpha + 7) = 0$$

So, $$\alpha = 1 \text{ or } -\frac{7}{2}$$

When $\alpha = 1, \beta = -2$ and when $\alpha = -\frac{7}{2}, \beta = \frac{19}{4}$

If we take $\alpha = 1, \beta = -2$, then x and y are roots of

$$t^2 + t - 2 = 0$$

$\Rightarrow$ $$(t + 2)(t - 1) = 0$$

$\Rightarrow$ $$t = -2 \text{ and } 1$$

i.e., $$x = -2 \text{ and } y = 1 \text{ or } x = 1 \text{ and } y = -2$$

If we take $\alpha = -\frac{7}{2}$ and $\beta = \frac{19}{4}$, then x, y are roots of $4t^2 + 14t + 19 = 0$

Here, discriminant $14^2 - 4 \times 4 \times 19 < 0$. There are no real roots.

Real values of x, y satisfying the given equation are $(2, -1)$ or $(-1, 2)$.

Example 75. *If x_1 and x_2 are non-zero roots of the equation $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$ respectively. Prove that $\frac{a}{2}x^2 + bx + c = 0$ has a root between x_1 and x_2.*

Solution If x_1 and x_2 are roots of

$$ax^2 + bx + c = 0 \quad \text{...(i)}$$

$$-ax^2 + bx + c = 0 \quad \text{...(ii)}$$

We have,

$$ax_1^2 + bx_1 + c = 0 - ax_2^2 + bx_2 + c = 0$$

Let $$f(x) = \frac{a}{2}x^2 + bx + c$$

Thus, $$f(x_1) = \frac{a}{2}x_1^2 + bx_1 + c \quad \text{...(iii)}$$

$$f(x_2) = \frac{a}{2}x_2^2 + bx_2 + c \quad \text{...(iv)}$$

Adding $\frac{1}{2}ax_1^2$ in Eq. (iii), we get

$$f(x_1) + \frac{1}{2}ax_1^2 = ax_1^2 + bx_1 + c = 0$$

$\Rightarrow$ $$f(x_1) = -\frac{1}{2}ax_1^2 \quad \text{...(v)}$$

Subtracting $\frac{3}{2}ax_2^2$ from Eq. (iv), we get

$$f(x_2) - \frac{3}{2}ax_2^2 = -ax_2^2 + bx_2 + c = 0$$

$$\Rightarrow \qquad f(x_2) = \frac{3}{2}ax_2^2$$

Thus, $f(x_1)$ and $f(x_2)$ have opposite signs.

Hence, $f(x)$ must have a root between x_1 and x_2.

Example 76. *Find all +ve integers x, y, z satisfying $x^{y^z} \cdot y^{z^x} \cdot z^{x^y} = 5xyz$.*

Solution x, y, z are integers and 5 is a prime number and given equation is $x^{y^z} \cdot y^{z^x} \cdot z^{x^y} = 5xyz$

Dividing both sides of the equation by xyz.

$$x^{y^z - 1} \cdot y^{z^x - 1} \cdot z^{x^y - 1}$$

So, different possibilities are

$$\left.\begin{aligned} x^{y^z-1} &= 5 \\ y^{z^x-1} &= 1 \\ z^{x^y-1} &= 1 \end{aligned}\right| \quad \text{or} \quad \left.\begin{aligned} x^{y^z-1} &= 1 \\ y^{z^x-1} &= 5 \\ z^{x^y-1} &= 1 \end{aligned}\right| \quad \text{or} \quad \begin{aligned} x^{y^z-1} &= 1 \\ y^{z^x-1} &= 1 \\ z^{x^y-1} &= 1 \end{aligned}$$

Taking Ist column

$$x = 5, y^z - 1 = 1; y^z = 2, y = 2 \text{ and } z = 1$$

and these values are satisfying the other expressions in first column.

Similarly, from IInd column, we get

$$y = 5, z = 2 \text{ and } x = 1$$

From IIIrd column, we get $z = 5, x = 2$ and $y = 1$

Example 77. *Show that*

$$f(x) = x^{1000} - x^{500} + x^{100} + x + 1 = 0 \textit{ has no rational roots.}$$

Solution If there a rational root, let it be $\frac{p}{q}$, where $(p, q) = 1, q \neq 0$. Then, q should divide the coefficient of the leading term and p should divide the constant term.

Thus, $\qquad q \mid 1 \Rightarrow q = \pm 1$

and $\qquad p \mid 1 = p = \pm 1$

Thus, $\qquad \frac{p}{q} = \pm 1$

If the root $\frac{p}{q} = 1$, then

$$f(1) = 1 - 1 + 1 + 1 + 1 = 3 \neq 0$$

So, 1 is not a root.

If $\frac{p}{q} = -1$, then $f(-1) = 1 \neq 0$

Hence, -1 is not a root.

Thus, there exists no rational roots for given polynomial.

Example 78. *Find all integers x for which*

$$x^4 + x^3 + x^2 + x + 1 \text{ is a perfect square.}$$

Solution If $x^4 + x^3 + x^2 + x + 1$ is a perfect square, then

Let $$y^2 = x^4 + x^3 + x^2 + x + 1$$

Consider $$\left(x^2 + \frac{x}{2}\right)^2 = x^4 + x^3 + \frac{x^2}{4}$$

$$= x^4 + x^3 + x^2 + x + 1 - \left(\frac{3}{4}x^2 + x + 1\right)$$

$$= y^2 - \frac{1}{4}(3x^2 + 4x + 4)$$

As discriminant of $(3x^2 + 4x + 4)$ is negative.

Therefore, $3x^2 + 4x + 4$ is always greater than zero.

Thus, $$\left(x^2 + \frac{x}{2}\right)^2 < y^2$$

or $$\left|x^2 + \frac{x}{2}\right| < |y|$$

But $x^2 + \frac{x}{2} = \left(x + \frac{1}{2}\right)x$ is non-negative, $\forall x \in Z$

$\therefore$ $$\left|x^2 + \frac{x}{2}\right| = x^2 + \frac{x}{2} < |y|$$

If x is even, then $|y| \geq x^2 + \frac{x}{2} + 1$

$$y^2 \geq x^4 + x^3 + x^2 + x + 1 + \frac{5}{4}x^2$$

$\Rightarrow$ $$y^2 \geq y^2 + \frac{5}{4}x^2$$

Which is not possible,

If $x \neq 0$, then $x = 0$ is the only solution when x is even.

If x is odd, then $x^2 + \frac{x}{2} + \frac{1}{2}$ is an integer.

So, $$|y| \geq \left(x^2 + \frac{x}{2}\right) + \frac{1}{2}$$

In this case $$y^2 \geq x^4 + x^3 + x^2 + x + 1 + \left(\frac{x^2}{4} - \frac{x}{2} - \frac{3}{4}\right)$$

i.e., $$y^2 \geq y^2 + \left(\frac{x^2}{4} - \frac{x}{2} - \frac{3}{4}\right)$$

$$= y^2 + \frac{1}{4}(x^2 - 2x - 3)$$

and hence $\frac{1}{4}(x^2 - 2x - 3) \leq 0$

$\Rightarrow$ $$x^2 - 2x - 3 \leq 0$$

$\Rightarrow$ $$(x - 3)(x + 1) \leq 0$$

$\therefore$ $$-1 \leq x \leq 3$$

Odd integral values of x are -1, 1 and 3 of which 1 does not give a perfect square.

There are exactly 3 integral values of x namely. 0, -1 and 3 for which the expression is a perfect square.

Example 79. *Let $p(x)$ be a real polynomial function*

$$p(x) = ax^3 + bx^2 + cx + d.$$

Prove, if $|p(x)| \le 1$ for all x such that $|x| \le 1$

then $$|a| + |b| + |c| + |d| \le 7.$$

Solution Considering the polynomials $\pm\, p(\pm\, x)$. We may assume without loss of generality that $a, b \ge 0$.

Case I If $c, d \ge 0$, then

$$p(1) = a + b + c + d \le 1 < 7$$

Case II If $d \le 0$ and $c \ge 0$, then

$$|a| + |b| + |c| + |d| = a + b + c + d = (a + b + c + d) - 2d$$
$$= p(1) - 2p(0) \le 1 + 2 = 3 < 7$$

Case III If $d \ge 0, c > 0$, then

$$|a| + |b| + |c| + |d| = a + b - c + d$$
$$= \frac{4}{3}p(1) - \frac{1}{3}p(-1) - \frac{8}{3}p\left(\frac{1}{2}\right) + \frac{8}{3}p\left(\frac{-1}{2}\right) \le \frac{4}{3} + \frac{1}{3} + \frac{8}{3} + \frac{8}{3} = \frac{21}{3} = 7$$

Case IV $d < 0, c < 0$, then

$$|a| + |b| + |c| + |d| = a + b - c - d$$
$$= \frac{5}{3}p(1) - 4p\left(\frac{1}{2}\right) + \frac{4}{3}p\left(-\frac{1}{2}\right) \le \frac{5}{3} + 4 + \frac{4}{3} = \frac{21}{3} = 7$$

Example 80. *If $x^5 - x^3 + x = a$. Prove that $x^6 \ge 2a - 1$.*

Solution $x^6 + 1 = (x^2 + 1)(x^4 - x^2 + 1) \ge 2x(x^4 - x^2 + 1)$ $[\because x^2 + 1 \ge 2x$ and $x^4 - x^2 + 1 = (x^2 - 1)^2 + x^2 > 0]$

$$x^6 + 1 \ge 2a$$

Hence, $$x^6 \ge 2a - 1$$

Example 81. *Find the real points (x, y) satisfying*

$$3x^2 + 3y^2 - 4xy + 10x - 10y + 10 = 0.$$

Solution It can be considered as a quadratic in x

$$3x^2 + (10 - 4y)x + (3y^2 - 10y + 10) = 0$$

Solving for x we get

$$x = \frac{1}{6}\{(4y - 10)\} \pm \sqrt{(10 - 4y)^2 - 12(3y^2 - 10y + 10)}$$

Since x, y are real

$$(10 - 4y)^2 - 12(3y^2 - 10y + 10) \ge 0$$

Which on simplification gives $(y - 1)^2 \le 0$

i.e., $y = 1$ and so $x = -1$

Example 82. *Solve*

$$(12x-1)(6x-1)(4x-1)(3x-1)=5.$$

Solution We can write the equation in the form

$$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right)=\frac{5}{12\cdot 6\cdot 4\cdot 3} \quad \ldots(\text{i})$$

$$\because \quad \frac{1}{12}<\frac{1}{6}<\frac{1}{4}<\frac{1}{3} \text{ and } \frac{1}{6}-\frac{1}{12}=\frac{1}{3}-\frac{1}{4}$$

We can introduce a new variable

$$y=\frac{1}{4}\left[\left(x-\frac{1}{12}\right)+\left(x-\frac{1}{6}\right)+\left(x-\frac{1}{4}\right)+\left(x-\frac{1}{3}\right)\right]$$

$$=x-\frac{5}{24}$$

Substitute $x=y+\frac{5}{24}$ in Eq. (i), we get

$$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right)=\frac{5}{12\cdot 6\cdot 4\cdot 3}$$

$$\left(y^2-\left(\frac{1}{24}\right)^2\right)\left(y^2-\left(\frac{3}{24}\right)^2\right)=\frac{5}{12\cdot 6\cdot 4\cdot 3}$$

So, $$y^2=\frac{49}{24^2}$$

i.e., $$y_1=\frac{7}{24}$$

and $$y_2=-\frac{7}{24}$$

Corresponding roots are $-\frac{1}{12}$ and $\frac{1}{2}$.

Example 83. *Find the value of a, b, c which will make each of the expressions* $x^4+ax^3+bx^2+cx+1$ *and* $x^4+2ax^3+2bx^2+2cx+1$ *a perfect square.*

Solution Suppose

$$x^4+ax^3+bx^2+cx+1\equiv\left(x^2+\frac{ax}{2}+1\right)^2 \quad \ldots(\text{i})$$

and $$x^4+2ax^3+2bx^2+2cx+1\equiv(x^2+ax+1)^2 \quad \ldots(\text{ii})$$

Equating like coefficients from Eqs. (i) and (ii), we get

$$b=\frac{a^2}{4}+2 \text{ and } c=a$$

$$2b=a^2+2, 2c=2a$$

$$\therefore \quad a^2+2=2\left(\frac{a^2}{4}+2\right)=\frac{a^2}{2}+4$$

$$2a^2+4=a^2+8 \text{ or } a^2=4$$

$$\therefore \quad a=\pm 2, c=\pm 2, b=3$$

Example 84. *Show that the expression*

$$(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - zx)(z^2 - xy)$$

is a perfect square. Find its square root.

Solution $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - zx)(z^2 - xy)$

Putting $x^2 - yz = a, y^2 - zx = b$ and $z^2 - xy = c$,

we get

$$a^3 + b^3 + c^3 - 3abc$$
$$= (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$$
$$= (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$$

Now,

$$a + b + c = x^2 - yz + z^2 - zx + z^2 - xy$$
$$= (x + \omega y^2 + \omega^2 z)(x + \omega^2 y + \omega x) \quad \ldots\text{(i)}$$
$$(a + \omega b + \omega^2 c) = x^2 - yz + \omega(y^2 - zx) + \omega^2(z^2 - xy)$$
$$= x^2 + \omega y^2 + \omega^2 z^2 - yz - \omega zx - \omega^2 xy$$
$$= (x + y + z)(x + \omega y + \omega^2 z) \quad \ldots\text{(ii)}$$
$$(a + \omega^2 b + \omega c) = y^2 - zx + \omega^2(z^2 - xy) + \omega(x^2 - yz)$$
$$= y^2 + \omega^2 z^2 + \omega x^2 - zx - \omega^2 xy - \omega yz$$
$$= (x + y + z)(x + \omega^2 y + \omega z) \quad \ldots\text{(iii)}$$

From Eqs. (i), (ii) and (iii) it is clear that given expression is a perfect square.

Required square root

$$= (x + y + z)(x + \omega y - \omega^2 z)(x + \omega^2 y + \omega z)$$
$$= x^3 + y^3 + z^3 - 3xyz$$

Let us Practice

Let us Practice

Level 1

1. Prove that the equation $ax^5 + bx^3 + cx^2 + d = 0$ will have three equal roots, if $-\frac{c}{b} = \frac{b^2}{5ac} = \frac{5bd}{c^2}$ each of the quantities being equal to the repeated roots.

2. Find k so that the equation $2x^4 - 3x^2 - 2x + k = 0$ may have a double root and solve the equation.

3. Show that the equation $x^4 + x^2 + 6 = 0$ cannot have three equal roots.

4. Given that the equation $x^5 - x^4 + 2x^3 + 4 = 0$ has one root of the form $1 + \alpha i$, find all the roots.

5. For what values of a and b is the polynomial $f(x) = x^2 + (a^2 + b)x + b - 9$ negative of the polynomial $g(x) = -x^2 - 5ax + b - 3$.

6. If $f(x) = x^3 + 2x^2 + 5$ and $g(x) = x^4 - 3x^2 + 2x + 1$, calculate their gcd. Find polynomials $a(x)$, $b(x)$ such that $(f(x), g(x)) = a(x)f(x) + b(x)g(x)$.

7. Give an example of two (non-constant) polynomials $f(x), g(x)$ such that $(f(x), g(x)) = 1$.

8. Let $f(x), g(x)$ and $h(x)$ be three polynomials such that $f(x)/h(x), g(x)/h(x)$ and $(f(x), g(x)) = 1$. Prove that $f(x)g(x)/h(x)$.

9. If -2 is a root of $x^4 - (2a + 3)x^2 - 2(a - 1)x + 12 = 0$. Prove that it is a repeated root. Find its multiplicity. Solve the equation.

10. Verify that $\frac{1}{2}$ is a root of $4x^3 + 20x^2 - 23x + 6 = 0$. Find its multiplicity. Solve the equation.

11. Find number a, b, c such that $(x^2 + x + 5)(ax + b) + c = x^3 + 7x^2 + 3x + 5$.

12. If m, n are integers ≥ 0. $f(x), g(x)$ are polynomial such that $(x - \alpha)^m f(x) = (x - \alpha)^n g(x)$ with $f(\alpha) \neq 0$, $g(\alpha) \neq 0$. Prove that $m = n$ and $f(x) = g(x)$.

13. Solve $x^4 + 2x^3 - 2x - 1 = 0$, given it has repeated roots.

14. Solve the equation
$$6x^4 - 13x^3 - 35x^2 - x + 3 = 0$$
which has $2 - \sqrt{3}$ as a root.

15. Solve the equation $x^5 - 5x^4 - 5x^3 + 25x^2 + 4x - 20 = 0$, whose roots are given to be of the form $\pm\alpha, \pm\beta, \gamma$.

16. Solve $x^3 + x^2 - 16x + 20 = 0$, the difference between two of its roots being 7.

17. Solve $x^3 - 13x^2 + 15x + 189 = 0$, having given that one root exceeds the other by 2.

18. Solve $x^4 - 8x^3 + 7x^2 + 36x - 36 = 0$ given that product of two of the roots is negative of the product of the remaining two.

19. Solve the equation $18x^3 + 81x^2 + 121x + 60 = 0$ given that, one of its roots is equals to half the sum of the other two.

20. The sum of two roots of
$$x^4 - 8x^3 + 19x^2 + 4\lambda x + 2 = 0,$$
is equal to the sum of the other two roots, find λ. Solve the equation.

21. Find the condition that the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, be connected by the relation $\alpha\beta + \gamma\delta = 0$.

22. Given that, two of the roots of $45x^4 - 54x^3 - 98x^2 + 150x - 75 = 0$ are equal in absolute value but opposite in sign. Solve the equation completely.

23. Solve $2x^3 - x^2 - 22x - 24 = 0$ two of the roots being in the ratio 3 : 4.

24. The roots of $2x^3 - 15x^2 + 37x - 30 = 0$ are in AP. Find them.

25. Solve the equation $x^3 - 13x^2 + 15x + 189 = 0$, having given that one root exceeds the other by 2.

26. Solve the equation $x^3 - 7x^2 + 36 = 0$ given that
(i) one root is double of another
(ii) difference of two roots is 5.

27. Solve the equation $4x^3 + 20x^2 - 23x + 6 = 0$ two of its roots being equal.

28. Solve the equation
$$x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$$

29. Form equations, whose roots are the roots of the following equations with their sign changed.
(i) $x^3 - 5x^2 - 7x + 3 = 0$
(ii) $-4x^3 + 2x^2 - 3x - 5 = 0$

30. Find the equation whose roots are the roots of $x^5 - 4x^4 + 3x^2 - 4x + 6 = 0$ each diminished by 3.

31. Diminish the roots of equation $5x^3 - 13x^2 - 12x + 7 = 0$ by 2.

32. Find the equation whose roots are the squares of the roots of the equation
$$x^5 + x^3 + x^2 + 2x + 3 = 0$$

33. Form the equation whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.

34. Find the equation whose roots are the cubes of the roots of $2x^3 - x^2 + 2x - 3 = 0$. Obtain value of $\Sigma\alpha^3\beta^3$.

35. Show that the roots of the equation
$$x^3 - \frac{1}{3}q^2x - \frac{2}{27}q^3 - r^2 = 0$$
differ by a constant from the squares of roots of
$$x^3 + qx + r = 0.$$

36. Show that the cubes of the roots of $x^3 + ax^2 + bx + ab = 0$ are given by
$$x^3 + a^3x^2 + b^3x + a^3b^3 = 0.$$

37. If α, β, γ are the roots of $z^3 + 3Hz + G = 0$, find the equation whose roots are
$$\frac{\alpha + 1}{\beta + \gamma - \alpha}, \frac{\beta + 1}{\gamma + \alpha - \beta}, \frac{\gamma + 1}{\alpha + \beta - \gamma}$$

38. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ for the equation whose roots are
(a) $\alpha - \frac{1}{\beta\gamma}, \beta - \frac{1}{\gamma\alpha}, \gamma - \frac{1}{\alpha\beta}$
(b) $\alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta)$
(c) $\beta\gamma + \frac{1}{\alpha}, \gamma\alpha + \frac{1}{\beta}, \alpha\beta + \frac{1}{\gamma}$

39. If α, β, γ are the roots of the equation $x^3 + x^2 + 2x + 3 = 0$, from the equation whose roots are $\beta + \gamma - \alpha, \gamma + \alpha - \beta, \alpha + \beta + \gamma$.

40. If α, β, γ be the roots of $x^3 - ax^2 + bx - c = 0$, find an equation whose roots are $2\alpha - \beta - \gamma$, $2\beta - \gamma - \alpha, 2\gamma - \alpha - \beta$.

Hence, evaluate $(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta)$.

41. Solve the system
$$x_1 + x_2 + x_3 + x_4 = 2a_1$$
$$x_1 + x_2 - x_3 - x_4 = 2a_2$$
$$x_1 - x_2 + x_3 - x_4 = 2a_3$$
$$x_1 - x_2 - x_3 + x_4 = 2a_4.$$

42. Solve the system
$$ax + m(y + z + v) = k$$
$$by + m(x + z + v) = l$$
$$cz + m(x + y + v) = p$$
$$dv + m(x + y + z) = q.$$

43. Solve the system
$$(b + c)(y + z) - ax = b - c$$
$$(c + a)(x + z) - by = c - a$$
$$(a + b)(x + y) - cz = a - b$$
if $\quad a + b + c \neq 0.$

44. Solve the system
$$z + ay + a^2x + a^3t + a^4 = 0$$
$$z + by + b^2x + b^3t + b^4 = 0$$
$$z + cy + c^2x + c^3t + c^4 = 0$$
$$z + dy + d^2x + d^3t + d^4 = 0.$$

45. Solve the system
$$yz = ax$$
$$zx = by \quad (a > 0, b > 0, c > 0)$$
$$xy = cz.$$

46. Solve

$$x(y+z)=a^2$$
$$y(x+z)=b^2$$
$$z(x+y)=c^2.$$

47. Solve the system

$$x^2+y^2=cxyz$$
$$x^2+z^2=bxyz$$
$$y^2+z^2=axyz.$$

48. Solve the system

$$\frac{b(x+y)}{x+y+cxy}+\frac{c(z+x)}{x+z+bxz}=a$$
$$\frac{c(y+z)}{y+z+ayz}+\frac{a(x+y)}{x+y+cxy}=b$$
$$\frac{a(x+z)}{x+z+bxz}+\frac{b(y+z)}{y+z+ayz}=c.$$

49. Solve the system

$$x^2-yz=a$$
$$y^2-xz=b$$
$$z^2-xy=c.$$

50. Solve

$$x^2+y^2+z^2=84$$
$$x+y+z=14$$
$$xz=y^2.$$

51. Solve

$$x+y+z=13$$
$$x^2+y^2+z^2=65$$
$$xy=10.$$

52. Solve the equation for all possible values of x, y and z.

53. Solve $x+y+z=a\,;x^2+y^2+z^2=b^2$, a and b are real numbers and $xy=z^2$.

What conditions a and b satisfy for x, y, z to be all +ve and distinct?

54. Solve

$$y^2+yz+z^2=ax$$
$$z^2+zx+x^2=ay$$
$$x^2+xy+y^2=az.$$

55. Solve

$$x(y+z-x)=a$$
$$y(z+x-y)=b$$
$$z(x+y-z)=c.$$

56. Solve the equations

(a) $ax+by+z=zx+ay+b=yz+bz+a=0$

(b) $x+y+z-u=12\,;x^2+y^2-z^2-u^2=6;$
$x^3+y^3-z^3+u^3=218\,;xy+zu=6.$

57. Eliminate p and q from the equations
$$x(p+q)=y\,;p-q=k(1+pq);xpq=a.$$

58. Eliminate x, y from the equations $4(x^2+y^2)=ax+by;2(x^2-y^2)=ax-by$ and $xy=c^2$.

59. Eliminate x, y, z from the equations
$$(y+z)^2=4a^2yz\,;(z+x)^2=4b^2zx;$$
$$(x+y)^2=4c^2xy.$$

60. Eliminate x, y from the equations
$$x^2y=a\,;x(x+y)=b\,;2x+y=c.$$

61. Eliminate x, y from the equations
$$ax^2+by^2=ax+by=\frac{xy}{x+y}=c.$$

62. Show that $b^3c^3+c^3a^3+a^3b^3=5a^2b^2c^2$ is the eliminant of $ax+yz=bc\,;$ $by+zx=ca;$ $cz+xy=ab$ and $xyz=abc$.

63. Solve the equation $y^3=x^3+8x^2-6x+8$, for positive integers x and y. **(RMO 2000)**

64. Find all real values of a for which the equation $x^4-2ax^2+x+a^2-a=0$ has all its roots real. **(RMO 2000)**

65. Solve the following equation for real x $(x^2+x-2)^3+(2x^2-x-1)^3=27(x^2-1)^3$. **(RMO 2002)**

66. Find all real numbers a for which the equation $x^2+(a-2)x+1=3|x|$ has exactly three distinct real solutions in x. **(RMO 2003)**

67. Let α and β be the roots of the quadratic equation $x^2+mx-1=0$, where m is an odd integer. Let $\lambda_n=\alpha^n+\beta^m$, for $n\geq 0$. Prove that for $n\geq 0$.

(a) λ_n is an integer.

(b) GCD $(\lambda_n, \lambda_{n+1})=1$. **(RMO 2004)**

68. Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2 + ax + b = 0, \alpha^2 - 2$ is also a root of the equation. **(RMO 2007)**

69. Suppose a and b are real numbers such that the roots of the cubic equation $ax^3 - x^2 + bx - 1 = 0$ are all positive real numbers. Prove that
(a) $0 < 3ab \le 1$ (b) $b \ge \sqrt{3}$. **(RMO 2008)**

70. Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, $P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$. **(RMO 2010)**

Level 2

1. Eliminate x, y, z from
$$x^2 + y^2 + z^2 = x + y + z = 1$$
$$\frac{a}{x}(x - p) = \frac{b}{y}(y - q) = \frac{c}{z}(z - r).$$

2. Eliminate x and y from the equations
$$x + y = a;\ x^2 + y^2 = b;\ x^3 + y^3 = c.$$

3. If the roots of the equation
$$2x^3 + x^2 + x + 1 = 0$$
be α, β, γ. Find the equation whose roots are
$$\frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{1}{\alpha^2}; \frac{1}{\alpha^2} + \frac{1}{\gamma^2} - \frac{1}{\beta^2}; \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{1}{\gamma^2}.$$

4. Find the equation whose roots are the ratios of the roots α, β, γ of the cubic $x^3 + qx + r = 0$.

5. If the roots of the equation
$$x^3 - 6x^2 + 11x - 6 = 0$$
be α, β, γ. Find the equation whose roots are
$$\beta^2 + \gamma^2, \gamma^2 + \alpha^2, \alpha^2 + \beta^2.$$

6. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $\beta\gamma - \alpha^2, \gamma\alpha - \beta^2, \alpha\beta - \gamma^2$.

7. The roots of the equation
$$x^3 - ax^2 + bx - c = 0$$
are α, β, γ form the equation whose roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$. Also, express $\frac{1}{\alpha + \beta} + \frac{1}{\beta + \gamma} + \frac{1}{\gamma + \alpha}$ in terms of a, b, c.

8. If α, β, γ are the roots of the cubic $x^3 + 3x + 2 = 0$, form an equation whose roots are $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$ and hence show that $x^3 + 3x + 2 = 0$ has imaginary roots.

9. If the roots of the equation
$$x^3 + px^2 + qx + r = 0$$
be α, β, γ. Find the equation whose roots are
(a) $\beta^2 + \gamma^2 - \alpha^2, \gamma^2 + \alpha^2 - \beta^2, \alpha^2 + \beta^2 - \gamma^2$
(b) $\beta^2 + \gamma^2, \gamma^2 + \alpha^2, \alpha^2 + \beta^2$.

10. Find the sum of the fifth powers of the roots of the following equation.
(a) $x^4 - 3x^3 + 5x^2 - 12x + 4 = 0$
(b) $x^3 + x^2 + x + 5 = 0$

11. Find the value of $\alpha^5 + \beta^5 + \gamma^5$, where α, β, γ are the roots of $x^3 + 3x + 3 = 0$.

12. If α, β, γ are the roots of the cubic $x^3 + 2x + 6 = 0$, prove that $s_7 = 2(s_4 - s_6)$.

13. Show that the set of polynomials
$P = \{pk \mid pk(x) = x^{5k+4} + x^3 + x^2 + x + 1, k \in N\}$
has a common non-trivial polynomial divisor.

14. Show that the polynomial $x(x^{n-1} - na^{n-1} + a^n(n - 1)$ is divisible by $(x - a)^2$.

15. If $x^3 + 3px + q$ has a factor of the form $(x - a)^2$. Show that $q^2 + 4p^3 = 0$.

16. Prove that
(a) The polynomial $x(x^{n-1} - na^{n-1}) + a^n(n - 1)$ is divisible by $(x - a)^2$.
(b) The polynomial
$(1 - x^n)(1 + x) - 2nx^n(1 - x) - n^2x^n(1 - x)^2$
is divisible by $(1 - x)^3$.

17. Prove that the polynomial
$$x^n \sin\psi - e^{n-1}x\sin n\psi + e^n \sin(n - 1)\psi$$
is divisible by $x^2 - 2ex\cos\psi + e^2$.

18. Prove that if $x_1, x_2, \ldots, x_m$ are m different arbitrary quantities. $f(x)$ is a polynomial of degree less than m, then there exists the identity.

$$f(x)=f(x_1)\frac{(x-x_2)(x-x_3)\ldots(x-x_m)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)}+\ldots+$$
$$f(x_2)\frac{(x-x_1)(x-x_3)\ldots(x-x_m)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_m)}+\ldots+$$
$$f(x_m)\frac{(x-x_1)(x-x_2)\ldots(x-x_{m-1})}{(x_m-x_1)(x_m-x_2)\ldots(x_m-x_{m-1})}.$$

19. If x and y are real, solve the inequality.
$$\log_2 x + \log_x 2 + 2\cos y \le 0.$$

20. Show that $(a-b)^2 + (a-c)^2 = (b-c)^2$ is not solvable, when a, b, c are all distinct.

21. Find all pairs of integers x, y such that
$$(xy-1)^2 = (x+1)^2 + (y+1)^2.$$

22. Find all the solutions of the system of equations
$$y = 4x^3 - 3x;\ z = 4y^3 - 3y \text{ and}$$
$$x = 4z^3 - 3z.$$

23. The roots x_1, x_2, x_3 of the equation $x^3 + ax + a = 0$, where a is a non-zero real, satisfy
$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \frac{x_3^2}{x_1} = -8, \text{ find } x_1, x_2, x_3.$$

24. Find all values of a for which the system of equation $2^{|x|} + |x| = y + x^2 + a$, $x^2 + y^2 = 1$ has only one solution (a, x, y are real numbers).

25. If $p(x) = ax^2 + bx + c$ and $q(x) = -ax^2 + bx + c, ac \ne 0$. Show that $p(x)\cdot q(x) = 0$ has at least two real roots.

26. Find the real roots of the equation
$$x^2 + 2ax + \frac{1}{16} = -a + \sqrt{\left(a^2 + x - \frac{1}{16}\right)}.$$

27. If $a, b, c \in R$, $a \ne 0$. Solve the system of equations.
$$ax_1^2 + bx_1 + c = x_2$$
$$ax_2^2 + bx_2 + c = x_3$$
$$\ldots\ \ldots\ \ldots$$
$$\ldots\ \ldots\ \ldots$$
$$\ldots\ \ldots\ \ldots$$
$$ax_{n-1}^2 + bx_{n-1} + c = x_n$$
$$ax_n^2 + bx_n + c = x_1$$
is n unknowns $x_1, x_2, \ldots, x_n$, then

(a) $(b-1)^2 < 4ac$ (b) $(b-1)^2 = 4ac$

(c) $(b-1)^2 > 4ac$

28. Let $p(x) = 0$ be a fifth degree polynomial equation with integer coefficients that has at least one integral root. If $p(2) = 13$ and $p(10) = 5$. Compute a value of x that must satisfy $p(x) = 0$.

29. If a, b, c, x are real numbers such that $abc \ne 0$ and
$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c},$$
then prove that either $a + b + c = 0$ or $a = b = c$.

(INMO 2000)

30. Solve for integers x, y, z
$$x + y = 1 - z, x^3 + y^3 = 1 - z^2.$$ (INMO 2000)

31. Show that the equation
$$x^2 + y^2 + z^2 = (x-y)(y-z)(z-x)$$
has infinitely many solutions in integers x, y, z. (INMO 2001)

32. Show that for every real number a the equation
$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0$$
has atleast one non-real root and find the sum of all the non-real roots of the equation.

(INMO 2003)

33. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$. (For any real number a, we denote by $[a]$ the greatest integer not exceeding a.) (INMO 2004)

34. Suppose p is a prime greater than 3. Find all the pairs of integers (a, b) satisfying the equation $a^2 + 3ab + 2p(a+b) + p^2 = 0$.

(INMO 2004)

35. Let p, q, r be positive real numbers, not all equal, such that some two of the equations
$$px^2 + 2qx + r = 0$$
$$qx^2 + 2rx + p = 0$$
$$rx^2 + 2px + q = 0$$
have a common root, say α. Prove that

(a) α is real and negative,

(b) the third equation has non-real roots.

(INMO 2005)

36. Let m and n be positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α and β are integers if and only if $[m\alpha] + [m\beta]$ is the square of an integer. (Here, $[x]$ denotes the largest integer not exceeding x.) **(INMO 2007)**

37. Find all triples (p, x, y) such that $p^x = y^4 + 4$, where p is a prime and x, y are natural numbers. **(INMO 2008)**

38. Let $P(x)$ be a given polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that (a) $P(x)Q(x)$ is a polynomial in x^2 (b) $P(x)R(x)$ is a polynomial in x^3. **(INMO 2008)**

39. Find all the real numbers x such that

$$[x^2 + 2x] = [x]^2 + 2[x].$$

(Here, $[x]$ denotes the largest integer not exceeding x) **(INMO 2009)**

40. Find all non-zero real numbers x, y, z which satisfy the system of equations

$$(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) = xyz,$$

$$(x^4 + x^2y^2 + y^4)(y^4 + y^2z^2 + z^4)(z^4 + z^2x^2 + x^4) = x^3y^3z^3.$$ **(INMO 2010)**

41. Consider two polynomials $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ and $Q(x) = b_nx^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0$ with integer coefficients such that $a_n - b_n$ is a prime, $a_{n-1} = b_{n-1}$ and $a_nb_0 - a_0b_n \neq 0$. Suppose there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that r is an integer. **(INMO 2011)**

Solutions

Solutions

Level 1

1. Let $f(x) = ax^5 + bx^3 + cx^2 + d$

$$f'(x) = 5ax^4 + 3bx^2 + 2cx$$

$$f''(x) = 20ax^3 + 6bx + 2c$$

$f(x) = 0$ will have three equal roots α, α, α (say)

If $f'(x) = 0$ has two equal roots α, α

and $f''(x) = 0$ has a root α.

$$\therefore \quad a\alpha^5 + b\alpha^3 + c\alpha^2 + d = 0 \quad \ldots(i)$$

$$5a\alpha^4 + 3b\alpha^2 + 2c\alpha = 0$$

or $$5a\alpha^3 + 3b\alpha + 2c = 0 \quad (\because \alpha \neq 0) \ldots(ii)$$

$$20a\alpha^3 + 6b\alpha + 2c = 0$$

$$\Rightarrow \quad 10a\alpha^3 + 3b\alpha + c = 0 \quad \ldots(iii)$$

Multiply Eq. (ii) by 2 and subtract from Eq. (iii), we get

$$-3b\alpha - 3c = 0$$

$$\Rightarrow \quad \alpha = -\frac{c}{b}$$

Putting $-\alpha = \frac{c}{b}$ in Eq. (iii), we have

$$-\frac{10ac^3}{b^3} - 3c + c = 0$$

or $$-\frac{10ac^3}{b^3} = 2c \quad \text{or} \quad -\frac{5ac^2}{b^3} = 1$$

or $$-\frac{c}{b} = \frac{b^2}{5ac} \Rightarrow \alpha = \frac{b^2}{5ac}$$

Putting $\alpha = -\frac{c}{b}$ in Eq. (ii), we have

$$-\frac{5ac^3}{b^3} - 3c + 2c = 0$$

or $$-\frac{5ac^3}{b^3} = c \quad \text{or} \quad b^3 = -5ac^2$$

Putting $\alpha = -\frac{c}{b}$ in Eq. (i), we get

$$-\frac{ac^5}{b^5} - \frac{c^3}{b^2} + \frac{c^3}{b^2} + d = 0$$

or $$\frac{ac^5}{b^2 \cdot b^3} = d \quad \text{or} \quad \frac{ac^5}{b^2(-5ac^2)} = d$$

$$[\because b^3 = -5ac^2]$$

or $$-\frac{c^3}{5b^2} = d \quad \text{or} \quad -\frac{c}{b} = \frac{5bd}{c^2}$$

or $$\alpha = \frac{5bd}{c^2}$$

Hence, $$-\frac{c}{b} = \frac{b^2}{5ac} = \frac{5bd}{c^2}, \text{ each} = \alpha$$

2. Let $f(x) = 2x^4 - 3x^2 - 2x + k$

$$f'(x) = 8x^3 - 6x - 2 = 2(4x^3 - 3x - 1)$$

Since, $f(x) = 0$ has a double root $f(x) = 0$ and $f'(x) = 0$ have a common root.

$\therefore$ HCF of $f(x)$ and $f'(x)$ is a linear polynomial.

HCF of $f(x)$ and $f'(x)$ will be the linear polynomial $(x - 1)$ only, when $6 - 2k = 0$ *i.e.*, $k = 3$.

Now, $$\text{HCF } \phi(x) = x - 1$$

$$\phi(x) = 0 \text{ gives } x = 1$$

$\therefore$ Two roots of the equation

$$2x^4 - 3x^2 - 2x + 3 = 0 \quad (\because k = 3)$$

are 1, 1.

4. If we put $x = -1$ in the given equation, it is satisfied and as such $x + 1$ is a factor and the equation when divided by $x + 1$ by synthetic division gives the quotient as under

-1	1	-1	2	0	0	4
		-1	2	-4	4	-4
	1	-2	4	-4	4	0

Quotient is $x^4 - 2x^3 + 4x^2 - 4x + 4 = 0 \quad \ldots(i)$

It has a root $1 + i\alpha$ and hence, $1 - i\alpha$ must also be its root and product of factors corresponding to these is

$$(x - 1)^2 + \alpha^2 \text{ or } (x^2 - 2x + 1 + \alpha^2)$$

Let the remaining factor be $(x^2 + px + q)$

Hence, we have

$$(x^4 - 2x^3 + 4x^2 - 4x + 4)$$
$$= (x^2 - 2x + 1 + \alpha^2)(x^2 + px + q)$$

Comparing the coefficients of like powers of x in both sides, we get

$$-2 = -2 + p, \therefore p = 0,$$

$$(1 + \alpha^2) + q - 2p = 4; p(1 + \alpha^2) - 2q = -4;$$

$$q(1 + \alpha^2) = 4;$$

Putting $p = 0$, we get $q = 2$

$\therefore \quad 1 + \alpha^2 = 2$ or $\alpha = \pm 1$

Hence, the factors are

$$(x + 1)(x^2 - 2x + 2)(x^2 + 2) = 0$$

$\therefore$ Roots are $-1, 1 \pm i, \pm i\sqrt{2}$.

5. $f(x) = -g(x)$

$\therefore x^2 + (a^2 + b)x + b - 9 = -(-x^2 - 5ax + b - 3)$

$\Rightarrow \quad x^2 + (a^2 + b)x + b - a = x^2 + 5ax - b + 3$

On comparing coefficient of like powers of x on both sides

$$a^2 + b = 5a \qquad \ldots(\text{i})$$

and $$b - 9 = -b + 3 \qquad \ldots(\text{ii})$$

$$2b = 12 \Rightarrow b = 6$$

Putting in Eq. (i), we get

$$a^2 + 6 = 5a$$

$$a^2 - 5a + 6 = 0$$

$\therefore \quad (a - 2)(a - 3) = 0 \Rightarrow a = 2, 3$

Hence, $a = 2$ or 3 and $b = 6$.

6. We first, find GCD of $f(x), g(x)$

GCD of $f(x), g(x) = 1$

$\therefore f(x), g(x)$ are coprime polynomials.

$$\begin{array}{r|l} & x - 2 \\ x^3 + 2x^2 + 5 & x^4 + 0x^3 - 3x^2 + 2x + 1 \\ & x^4 + 2x^3 + 0x^2 + 5x \\ \hline & -2x^3 - 3x^2 - 3x + 1 \\ & -2x^3 - 4x^2 - 0x - 10 \\ \hline & x^2 - 3x + 11 \end{array}$$

$$\begin{array}{r|l} & x + 5 \\ x^2 - 3x + 11 & x^3 + 2x^2 + 0x + 5 \\ & x^3 - 3x^2 + 11x \\ \hline & 5x^2 - 11x + 5 \\ & 5x^2 - 15x + 55 \\ \hline & 4x - 50 \end{array}$$

$$\begin{array}{r|l} & x + 19/2 \\ 4x - 50 & x^2 - 3x + 11 \\ & \times 4 \\ \hline & 4x^2 - 12x + 44 \\ & 4x^2 - 50x \\ \hline & 38x + 44 \\ & 38x - 475 \\ \hline & 519/519 \end{array}$$

$$\begin{array}{r|l} & 4x - 50 \\ 1 & 4x - 50 \\ & 4x - 50 \\ \hline & \times \end{array}$$

In the first stage of above division, we have

$$g(x) = f(x)(x - 2) + x^2 - 3x + 11$$

$\therefore \quad x^2 - 3x + 11 = g(x) - (x - 2)f(x) \qquad \ldots(\text{i})$

In the second stage of above division

$$f(x) = (x^2 - 3x + 11)(x + 5) + 4x - 50$$

$$= \{g(x) - (x - 2)f(x)\}(x + 5) + 4x - 50$$

$$4x - 50 = f(x) + (x - 2)(x + 5)f(x) - (x + 5)g(x) \qquad \ldots(\text{ii})$$

In the third stage of division

$$4(x^2 - 3x + 11) = (4x - 50)\left(x + \frac{19}{2}\right) + 519$$

$$519 = 4(x^2 - 3x + 11) - (4x - 50)\left(x + \frac{19}{2}\right)$$

$$= 4[g(x) - (x - 2)f(x)] - [(x^2 + 3x - 9)f(x) - (x + 5)g(x)]\left[x + \frac{19}{2}\right]$$

$\therefore \quad 519 = 4g(x) - (4x - 8)f(x) - (x^2 + 3x - 9)\left(x + \frac{19}{2}\right)f(x) + (x + 5)\left(x + \frac{19}{2}\right)g(x)$

$$= -\left[4x - 8 + x^3 + \frac{19x^2}{2} + 3x^2 + \frac{57x}{2} - 9x - \frac{171}{2}\right]f(x) + \left[x^2 + \frac{19x}{2} + 5x + \frac{95}{2} + 4\right]g(x)$$

$$= -\frac{f(x)}{2}[2x^3 + 25x^2 + 47x - 187] + \frac{g(x)}{2}[2x^2 + 29x + 103]$$

$$\therefore \quad 1 = \left[-\frac{2x^3 + 25x^2 + 47x - 187}{1038}\right] f(x) + \frac{[2x^2 + 29x + 103]}{1038} g(x)$$

Hence, $1 = a(x)f(x) + b(x)g(x)$, where

$$a(x) = -\frac{1}{1038}(2x^3 + 25x^2 + 47x - 187)$$

and $b(x) = \dfrac{1}{1038}(2x^2 + 29x + 103)$

7. Let $f(x) = 3x + 1; g(x) = x;$

To find GCD, we proceed as follows

$$x\overline{)3x+1}(3$$
$$\frac{3x}{1}$$
$$1\overline{)x}(x$$
$$\frac{x}{\times}$$

GCD = 1 *i.e.*, $(f(x), g(x)) = 1$

8. Since, $f(x) \mid h(x)$

$\therefore$ $\exists$ s a polynomial $p(x)$ such that

$$h(x) = g(x)p(x) \quad \ldots\text{(i)}$$

Again $\quad g(x) \mid h(x)$

$\therefore$ $\exists$s a polynomial $q(x)$ such that

$$h(x) = q(x) \cdot g(x) \quad \ldots\text{(ii)}$$

Further $\because \quad (f(x), g(x)) = 1$

$\therefore$ $\exists$ polynomial $a(x)$ and $b(x)$

Such that $f(x)a(x) + g(x)b(x) = 1 \quad \ldots$(iii)

Multiplying by $h(x)$ on both sides, we get

$$f(x)h(x)a(x) + g(x)h(x)b(x) = h(x)$$

Using Eqs. (ii) and (i), we get

$$\Rightarrow \quad f(x)[g(x)q(x)]a(x) + g(x)[f(x)p(x)] b(x) = h(x)$$

$$\Rightarrow \quad f(x)g(x)[q(x)a(x) + p(x)b(x)] = h(x)$$

$$\Rightarrow \quad f(x)g(x) \mid h(x)$$

Hence, the result.

9. Let $f(x) = x^4 - (2a + 3)x^2 - 2(a - 1)x + 12 \quad \ldots$(i)

$$f(-2) = (-2)^4 - (2a + 3)(-2)^2 - 2(a - 1)(-2) + 12$$
$$= 16 - 8a - 12 + 4a - 4 + 12 = 12 - 4a$$

Now, it is given that -2 is a root of $f(x) = 0$

$\therefore \quad f(-2) = 0 \Rightarrow 12 - 4a = 0 \Rightarrow a = 3$

Putting $a = 3$ in Eq. (i), we get

$$f(x) = x^4 - 9x^2 - 4x + 12 = 0$$

By Horner's method

-2	1	0	-9	-4	12
		-2	4	10	-12
-2	1	-2	-5	6	0
		-2	8	-6	(Remainder)
-2	1	-4	3	0	
		-2	12		
-2	1	-6	15		
		-2			
	1	-8			
	1				

$\therefore$ Transformed equation is

$$y^4 - 8y^3 + 15y^2 = 0$$

where $\quad y = x - h = x - (-2)$

$\Rightarrow \quad y^2(y^2 - 8y + 15) = 0$

$\Rightarrow \quad y^2 = 0, (y - 3)(y - 5) = 0$

$$y = 0, 0, 3, 5$$
$$y = x + 2 \Rightarrow x = y - 2$$
$$x = 0 - 2, 0 - 2, 3 - 2, 5 - 2$$
$$= -2, -2, 1, 3$$

$\therefore$ Roots of $f(x) = 0$ are $-2, -2, 1, 3$

10. Let $f(x) = 4x^3 + 20x^2 - 23x + 6$

By Horner's method

$\frac{1}{2}$	4	20	-23	6
		2	11	-6
	4	22	-12	0
		2	12	
	4	24	0	
		2		
	4	26		
	4			

$\Rightarrow$ Remainder = 0

$\therefore \quad \frac{1}{2}$ is a root of $4x^3 + 20x^2 - 23x + 6 = 0$

Also, it is clear, when $f(x)$ is divided by $x - \frac{1}{2}$.

Quotient is $4x^2 + 22x - 12$

$$\therefore \quad f(x) = \left(x - \frac{1}{2}\right)(4x^2 + 22x - 12)$$
$$= \left(x - \frac{1}{2}\right)4\left(x^2 + \frac{11}{2}x - 3\right)$$
$$= 4\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)(x + 6)$$
$$= 4\left(x - \frac{1}{2}\right)^2 (x + 6)$$

$\Rightarrow x = \frac{1}{2}$ is a root of $f(x)$ with multiplicity 2 and third root is given by factor $x + 6$ *i.e.*, -6

$\therefore$ Roots are 1/2, 1/2, -6.

11. Since, $(x^2 + x + 5)(ax + b) + c$
$$= x^3 + 7x^2 + 3x + 5$$
$$\therefore \quad ax^3 + (a + b)x^2 + (5a + b)x + 5b + c = x^3 + 7x^2 + 3x + 5$$

$\therefore$ By equality of two polynomials, we have $a = 1$

$$a + b = 7$$
$$\Rightarrow \quad b = 7 - a = 7 - 1 = 6, 5b + c = 5$$
$$5a + b = 3$$
$$\Rightarrow \quad 5 + 6 = 3$$

which is not true.

Hence, there do not exist a, b, c for which the two polynomials are equal.

12. Given, $(x - \alpha)^m f(x) = (x - \alpha)^n g(x)$,
$$f(\alpha) \neq 0, g(\alpha) \neq 0$$

we want to prove that
$$m = n \text{ and } f(x) = g(x)$$

If possible let $m \neq n$

Without any loss of generality, let $m < n$

$\therefore$ $n - m$ is a + ve integer, so that
$$(x - \alpha)^m f(x) = (x - \alpha)^n g(x)$$

i.e., $\quad f(x) = (x - \alpha)^{n-m} g(x)$

i.e., $\quad (x - \alpha) | f(x)$

$\Rightarrow \alpha$ is a root of $f(x) = 0$.

$\Rightarrow f(\alpha) = 0$ which is contrary to the given hypothesis.

$\therefore$ Our supposition is wrong.

Hence, $m = n$

and $\quad (x - \alpha)^m f(x) = (x - \alpha)^n g(x)$
$$\Rightarrow \quad f(x) = g(x)$$

13. $f(x) = x^4 + 2x^3 - 2x - 1 = 0 \quad \ldots$(i)
$$f'(x) = 4x^3 + 6x^2 - 2 = 0$$

i.e., $\quad 2x^3 + 3x^2 - 1 = 0 \quad \ldots$(ii)

Since, $f(x) = 0$ has repeated roots (i) and (ii) has common roots which can be obtained by solving $d(x) = 0$ where $d(x)$ is HCF of $f(x)$ and $f'(x)$.

$$2x^3 + 3x^2 - 1)\overline{x^4 + 2x^3 - 2x - 1}(\frac{x}{2} + \frac{1}{4}$$
$$\underline{x^4 + \frac{3x^3}{2} - \frac{x}{2}}$$
$$\frac{x^3}{2} - \frac{3x}{2} - 1$$
$$\underline{\frac{x^3}{2} + \frac{3x^2}{4} - \frac{1}{4}}$$
$$\underline{-\frac{3}{4}x^2 - \frac{3x}{2} - \frac{3}{4}}$$
$$x^2 + 2x + 1)2x^3 + 3x^2 - 1\,(2x - 1$$
$$\underline{2x^3 + 4x^2 + 2x}$$
$$-x^2 - 2x - 1$$
$$\underline{-x^2 - 2x - 1}$$
$$\times$$

$\therefore$ HCF of $f(x) = 0$ and $f'(x) = 0$ is
$$d(x) = x^2 + 2x + 1 \; d(x) = 0 \Rightarrow x^2 + 2x + 1 = 0$$
$$\Rightarrow \quad (x + 1)^2 = 0 \Rightarrow x = -1, -1.$$

$\therefore$ Roots of $f(x) = 0$ are $-1, -1$.

Let us use synthetic division method.

-1	1	2	0	-2	-1
		-1	-1	1	1
	1	1	-1	-1	0
		-1	0	1	
	1	0	-1	0	

$\therefore$ Reduced equation is $x^2 + 0x - 1 = 0, x = \pm 1$

$\therefore$ Roots of $f(x) = 0$ are $-1, -1, -1, 1$.

14. $f(x) = 6x^4 - 13x^3 - 35x^2 - x + 3$ ...(i)

All its coefficients are rational since $2 - \sqrt{3}$ is a root of Eq. (i)

$\therefore$ $2 + \sqrt{3}$ is also a root of Eq. (i)

$\therefore$ $(x - 2 + \sqrt{3})(x - 2 - \sqrt{3})$ divides $f(x)$

i.e., $(x - 2)^2 - 3$ divides $f(x)$ *i.e.*, $x^2 - 4x + 1$ divides $f(x)$

$\therefore$ We have, $f(x) = 6x^4 - 13x^3 - 35x^2 - x + 3$

$$= (x^2 - 4x + 1)(6x^2 + 11x + 3)$$

$\therefore$ The other two roots are given by

$$6x^2 + 11x + 3 = 0$$

i.e., $x = \dfrac{-11 \pm \sqrt{121-72}}{12} = \dfrac{-11 \pm \sqrt{49}}{12}$

$$= -\frac{18}{12} \text{ or } -\frac{4}{12}$$

$$= -\frac{3}{2} \text{ or } -\frac{1}{3}$$

Hence, all the roots of Eq. (i) are

$$2 \pm \sqrt{3}, -3/2, -1/3$$

15. Sum of the roots $= \alpha - \alpha + \beta - \beta + \gamma = 5$

$\therefore$ $\gamma = 5$

But γ is a root of the given equation.

$\therefore$ $x - 5$ is a factor of

$$x^5 - 5x^4 - 5x^3 + 25x^2 + 4x - 20$$

5	1	-5	-5	25	4	-20
		5	0	-25	0	20
	1	0	-5	0	4	0

Depressed equation is $x^4 - 5x^2 + 4 = 0$

$(x^2 - 1)(x^2 - 4) = 0$, $\therefore x = \pm 1, \pm 2$

Roots are $\pm 1, \pm 2, 5$.

16. Let the roots be $\alpha, \alpha + 7, \beta$, then

$$\alpha + (\alpha + 7) + \beta = -1$$

or $2\alpha + \beta = -8$ ...(i)

Also, $\alpha(\alpha + 7) + (\alpha + 7)\beta + \alpha\beta = -16$

$\alpha(\alpha + 7) + \beta(2\alpha + 7) = -16$ ...(ii)

Eliminating β from Eqs. (i) and (ii)

$$\alpha(\alpha + 7) + (-8 - 2\alpha)(2\alpha + 7) = -16$$

$$\alpha^2 + 7\alpha - 30\alpha - 4\alpha^2 - 56 + 16 = 0$$

$$3\alpha^2 + 23\alpha + 40 = 0$$

$\therefore$ $\alpha = \dfrac{-23 \pm 7}{6} = -5, -\dfrac{8}{3}$

But $\alpha = -\dfrac{8}{3}$ does not satisfy the given equation.

$\therefore$ $\alpha = -5, \beta = -8 - 2\alpha = -8 + 10 = 2$

Roots of the given equation are $\alpha, \alpha + 7, \beta$

i.e., $-5, -5 + 7, 2$ or $-5, 2, 2$.

17. Let roots be $\alpha, \alpha + 2, \beta$ then

$2\alpha + \beta = 11$ ...(i)

Also, $\alpha(\alpha + 2) + (\alpha + 2)\beta + \alpha\beta = 15$

or $\alpha(\alpha + 2) + \beta(2\alpha + 2) = 15$ ...(ii)

Eliminating β from Eqs. (i) and (ii)

$$\alpha(\alpha + 2) + (11 - 2\alpha)(2\alpha + 2) = 15$$

$$\alpha^2 + 2\alpha + 22 + 18\alpha - 4\alpha^2 = 5$$

$$3\alpha^2 - 20\alpha - 7 = 0$$

$\therefore$ $\alpha = \dfrac{20 \pm \sqrt{400 + 84}}{6}$

$$= \frac{20 \pm 22}{6} = 7, \frac{-1}{3}$$

But $\alpha = \dfrac{-1}{3}$ does not satisfy equation.

$\therefore$ $\alpha = 7, \beta = 11 - 2\alpha = 11 - 14 = -3$

Roots of given equation are $\alpha, \alpha + 2, \beta$

i.e., 7, 9, -3.

18. Let roots be $\alpha, \beta, \gamma, \delta$ such that $\alpha\beta = -\gamma\delta$

Now, $\alpha\beta\gamma\delta = -36 \Rightarrow (-\gamma\delta)\gamma\delta = -36$

$\Rightarrow$ $\gamma^2\delta^2 = 36 \Rightarrow \gamma\delta = 6$ or -6

and $\alpha\beta = -6$ or 6

The factors corresponding to these roots are of the types $x^2 - px - 6$ and $x^2 - qx + 6$

We have,

$$x^4 - 8x^3 + 7x^2 + 36x - 36$$

$$\equiv (x^2 - px - 6)(x^2 - qx + 6)$$

Equating coefficients of like powers of x

Coefficients of x^3; $-8 = -p - q$

$\therefore$ $p + q = 8$ ...(i)

Coefficients of x; $36 = -6p + 6q$

$\therefore$ $p - q = -6$ ...(ii)

From Eqs. (i) and (ii), we get

$$p = 1 \text{ and } q = 7$$

$\therefore$ Given equation may be written as

$$(x^2 - x - 6)(x^2 - 7x + 6) = 0$$

or $(x - 3)(x + 2)(x - 1)(x - 6) = 0$

$\therefore$ $x = 3, -2, 1, 6$

19. Since, one root is equal to half the sum of the other two.

$\therefore$ The roots are in AP.

Let the roots be $\alpha - \delta, \alpha, \alpha + \delta$

$$\therefore \quad \alpha = \frac{1}{2}(\alpha - \delta + \alpha + \delta)$$

Sum of roots $= \alpha - \delta + \alpha + \alpha + \delta = -\frac{81}{18}$

$$\Rightarrow \quad 3\alpha = -\frac{9}{2}$$

$$\therefore \quad \alpha = -\frac{3}{2}$$

$\therefore$ $-\frac{3}{2}$ is a root of given equation.

$-3/2$	18	81	121	60
		-27	-81	-0
	18	54	40	0

Depressed equation is $18x^2 + 54x + 40 = 0$

$$\Rightarrow \quad 9x^2 + 27x + 20 = 0$$

Its roots are

$$\frac{-27 \pm \sqrt{729 - 720}}{18} = \frac{-27 \pm 3}{18} = -\frac{4}{3} \text{ or } -\frac{5}{3}$$

Roots of the given equation are $-4/3$, $-3/2$, $-5/3$.

20. Sum of all the four roots = 8

$\because$ Sum of two roots = Sum of other two roots

$\therefore$ Each sum = 4

Factors corresponding to these roots are of the types

$$x^2 - 4x + p$$

and $x^2 - 4x + q$

Thus, we have

$$x^5 - 8x^3 + 19x^2 + 4\lambda x + 2 \equiv (x^2 - 4x + p)(x^2 - 4x + q)$$

Equating the coefficients of like powers of x

Coefficient of x^2, $19 = p + q + 16$

$$\Rightarrow \quad p + q = 3 \qquad \text{...(i)}$$

Coefficient of x, $4\lambda = -4(p + q)$

$$\Rightarrow \quad p + q = -\lambda \qquad \text{...(ii)}$$

Constant terms $2 = pq$ ...(iii)

From Eqs. (i) and (iii), it is evident that p, q are the roots of $t^2 - 3t + 2 = 0$

or $(t - 1)(t - 2) = 0$; $\therefore$ $t = 1, 2$

Take, $p = 1, q = 2$

Given equation can be written as

$$(x^2 - 4x + 1)(x^2 - 4x + 2) = 0$$

$$\therefore \quad x = \frac{4 \pm \sqrt{16 - 4}}{2}, \frac{4 \pm \sqrt{16 - 8}}{2}$$

or $2 \pm \sqrt{3}, 2 \pm \sqrt{2}$

21. $\alpha\beta + \gamma\delta = 0$ ...(i)

Let $\alpha\beta = k$, then $\gamma\delta = -k$

If $\alpha + \beta = l$ and $\gamma + \delta = m$, then

$$(x^4 + px^3 + qx^2 + rx + s) \equiv (x^2 - lx + k)(x^2 - mx - k)$$

Comparing the coefficients of like powers of x, we get

$$l + m = -p; lm = q; k(l - m) = r$$

$$\therefore \quad l - m = \frac{r}{k} \text{ and } -k^2 = s$$

In order to find the condition, we have to eliminate l, m and k between the above four relations.

$$(l - m)^2 = (l + m)^2 - 4lm$$

$$\therefore \quad \frac{r^2}{k^2} = p^2 - 4q \text{ or } r^2 = -s(p^2 - 4q)$$

or $p^2 s + r^2 = 4qs$ is the required condition.

22. $\alpha = -\beta$, *i.e.*, $\alpha + \beta = 0$

$$\therefore \quad \alpha + \gamma = 54/45$$

$$\therefore f(x) = \left(x^4 - \frac{54}{45}x^3 - \frac{98}{45}x^2 + \frac{150}{45}x - \frac{75}{45}\right) = 0$$

$$\equiv (x^2 + l)\left(x^2 - \frac{54}{45}x + m\right)$$

On comparing, we get $l + m = -\frac{98}{45}$

and $-\frac{54}{45}l = \frac{150}{45}$

$$\therefore \quad l = -\frac{25}{9}; m = -l - \frac{98}{45}$$

$$m = \frac{25}{9} - \frac{98}{45} = \frac{27}{45} = \frac{3}{5}$$

$\therefore \quad f(x) = 0$ is $\left(x^2 - \frac{25}{9}\right)\left(x^2 - \frac{54}{45}x + \frac{3}{5}\right) = 0$

Solving the above, we get

$$x = \pm\frac{5}{3}, \frac{3 \pm 5\sqrt{6}}{5}$$

23. Let the roots be 3α, 4α and β.

$$\therefore \quad \Sigma\alpha = 7\alpha + \beta = \frac{1}{2}$$

$$\therefore \quad \beta = \left(\frac{1}{2} - 7\alpha\right) \quad \text{...(i)}$$

$\Sigma\alpha\beta = 12\alpha^2 + 4\alpha\beta + 3\alpha\beta = -(22)/2 = -11$

or $\quad 12\alpha^2 + 7\alpha\left(\frac{1}{2} - 7\alpha\right) = -11 \quad$ [by Eq. (i)]

$$74\alpha^2 - 7\alpha - 22 = 0$$

or $\quad (2\alpha + 1)(37\alpha - 22) = 0$

$\therefore \ \alpha = -\frac{1}{2}$ or $\frac{22}{37}$ and corresponding values of β

From Eq. (i) are 4 and $-171/37$

$$\alpha\beta\gamma = 12\alpha^2\beta = -12$$

The set of values $\alpha = -1/2$ and $\beta = 4$ satisfy the given equation.

Roots are $3\alpha, 4\alpha, \beta$ *i.e.*,

$$-3/2, -2, 4.$$

24. Let the roots be $a - d, a, a + d$

$$\therefore \quad \Sigma\alpha = 3a = \frac{15}{2},$$

$$\therefore \quad a = \frac{5}{2}, \text{ which is a root.}$$

Dividing the given equation by $2x - 5$ by synthetic division, we get quotient

$$2x^2 - 10x + 12 = 0$$

or $\quad 2(x-2)(x-3) = 0$

Other roots are 2 and 3.

$\therefore$ Roots are $3, 5/2, 2$.

25. Let roots be $\alpha, \alpha + 2, \beta$

$$\therefore \quad \Sigma\alpha = \alpha + \alpha + 2 + \beta = 13$$

$$\therefore \quad \beta = 11 - 2\alpha \quad \text{...(i)}$$

$$\Sigma\alpha\beta = \alpha(\alpha + 2) + (\alpha + 2)\beta + \alpha\beta = 15$$

or $\quad \alpha^2 + 2\alpha + (2\alpha + 2)(11 - 2\alpha) = 15$ [by Eq. (i)]

or $\quad 3\alpha^2 - 20\alpha - 7 = 0$

$$(3\alpha + 1)(\alpha - 7) = 0; \alpha = 7, 1/3$$

when $\alpha = 7, \beta = -3$ and $\alpha = \frac{1}{3}, \beta = 11\frac{2}{3}$

[by Eq. (i)]

We shall choose that set of values which satisfies third relation

$$\alpha\beta\gamma = \alpha(\alpha + 2)\beta = -189$$

The value $\alpha = 7, \beta = -3$, satisfies it.

Roots are 7, 9, -3.

26. (a) Let roots be α, 2α and β.

$$\Sigma\alpha = \alpha + 2\alpha + \beta = 7$$

$$\therefore \quad \beta = 7 - 3\alpha \quad \text{...(i)}$$

$$\Sigma\alpha\beta = \alpha 2\alpha + 2\alpha\beta + \beta\alpha = 0$$

as the coefficient of x is missing.

or $\quad 2\alpha^2 + 3\alpha(7 - 3\alpha) = 0 \quad$ [by Eq. (i)]

or $\quad \alpha(21 - 7\alpha) = 0; \alpha = 0$ or 3

when $\alpha = 3, \ \beta = 7 - 3 \cdot 3 = -2$

when $\alpha = 0, \ \beta = 7$

Also, $\quad \alpha\beta\gamma = \alpha \Rightarrow 2\alpha\beta = -36$

$\Rightarrow \quad \alpha^2\beta = = -18$

Roots are $\alpha, 2\alpha, \beta$ *i.e.*, 3, 6, -2.

Roots for (b) part are 6, 3, -2.

27. Let the root be α, α, β.

$$\Sigma\alpha = 2\alpha + \beta = -5$$

$$\therefore \quad \beta = -5 - 2\alpha \quad \text{...(i)}$$

$$\Sigma\alpha\beta = \alpha^2 + \alpha\beta + \alpha\beta = -23/4$$

or $\quad \alpha^2 + 2\alpha(-5 - 2\alpha) = -23/4$ [by Eq. (i)]

or $\quad 12\alpha^2 + 40\alpha - 23 = 0$

$$(2\alpha - 1)(16\alpha + 23) = 0$$

$$\alpha = 1/2 \text{ and } -23/6$$

$\alpha = 1/2$ satisfies above.

Hence, $\alpha = -23/6$ will be rejected.

Putting $\alpha = 1/2$ in Eq. (i), we get

$$\beta = -6$$

Roots are 1/2, 1/2, -6.

28. $x = 1$ is a root of this equation

It can be written as

$$(x - 1)(x^4 - 4x^3 + 5x^2 - 4x + 1) = 0$$

Solve the second factor, we get answer

$$1, \frac{3 \pm \sqrt{5}}{2}, \frac{1 \pm i\sqrt{3}}{2}$$

29. (a) Answer $x^3 + 5x^2 - 7x - 3 = 0$

(b) Answer $-4x^3 - 2x^2 - 3x + 5 = 0$

or $\quad 4x^3 + 2x^2 + 3x - 5 = 0$

30. $x^5 + 11x^4 + 42x^3 + 57x^2 - 13x - 60 = 0$

31. $5x^3 + 17x^2 - 4x - 29 = 0$

32. $x^5 + x^3 + x^2 + 2x + 3 = 0 \quad \ldots\text{(i)}$

If y be a root of the transformed equation, then

$$y = x^2 \quad \ldots\text{(ii)}$$

We have to eliminate x between Eqs. (i) and (ii), from Eq. (i)

$$x^5 + x^3 + 2x = -(x^2 + 3)$$

On squaring both sides

$$x^{10} + x^6 + 4x^2 + 2x^8 + 4x^4 + 4x^6 = x^4 + 6x^2 + 9$$

or $$y^5 + y^3 + 4y + 2y^4 + 4y^2 + 4y^3 = y^2 + 6y + 9$$

or $$y^5 + 2y^4 + 5y^3 + 3y^2 - 2y - 9 = 0$$

which is required equation.

33. $$x^3 + 3x^2 + 2 = 0 \quad \ldots\text{(i)}$$

If y be a root of transformed equation, then

$$y = x^3 \quad \ldots\text{(ii)}$$

we have to eliminate x between Eqs. (i) and (ii) from Eq. (i),

$$x^3 + 2 = -3x^2$$

Cubing both sides

$$x^9 + 8 + 3x^3 2(x^3 + 2) = -27x^6$$

or $$x^9 + 33x^6 + 12x^3 + 8 = 0$$

Putting $x^3 = y$,

$$y^2 + 33y^2 + 12y + 8 = 0$$

which is the required equation.

34. $$2x^3 - x^2 + 2x - 3 = 0 \quad \ldots\text{(i)}$$

If y be a root of transformed equation, then

$$y = x^3 \quad \ldots\text{(ii)}$$

We have to eliminate x between Eqs. (i) and (ii), from Eq. (i)

$$2x^3 - 3 = x^2 - 2x$$

On cubing both sides, we get

$$(2x^3 - 3)^3 = x^6 - 8x^3 - 3x^2 \cdot 2x(x^2 - 2x)$$

or $$8x^9 - 36x^6 + 54x^3 - 27 = x^6 - 8x^3 - 6x^3(2x^3 - 3)$$

Putting $x^3 = y$,

$$8y^3 - 36y^2 + 54y - 27 = y^2 - 8y - 6y(2y - 3)$$

or $$8y^3 - 25y^2 + 44y - 27 = 0 \quad \ldots\text{(iii)}$$

which is required transformed equation

$\because$ Roots of Eq. (i) are α, β, γ

$\therefore$ Roots of Eq. (iii) are $\alpha^3, \beta^3, \gamma^3$

$$\Sigma\alpha^3\beta^3 = \frac{44}{8} = \frac{11}{2}$$

35. Let α, β, γ are roots of $x^3 + qx + r = 0$, equation in y whose roots are $\alpha^2, \beta^2, \gamma^2$ is obtained by putting $y = x^2$,

$$x(x^2 + q) = -r \text{ or } \sqrt{y}(y + q) = -r$$

On squaring both sides, we get

$$y(y^2 + 2qy + q^2) = r^2$$

or $$y^3 + 2qy^2 + q^2y - r^2 = 0$$

Comparing this with given equation, we conclude that its second term should be removed by h where $na_0 h + a_1 = 0$

or $$3h + 2q = 0$$

$\therefore$ $$h = -2q/3$$

$$\begin{array}{r|cccc}
-2q/3 & 1 & 2q & q^2 & -r^2 \\
 & & -2q/3 & -8q^2/9 & -2q^3/27 \\
\hline
 & & -4q/3 & q^2/9 & (-2q^3/27 - r^2) \\
 & & -2q/3 & -4q^2/9 & \\
\hline
 & & 2q/3 & -q^2/3 & \\
\hline
 & & 0 & &
\end{array}$$

Transformed equation is

$$z^3 - \frac{1}{3}q^2 z - \frac{2}{27}q^3 - r^2 = 0$$

Whose roots are $\alpha^2 - h, \beta^2 - h, \gamma^2 - h$

or $$\alpha^2 + 2q/3, \beta^2 + 2q/3, \gamma^2 + 2q/3$$

which differ from the squares of the roots of $x^3 + qx + r = 0$ by a constant $-2q/3$.

36. Put $x^3 = y$,

$\therefore$ $$y + ab = -(ay^{2/3} + by^{1/3}) \quad \ldots\text{(i)}$$

On cubing both sides, we get

$$y^3 + 3y^2ab + 3ya^2b^2 + a^3b^3$$
$$= -[a^3y^2 + b^3y + 3aby\,(ay^{2/3} + by^{1/3})]$$
$$y^3 + 3aby^2 + 3a^2b^2y + a^3b^3$$
$$= -a^3y^2 - b^3y - 3aby\{-(y + ab)\}, \text{[by Eq. (i)]}$$
$$\therefore \quad y^3 + a^3y^2 + b^3y + a^3b^3 = 0$$

37. Since, α, β, γ are the roots of

$$z^3 + 3Hz + G = 0 \qquad \ldots\text{(i)}$$
$$\therefore \quad \alpha + \beta + \gamma = 0$$

If the transformed equation is in terms of y, then

$$y = \frac{\alpha + 1}{\beta + \gamma - \alpha} = \frac{\alpha + 1}{\alpha + \beta + \gamma - 2\alpha} = \frac{\alpha + 1}{0 - 2\alpha}$$
$$\therefore \quad y = \frac{z + 1}{-2z} \text{ or } -2yz = z + 1$$
$$\text{or} \quad (1 + 2y)z = -1, \ \therefore \ z = -\frac{1}{1 + 2y}$$

Putting the value of z in Eq. (i), we get

$$-\frac{1}{(1 + 2y)^3} - \frac{3H}{(1 + 2y)} + G = 0$$
$$G(1 + 2y)^3 - 3H(1 + 2y)^2 - 1 = 0$$
$$\Rightarrow \quad G(1 + 6y + 12y^2 + 8y^3)$$
$$-3H(1 + 4y + 4y^2) - 1 = 0$$
$$\Rightarrow \quad 8Gy^3 + 12(G - H)y^2 + 6(G - 2H)y$$
$$+ (G - 3H + 1) = 0$$

which is the required equation.

38.
$$x^3 + px^2 + qx + r = 0 \qquad \ldots\text{(i)}$$
$$\therefore \quad \alpha + \beta + \gamma = -p, \alpha\beta + \beta\gamma + \gamma\alpha = q,$$
$$\alpha\beta\gamma = -r \qquad \ldots\text{(ii)}$$

(a) If the transformed equation is in terms of y.

$$y = \alpha - \frac{1}{\beta\alpha} = \alpha - \frac{\alpha}{\alpha\beta\gamma} = \alpha - \frac{\alpha}{-r} = \alpha + \frac{\alpha}{r}$$
$$[\because \text{Eq. (ii)}]$$
$$\therefore \quad y = x + \frac{x}{r} = \frac{(r + 1)x}{r}$$
$$\therefore \quad x = \frac{ry}{r + 1}$$

Putting this value of x in Eq. (i), we get

$$\frac{r^3y^3}{(r + 1)^3} + p \cdot \frac{r^2y^2}{(r + 1)^2} + q \cdot \frac{ry}{r + 1} + r = 0$$
$$\text{or} \quad r^2y^3 + pry^2(r + 1) + qy(r + 1)^2 + (r + 1)^3 = 0$$
$$\text{or} \quad r^2y^3 + pr(r + 1)y^2 + q(r + 1)^2y + (r + 1)^3 = 0$$

which is the required equation.

(b) If the transformed equation is in terms of y.

$$y = \alpha(\beta + \gamma) = (\alpha\beta + \beta\gamma + \gamma\alpha) - \beta\gamma$$
$$= q - \frac{\alpha\beta\gamma}{\alpha} = q + \frac{r}{\alpha} \qquad [\because \text{Eq. (ii)}]$$
$$\therefore \quad y = q + \frac{r}{x} \text{ or } y - q = \frac{r}{x}$$
$$\text{or} \quad x = \frac{r}{y - q}$$

Putting this value of x in Eq. (i), we get

$$\frac{r^3}{(y - q)^3} + p\frac{r^2}{(y - q)^2} + q\frac{r}{y - q} + r = 0$$
$$\text{or} \quad r^2 + pr(y - q) + q(y - q)^2 + (y - q)^3 = 0$$
$$\text{or} \quad y^3 - 2qy^2 + (q^2 + pr)y + (r^2 - pqr) = 0$$

which is the required equation.

(c) If the transformed equation is in terms of y.

$$y = \beta\gamma + \frac{1}{\alpha} = \frac{\alpha\beta\gamma + 1}{\alpha} = -\frac{r + 1}{\alpha} \qquad [\because \text{Eq. (ii)}]$$
$$\therefore \quad y = \frac{1 - r}{x} \text{ or } x = \frac{1 - r}{y}$$

Putting this value of x in Eq. (i), we get

$$\frac{(1 - r)^3}{y^2} + p\frac{(1 - r)^2}{y^2} + q\frac{1 - r}{y} + r = 0$$
$$\text{or} \quad ry^3 + q(1 - r)y^2 + p(1 - r)^2y + (1 - r)^3 = 0$$

which is the required equation.

39. $\because \alpha, \beta, \gamma$ are the roots of equation

$$x^3 + x^2 + 2x + 3 = 0$$
$$\therefore \quad \alpha + \beta + \gamma = -1 \qquad \ldots\text{(i)}$$

If y is a root of the transformed equation.

$$y = \beta + \gamma - \alpha = (\alpha + \beta + \gamma) - 2\alpha = -1 - 2\alpha$$
$$\therefore \quad y = -1 - 2x \text{ or } x = -\frac{1 + y}{2}$$

Putting this value of x in Eq. (i), we get

$$-\frac{(1 + y)^3}{8} + \frac{(1 + y)^2}{4} - (1 + y) + 3 = 0$$
$$(y + 1)^3 - 2(y + 1)^2 + 8(y + 1) - 24 = 0$$
$$\text{or} \quad y^3 + y^2 + 7y - 17 = 0$$

which is the required equation.

40. Hint

$$y = 2\alpha - \beta - \gamma = 3\alpha - (\alpha + \beta + \gamma) = 3\alpha - a$$

$$\therefore \quad y = 3x - a \quad \text{or} \quad x = \frac{y + a}{3}$$

Answer

$$y^3 + 3(3b - a^2)y + (9ab - 2a^3 - 27) = 0$$

and also the product of the roots is

$$2a^3 + 27c - ab$$

41. Adding all the four equations, we get

$$4x_1 = 2a_1 + 2a_2 + 2a_3 + 2a_4$$

$$x_1 = \frac{a_1 + a_2 + a_3 + a_4}{2}$$

Multiplying the last two equations by -1 and then adding all the four equations, we get

$$x_2 = \frac{a_1 + a_2 - a_3 - a_4}{2}$$

Similarly, $x_3 = \dfrac{a_1 - a_2 + a_3 - a_4}{2}$,

$$x_4 = \frac{a_1 - a_2 - a_3 + a_4}{2}$$

42. Put $x + y + z + v = s$, then the following system can be rewritten as

$$ax + m(s - x) = k,$$

$$by + m(s - y) = l$$

$$cz + m(s - z) = p$$

$$dv + m(s - v) = q$$

So that, $ms + x(a - m) = k$; $ms + y(b - m) = l$

$$ms + z(c - m) = p,$$

$$ms + v(d - m) = q$$

Hence, $x = \dfrac{k}{a - m} - \dfrac{m}{a - m}s$;

$$y = \frac{l}{b - m} - \frac{m}{b - m}s;$$

$$z = \frac{p}{c - m} - \frac{m}{c - m}s;$$

$$v = \frac{q}{d - m} - \frac{m}{d - m}s \qquad \ldots\text{(i)}$$

Adding these equalities termwise, we get

$$s = \frac{k}{a - m} + \frac{l}{b - m} + \frac{p}{c - m} + \frac{q}{d - m}$$

$$- ms\left(\frac{1}{a - m} + \frac{1}{b - m} + \frac{1}{c - m} + \frac{1}{d - m}\right)$$

$$\Rightarrow s\left[\left(1 + m\frac{1}{a - m} + \frac{1}{b - m} + \frac{1}{c - m} + \frac{1}{d - m}\right)\right]$$

$$= \frac{k}{a - m} + \frac{l}{b - m} + \frac{p}{c - m} + \frac{q}{d - m}$$

where from we find s and then from the equalities (i) we obtain the required values of x, y, z.

43. Adding all the three equations, we get

$$(x + y + z)(a + b + c) = 0$$

Hence, $x + y + z = 0$

$$\therefore \quad x = \frac{a - b}{a + b + c}; y = \frac{a - c}{a + b + c}; z = \frac{b - a}{a + b + c}$$

44. We find similarity,

$$t = -(a + b + c + d)$$

$$x = ab + ac + ad + bc + bd + cd$$

$$y = -(abc + abd + acd + bcd)$$

$$z = abcd$$

45. Multiplying the given equations, we get

$$(xyz)^2 = abcxyz$$

First of all we have an obvious solution $x = y = z = 0$, then $xyz = abc$

From the original equations, we find

$$xyz = ax^2; xyz = by^2; xyz = cz^2$$

Hence, $ax^2 = abc; by^2 = abc; cz^2 = abc$

$$x^2 = bc; y^2 = ac; z^2 = ab$$

Thus, we have following solution set

$$x = \sqrt{bc}; y = \sqrt{ac}; z = \sqrt{ab}$$

$$x = -\sqrt{bc}; y = -\sqrt{ac}; z = \sqrt{ab}$$

$$x = \sqrt{bc}; y = -\sqrt{ac}; z = -\sqrt{ab}$$

$$x = -\sqrt{bc}; y = \sqrt{ac}; z = -\sqrt{ab}$$

46. The system is reduced to the form

$$xy + xz = a^2$$

$$yz + yx = b^2$$

$$zx + zy = c^2$$

Adding these equations term by term, we get

$$xy + xz + yz = \frac{1}{2}(a^2 + b^2 + c^2)$$

Taking into consideration the first three equations

$$yz = \frac{b^2 + c^2 - a^2}{2}, \; zx = \frac{a^2 + c^2 - b^2}{2}$$

$$xy = \frac{a^2 + b^2 - c^2}{2}$$

Multiplying them, we have

$$(xyz)^2 = \frac{(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)(a^2 + b^2 - c^2)}{8}$$

i.e., $$xyz = \pm\sqrt{\frac{(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)(a^2 + b^2 - c^2)}{8}}$$

Now, we easily find

$$x = \pm\frac{\sqrt{(a^2 + c^2 - b^2)(a^2 + b^2 - c^2)}}{8(b^2 + c^2 - a^2)}$$

$$y = \pm\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)}}{8(a^2 + c^2 - b^2)}$$

$$z = \pm\sqrt{\frac{(a^2 + c^2 - b^2)(b^2 + c^2 - a^2)}{8(a^2 + b^2 - c^2)}}$$

47. Adding the first two equations and subtracting the third one, we get

$$2x^2 = (c + b - a)xyz$$

Likewise, we find

$$2y^2 = (c + a - b)xyz; 2z^2 = (a + b - c)xyz$$

Solving out the solution, we get

$$x = y = z = 0$$

We have,

$$2x = (c + b - a)yz; 2y = (c + a - b)xz;$$

$$2z = (a + b - c)xy$$

Now, proceed as in Q. 6.

48. Put $\dfrac{x + y}{x + y + cxy} = \gamma; \dfrac{y + z}{y + z + ayz} = \alpha$

and $$\frac{x + z}{x + z + bxz} = \beta$$

Then, the system takes the form

$$b\gamma + c\beta = a; c\alpha + a\gamma = b; a\beta + b\alpha = c$$

or $$\frac{\gamma}{c} + \frac{\beta}{b} = \frac{a}{bc}; \frac{\alpha}{a} + \frac{\gamma}{c} = \frac{b}{ac};$$

$$\frac{\beta}{b} + \frac{\alpha}{a} = \frac{c}{ab}$$

$\therefore$ $$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = \frac{1}{2}\frac{a^2 + b^2 + c^2}{abc}$$

and consequently

$$\alpha = \frac{b^2 + c^2 - a^2}{2bc}; \beta = \frac{a^2 + c^2 - b^2}{2ac};$$

$$\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

Further $$\frac{x + y + cxy}{x + y} = \frac{1}{\gamma}$$

or $$\frac{cxy}{x + y} = \frac{1}{\gamma} - 1$$

or $$\frac{x + y}{cxy} = \frac{\gamma}{1 - \gamma}$$

Finally $$\frac{1}{x} + \frac{1}{y} = \frac{c\gamma}{1 - \gamma}$$

Analogously, we get

$$\frac{1}{x} + \frac{1}{z} = \frac{b\beta}{1 - \beta}; \frac{1}{y} + \frac{1}{z} = \frac{a\alpha}{1 - \alpha}$$

Where from we find x, y, z.

49. Multiplying the first, second and third equations respectively by y, z and x, we get

$$cx + ay + bz = 0$$

Likewise multiplying these equations by z, x and y, we get

$$bx + cy + az = 0$$

Now, $$\frac{x}{a^2 - bc} = \frac{y}{b^2 - ac} = \frac{z}{c^2 - ab} = \lambda$$

i.e., $$x = (a^2 - bc)\lambda$$

$$y = (b^2 - ac)\lambda$$

$$z = (c^2 - ab)\lambda$$

Substituting these expressions into the third equation, we find

$$\lambda^2 = \frac{c}{(c^2 - ab)^2 - (a^2 - bc)(b^2 - ac)}$$

$$= \frac{1}{a^3 + b^3 + c^3 - 3abc}$$

Then, we can easily find x, y, z.

50. $$x^2 + y^2 + z^2 = 84 \quad \text{...(i)}$$

$$x + y + z = 14 \quad \text{...(ii)}$$

$$xz = y^2 \quad \text{...(iii)}$$

Squaring Eq. (ii), we have

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 196$$

Putting the values of $x^2 + y^2 + z^2$ and xz from Eqs. (i) and (iii), we get

$$84 + 2xy + 2yz + 2y^2 = 196$$
$$2xy + 2yz + 2y^2 = 112$$
$$xy + yz + y^2 = 56$$
$$y(x + y + z) = 56$$

Putting the values of $x + y + z$ from Eq. (ii), we get

$$14y = 56 \Rightarrow y = 4$$

When $y = 4 \Rightarrow xz = 16$ from Eq. (iii)

$$y = 4 \Rightarrow x + z = 10 \text{ from Eq. (ii)}$$

Hence, x and z are the roots of the quadratic equation $p^2 - 10p + 16 = 0$.

Roots of this equations are 8 and 2.

Hence, $x = 8$ or 2

$$z = 2 \text{ or } 8$$
$$\therefore \quad x = 8 \text{ or } 2$$
$$y = 4$$
$$z = 2 \text{ or } 1$$

51.
$$x + y + z = 13 \quad \text{...(i)}$$
$$x^2 + y^2 + z^2 = 65 \quad \text{...(ii)}$$
$$xy = 10 \quad \text{...(iii)}$$

Squaring Eq. (i)

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 169$$

Putting the values of $x^2 + y^2 + z^2$ and $2xy$, we get

$$65 + 20 + 2yz + 2zx = 169$$
or
$$2yz + 2zx = 84$$
$$yz + zx = 42$$
$$z(x + y) = 42$$

But $\quad x + y = 13 - z \quad$ [from Eq. (i)]

So, $\quad z(13 - z) = 42$

or $\quad z^2 - 13z + 42 = 0$

$\Rightarrow \quad z = 6$ or 7

When $z = 6, x + y = 7$ [from Eq. (i)]

Now, $\quad x + y = 7$ and $xy = 10$

Hence, x and y are the roots of $t^2 - 7t + 10 = 0$.

Roots of this equation are 5, 2.

Hence $\quad x = 5, y = 2$

$$x = 2, y = 5$$

Hence, the first set of values is

$$x = 5, 2$$
$$y = 2, 5$$
$$z = 6, 6$$

When $z = 7 \Rightarrow x + y = 6$ from Eq. (i) and $xy = 10$

x and y are now roots of $t^2 - 6t + 10 = 0$

or
$$t = \frac{6 \pm \sqrt{36 - 40}}{2}$$
$$t = 3 \pm i$$

Hence, $\quad x = 3 \pm i$

$$y = 3 \pm i$$
$$\therefore \quad x = 5, 2, 3 \pm i$$
$$y = 2, 5, 3 \pm i$$
$$x = 6, 6, 7, 7$$

52. We have,
$$xz + y = 7z \quad \text{...(i)}$$
$$yz + x = 8z \quad \text{...(ii)}$$
$$x + y + z = 12 \quad \text{...(iii)}$$

Adding Eqs. (i) and (ii), we get

$$z(x + y) + (x + y) = 15z$$
$$(z + 1)(x + y) = 15z \quad \text{...(iv)}$$

From Eq. (iii) $x + y = 12 - z$

Putting the value of $x + y$ in Eq. (iv), we get

$$(z + 1)(12 - z) = 15z$$
or
$$12z + 12 - z^2 - z = 15z$$
or
$$z^2 + 4z - 12 = 0$$
or
$$(z + 6)(z - 2) = 0$$
$$\therefore \quad z = -6 \text{ or } z = 2$$

When $\quad z = -6$,

$x + y = 18$ from Eq. (iii) and $-6x + y = -42$ from Eq. (i).

Multiplying first of these equations by 6 and adding to the later, we get

$$6x + 6y = 108$$
$$-6x + y = -42$$
$$7y = 66$$
or
$$y = \frac{66}{7}$$

But $\quad x + y = 18$

$$\therefore \quad x + \frac{66}{7} = 18 \text{ or } x = \frac{60}{7}$$

When we put $z = 2$ and solve for x and y in a similar way, we get $x = 4, y = 6$

Hence, $\quad x = \frac{60}{7}, y = \frac{66}{7}, z = -6, 2$

53.
$$x + y + z = a \quad \dots\text{(i)}$$
$$x^2 + y^2 + z^2 = b^2 \quad \dots\text{(ii)}$$
$$xy = z^2 \quad \dots\text{(iii)}$$

Subtract Eq. (ii) from the square of Eq. (i) and substitute xy by z^2

$$2xy + 2xz + 2yz = a^2 - b^2$$
$$2z^2 + 2z(x + y) = a^2 - b^2$$

Now, substitute $x + y$ with $a - z$

$$2z^2 + 2az - 2z^2 = a^2 + b^2$$

$a = 0$ implies $b = 0$, hence from Eq. (ii), we get $x = y = z = 0$ which according to additional hypothesis is not a solution.

Consequently we may assume that $a \neq 0$

$$z = \frac{a^2 - b^2}{2a},\ z^2 = \frac{(a^2 - b^2)^2}{4a^2}$$

Now, $x + y$ and xy can be expressed in terms of a and b

$$x + y = a - z = \frac{a^2 + b^2}{2a}$$

and $$xy = z^2 = \frac{(a^2 - b^2)^2}{4a^2} \quad \dots\text{(iv)}$$

This yield a quadratic equation in u

$$4a^2u^2 - 2a(a^2 + b^2)u + (a^2 - b^2)^2 = 0 \quad \dots\text{(v)}$$

Which is solved by x and y, solutions are

$$u_{1,2} = \frac{1}{8a^2}\{2a(a^2 + b^2)\} \pm \sqrt{4a^2(a^2 + b^2)^2 - 16a^2(a^2 - b^2)^2}$$
$$= \frac{1}{4a}(a^2 + b^2 \pm \sqrt{(3a^2 - b^2)(3b^2 - a^2)})$$

Hence, the solution of the system (when $a \neq 0$)

$$x = u_1;\ y = u_2;\ z = \frac{a^2 - b^2}{2a}$$
$$x = u_2;\ y = u_1;\ z = \frac{a^2 - b^2}{2a}$$

Sufficient conditions for the solutions to exist Eq. (i) implies $a > 0$; for $z > 0$, we also need $a > |b|$ Eq. (iv) implies that if the solutions exist, then both x and y are +ve, x and y are distinct, if the discriminant in Eq. (v) is +ve ; since $a > |b|$, this positively implies $|b|\sqrt{3} > a$. In this case neither x nor y is equal to z. If, for example, $x = z$ from Eq. (iii) we have $y = z$ so $x = y$ which is impossible. The system admits solutions with the desired properties, if $|b| < a < |b|\sqrt{3}$.

Remark

Let us sketch the geometric background of the problem (1) is simply the equation of a plane intersecting the axis in points of distance a from the origin, consequently its distance from the origin is $\frac{a}{\sqrt{3}}$. (2) is the equation of the sphere of radius $|b|$ around the origin. The two objects have common point if and only if

$$|b| \geq \frac{a}{\sqrt{3}} \text{ i.e., } a \leq |b|\sqrt{3}$$

(3)is the equation of a cone with the origin at its vertex.

54.
$$y^2 + yz + z^2 = ax \quad \dots\text{(i)}$$
$$z^2 + zx + x^2 = ay \quad \dots\text{(ii)}$$
$$x^2 + xy + y^2 = az \quad \dots\text{(iii)}$$

Multiplying Eq. (iii) by y and Eq. (ii) by z and then subtracting, we get

$$(z^3 - y^3) + x(z^2 - y^2) + x^2(z - y) = 0$$
$$(z - y)(z^2 + y^2 + zy + xz + xy + x^2) = 0 \quad \dots\text{(iv)}$$

Similarly, $$(x - y)(x^2 + y^2 + z^2 + yz + zx + xy) = 0 \quad \dots\text{(v)}$$

i.e., by Eqs. (iv) and (v) we have either

$$x = y = z$$

or $$x^2 + y^2 + z^2 + zx + xy + yz = 0$$

Now, if $x = y = z$ then by Eq. (i), $3x^2 = ax$

i.e., $$x = 0 \text{ or } \frac{a}{3}$$

If $x^2 + y^2 + z^2 + xy + yz + zx = 3$, then by Eq. (i)

$$x^2 + xy + xz + ax = 0$$

i.e., $$x = 0 \text{ or } (x + y + z) = 0$$

In this case the solution is indeterminate, for the given equations hold if the relation

$$x + y + z = -a$$

and $$x^2 + y^2 + z^2 + zx + xy + yz = 0$$

are satisfied.

55. Given equations can be rewritten as

$$\frac{a}{x} = y + z - x;\ \frac{b}{y} = z + x - y;$$
$$\frac{c}{z} = x + y - z$$

$$i.e., \quad \frac{\frac{a}{x}}{y+z-x} = \frac{\frac{b}{y}}{z+x-y} = \frac{\frac{c}{z}}{z+y-z}$$

$$= \frac{\left(\frac{b}{y}+\frac{c}{z}\right)}{2z} = \frac{bz+cy}{2xyz}$$

Hence, $bz + cy = cx + azx + ay + bx$

i.e., $\quad cx - cy + (a-b)z = 0$

and $\quad bx + (a-c)y - bz = 0$

$$\therefore \quad \frac{x}{a(-a+b+c)} = \frac{y}{b(a-b+c)} = \frac{z}{c(a+b-c)}$$

$$= k \text{ (say)}$$

Putting the values of x, y, z from above in $x(y+z-x) = a$

We have,

$$k^2 a(-a+b+c)(a-b+c)(a+b-c) = a$$

$$i.e., \quad k = \pm\sqrt{\frac{1}{(-a+b+c)(a-b+c)(a+b-c)}} \quad \text{...(i)}$$

$$x = ka(-a+b+c)$$

$$y = kb(a-b+c)$$

$$z = kc(a+b-ac)$$

while value of k is as in Eq. (i)

56. (a) $\quad ax + by + z = 0 \quad$ ...(i)

$zx + ay + b = 0 \quad$ ...(ii)

$yz + bx + a = 0 \quad$ ...(iii)

From Eq. (i), $\quad z = -(ax+by)$

Hence Eqs. (ii) and (iii) after removing z gives

$$x(ax+by) = ay + b$$

$$y(ax+by) = bx + a$$

$$x(ax+by) = ay + b$$

$$y = \frac{ax^2 - b}{a - bx}$$

$$\therefore \quad ax + by = ax + b \cdot \frac{ax^2 - b}{a - bx}$$

$$= \frac{a^2x - abx^2 + bax^2 - b^2}{a - bx}$$

$$= \frac{a^2x - b^2}{a - bx}$$

$\therefore \quad y(ax+by) = bx + a$ gives

$$\frac{ax^2 - b}{a - b}\left(\frac{a^2x - b^2}{a - bx}\right) = bx + a$$

or $\quad (ax^2 - b)(a^2x - b^2) = (bx + a)(a - bx)^2$

$$(a^3 - b^3)(x^3 - 1) = 0 \text{ i.e., } x^3 - 1 = 0$$

which gives three values of x as $1, \frac{1 \pm i\sqrt{3}}{2}$

$\therefore$ By $y = \dfrac{ax^2 - b}{a - bx}$, we have

For $x = 1$,

$$y = \frac{a-b}{a-b} = 1$$

For $x = \dfrac{-1 - i\sqrt{3}}{2}$

$$y = \frac{\frac{a(1 - 3 - 2i\sqrt{3})}{2} - b}{a - b\left(\frac{i + i\sqrt{3}}{2}\right)} = \frac{-1 - i\sqrt{3}}{2}$$

For $\quad x = \dfrac{-1 - i\sqrt{3}}{2}$,

$$y = \frac{-1 - i\sqrt{3}}{2}$$

If $\quad \omega = \dfrac{-1 - i\sqrt{3}}{2}$, then $\omega^2 = \dfrac{-1 - i\sqrt{3}}{2}$

i.e., $\quad x = 1, \omega, \omega^2$

$y = 1, \omega^2, \omega$

Then, by $z = -(ax + by)$

We have,

$$z = -(a+b), -(a\omega + b\omega^2), -(a\omega^2 + b\omega)$$

57. $\quad x(p+q) = y \quad$ ...(i)

$p - q = k(1 + pq) \quad$ ...(ii)

$xpq = a \quad$ ...(iii)

From Eq. (i) $\; p + q = \dfrac{y}{x}$

$$\therefore \quad p^2 + q^2 + 2pq = \frac{y^2}{x^2} \quad \left[\because pq = \frac{a}{x}\right]$$

$$p^2 + q^2 + \frac{2a}{x} = \frac{y^2}{x^2}$$

$$p^2 + q^2 = \frac{y^2}{x^2} - \frac{2a}{x} \quad \text{...(iv)}$$

Squaring and adding Eqs. (i) and (ii), we have

$$2(p^2 + q^2) = \frac{y^2}{x^2} + k^2\left(1 + \frac{a}{x}\right)^2$$

Hence, from Eq. (iv) expression becomes

$$2\left[\frac{y^2}{x^2}-\frac{2a}{x}\right]=\frac{y^2}{x^2}+k^2\left(1+\frac{a}{x}\right)^2$$

$$2(y^2x-2ax)=y^2+k^2(x+a)^2$$

$$y^2-4ax=k^2(a+x)^2$$

58.
$$4(x^2+y^2)=ax+by \quad \text{...(i)}$$
$$2(x^2-y^2)=ax-by \quad \text{...(ii)}$$
$$xy=c^2 \quad \text{...(iii)}$$

From Eqs. (i) and (ii)

$$ax=3x^2+y^2 \quad \text{...(iv)}$$
$$by=x^2+3y^2 \quad \text{...(v)}$$

Multiplying Eq. (iv) by y and Eq. (v) by x, we get

$$axy=3x^2y+y^3$$
$$ac^2=3x^2y+y^3 \quad \text{...(vi)}$$
$$bxy=3y^2x+x^3 \quad (\because xy=c^2)$$
$$bc^2=3y^2x+x^3 \quad \text{...(vii)}$$

Adding and subtracting Eqs. (vi) and (vii)

$$c^2(a+b)=(x+y)^3$$
$$(x+y)=(a+b)^{1/3}c^{2/3} \quad \text{...(viii)}$$
$$c^2(a-b)=(y-x)^3$$
$$(y-x)=(a-b)^{1/3}c^{2/3} \quad \text{...(ix)}$$

From Eqs. (viii) and (ix)

$$(y+x)^2-(y-x)^2=(a+b)^{2/3}c^{4/3}-(a-b)^{2/3}c^{4/3}$$
$$4xy=c^{4/3}[(a+b)^{2/3}-(a-b)^{2/3}]$$
$$4c^2=c^{4/3}[(a+b)^{2/3}-(a-b)^{2/3}]$$
$$4c^{2/3}=(a+b)^{2/3}-(a-b)^{2/3}$$

59. Multiplying these equations

$$(y+z)^2(z+x)^2(x+y)^2=64a^2b^2c^2(xyz)^2$$
$$\therefore \quad (y+z)(z+x)(x+y)=\pm 8abcxyz$$
$$(yz+xy+zx+z^2)(x+y)=\pm 8abcxyz$$
$$2xyz+y^2z+x^2y+xy^2+zx^2+z^2x+x^2y=\pm 8abcxyz$$
$$2+\frac{y}{x}+\frac{x}{z}+\frac{y}{z}+\frac{x}{y}+\frac{z}{y}+\frac{z}{x}=\pm 8abc \quad \text{...(i)}$$

On dividing by xyz, we get

$$y^2+z^2+2yz=4a^2yz$$

or
$$\frac{y}{z}+\frac{z}{y}+2=4a^2$$

Similarly,
$$\frac{z}{x}+\frac{x}{z}+2=4b^2$$

and
$$\frac{x}{y}+\frac{y}{x}+2=4c^2$$

$$4a^2+4b^2+4c^2=6+\Sigma\frac{x}{y}$$

Hence, from Eq. (i)

$$4a^2+4b^2+4c^2=6\pm 8abc-2$$
$$a^2+b^2+c^2=1\pm 2abc$$
$$a^2+b^2+c^2\pm 2abc=1$$

60.
$$x^2y=a\,;\,x(x+y)=b\,;\,y=c-2x$$

Putting the value of y in first two equations

$$x^2(c-2x)=0$$

or
$$2x^3-cx^2+a=0 \quad \text{...(i)}$$
$$x^2+x(c-2x)=b$$
$$x^2-cx+b=0 \quad \text{...(ii)}$$

Multiplying Eq. (ii) by $2x$ and subtracting from Eq. (i)

$$cx^2-2bx+a=0 \quad \text{...(iii)}$$

From Eqs. (i) and (iii) by cross multiplication

$$\frac{x^2}{ac-2b^2}=\frac{x}{a-bc}=\frac{1}{2b-c^2}$$

$$x=\frac{a-bc}{2b-c^2}\,;\,x^2=\frac{ac-2b^2}{2b-c^2}$$

$$\therefore \quad \left(\frac{a-bc}{2b-c^2}\right)^2=\frac{ac-2b^2}{2b-c^2} \quad [\because x^2=(x)^2]$$

$$(a-bc)^2=(ac-2b^2)(2b-c^2)$$
$$a^2+b^2c^2-2abc=(2abc-4b^3-ac^3+2b^2c^2)$$
$$a^2-4abc+ac^3+4b^3-b^2c^2=0$$

61. Given equations are

$$ax^2+by^2=c \quad \text{...(i)}$$
$$ax+by=c \quad \text{...(ii)}$$
$$\frac{xy}{x+y}=c \quad \text{...(iii)}$$

From Eqs. (i) and (ii), we get

$$c(ax^2+by^2)=c^2=(ax+by)^2$$

$$cax^2 + cby^2 = a^2x^2 + 2abxy + b^2y^2$$

$$a(a-c)x^2 + 2abxy + b(b-c)y^2 = 0 \quad \ldots\text{(iv)}$$

From Eqs. (ii) and (iii), we get

$$ax + by = \frac{xy}{x+y}$$

$$ax^2 + bxy + axy + by^2 = xy$$

$$ax^2 + xy(a+b-1) + by^2 = 0 \quad \ldots\text{(v)}$$

From Eqs. (iv) and (v) by the method of cross multiplication

$$\frac{x^2}{b\{2ab-(b-c)(a+b-1)\}} = \frac{xy}{-ab(a-b)}$$

$$= \frac{y^2}{a\{(a-c)(a+b-1)-2ab\}} = k \quad \text{(say)}$$

$\therefore \quad x^2y^2 = k^2 \cdot ab\{2ab-(b-c)(a+b-1)\}$

$\{(a-c)(a+b-1)-2ab\}$

Also $\quad xy = -kab(a-b)$

$\therefore \quad ab\{2ab-(b-c)(a+b-1)\}$

$\{(a-c)(a+b-1)-2ab\} = (ab)^2(a-b)^2$

or $\quad \{2ab-(b-c)(a+b-1)\}\{2ab-(a-c)$

$(a+b-1)\} = -ab(a-b)^2$

or $\quad 4a^2b^2 - 2ab(a+b-1)(a-c)(b-c)$

$+ (b-c)(a-c)(a+b-1)^2$

$= -ab(a-b)^2$

or $\quad 4a^2b^2 - 2ab(a+b-1)(ab-ab-bc+c^2)$

$+ (ab-bc-ac+c^2)(a+b-1)^2$

$= -ab(a-b)^2$

or $\quad 2ab(a+b-1)(ab-ac-bc+c^2)$

$-(a+b-1)^2(ab-bc-ac+c^2)$

$-4a^2b^2 = ab(a-b)^2$

or $\quad (a+b-1)(ab-ac-bc+c)$

$(2ab-a-b+1) - 4a^2b^2 = ab(a-b)^2$

or $\quad (a+b-c)(ab-ac-bc+c^2)$

$(2ab-a-b+1) = ab(a+b)^2$

or $\quad c^2(a+b-c)^2 - c(a+b-1)$

$(a^2-2ab+b^2-a-b) + ab = 0$

62.

$$ax + yz = bc \quad \ldots\text{(i)}$$

$$by + zx = ca \quad \ldots\text{(ii)}$$

$$cz + xy = ab \quad \ldots\text{(iii)}$$

$$xyz = abc \quad \ldots\text{(iv)}$$

Squaring Eq. (i) and with the help of Eq. (iv), we get

$$a^2x^2 + y^2z^2 + 2a^2bc = b^2c^2 \quad \ldots\text{(v)}$$

$$[\because 2axyz = 2a^2bc]$$

Similarly squaring Eqs. (ii) and (iii), and applying Eq. (iv)

$$b^2y^2 + z^2y^2 + 2b^2ca = c^2a^2 \quad \ldots\text{(vi)}$$

$$c^2z^2 + x^2y^2 + 2c^2ab = a^2b^2 \quad \ldots\text{(vii)}$$

Multiplying Eqs. (i), (ii) and (iii), we get

$$(ax+yz)(by+zx)(cz+xy) = a^2b^2c^2$$

or $\quad a^2b^2c^2 = (abxy + ax^2z + by^2z + xyz^2)$

$(cz+xy)$

$= abcxyz + cax^2z^2 + bcy^2z^2 + cxyz^3 + abx^2y^2$

$+ ax^2yz + bxy^3z + x^2y^2z^2$

$= a^2b^2c^2 + xyz(ax^2+by^2+cz^2) + abx^2y^2$

$+ bcy^2z^2 + cax^2z^2 + x^2y^2z^2$

With the help of Eqs. (iv), (v), (vi), (vii)

$= 2a^2b^2c^2 + abc(ax^2+by^2+cz^2) + abx^2y^2$

$+ bcy^2z^2 + cax^2z^2$

$= 2a^2b^2c^2 + bc(a^2x^2+y^2z^2) + ca(b^2y^2+z^2x^2)$

$+ ab(c^2z^2+x^2y^2)$

$= 2a^2b^2c^2 + bc(b^2c^2-2a^2cb)$

$+ ca(c^2a^2-2b^2ca) + ab(a^2b^2-2abc)^2$

or $\quad a^2b^2c^2 = 2a^2b^2c^2 + b^3c^3 - 2a^2b^2c^2 + c^3a^3$

$- 2a^2b^2c^2 + a^3b^3 - 2a^2b^2c$

or $\quad b^3c^3 + c^3a^3 + a^3b^3 = 5a^2b^2c^2$

63. We have, $y^3 - (x+1)^3 = x^3 + 8x^2 - 6x + 8$

$-(x^3+3x^2+3x+1) = 5x^2 - 9x + 7$

Consider the quadratic equation $5x^2 - 9x + 7 = 0$. The discriminant of this equation is $D = 9^2 - 4 \times 5 \times 7 = -59 < 0$ and hence the expression $5x^2 - 9x + 7$ is positive for all real values of x. We conclude that $(x+1)^3 < y^3$ and hence $x + 1 < y$.

On the other hand we have

$$(x+3)^3 - y^3 = x^2 + 9x^2 + 27x + 27 - (x^3 + 8x^2 - 6x + 8) = x^2 + 33x + 19 > 0$$

for all positive x. We conclude that $y < x + 3$. Thus, we must have $y = x + 2$. Putting this value of y, we get

$$0 = y^3 - (x+2)^3 = x^3 + 8x^2 - 6x + 8 - (x^3 + 6x^2 + 12x + 8) = 2x^2 - 18x$$

We conclude that $x = 0$ and $y = 2$ or $x = 9$ and $y = 11$.

64. Let us consider $x^4 - 2ax^2 + x + a^2 - a = 0$ as a quadratic equation in a. We see that the roots are

$$a = x^2 + x, a = x^2 - x + 1$$

Thus, we get a factorisation

$$(a - x^2 - x)(a - x^2 + x - 1) = 0$$

It follows that $x^2 + x = a$ or $x^2 - x + 1 = a$

Solving these, we get

$$x = \frac{-1 \pm \sqrt{1 + 4a}}{2}$$

or $$x = \frac{-1 \pm \sqrt{4a - 3}}{2}$$

Thus, all the four roots are real if and only if $a \geq \frac{3}{4}$.

65. By setting $u = x^2 + x - 2$ and $v = 2x^2 - x - 1$, we observe that the equation reduces to $u^3 + v^3 = (u+v)^3$. Since $(u+v)^3 = u^3 + v^3 + 3uv(u+v)$, it follows that $uv(u+v) = 0$. Hence, $u = 0$ or $v = 0$ or $u + v = 0$. Thus, we obtain $x^2 + x - 2 = 0$ or $2x^2 - x - 1 = 0$ or $x^2 - 1 = 0$. Solving each of them, we get $x = 1, -2$ or $x = 1, -1/2$ or $x = 1, -1$. Thus, $x = 1$ is a root of multiplicity 3 and the other roots are $-1, -2, -1/2$.

Aliter It can be seen that $x - 1$ is a factor of $x^2 + x - 2, 2x^2 - x - 1$ and $x^2 - 1$. Thus, we can write the equation in the form

$$(x-1)^3(x+2)^3 + (x-1)^3(2x+1)^3 = 27(x-1)^3(x+1)^3$$

Thus, it is sufficient to solve the cubic equation

$$(x+2)^3 + (2x+1)^3 = 27(x+1)^3$$

This can be solved as earlier or expanding every thing and simplifying the relation.

66. If $x \geq 0$, then the given equation assumes the form,

$$x^2 + (a-5)x + 1 = 0 \quad \ldots\text{(i)}$$

If $x < 0$, then it takes the form

$$x^2 + (a+1)x + 1 = 0 \quad \ldots\text{(ii)}$$

For these two equations to have exactly three distinct real solutions we should have,

(I) either $(a-5)^2 > 4$ and $(a+1)^2 = 4$

(II) or $(a-5)^2 = 4$ and $(a+1)^2 > 4$

Case I From $(a+1)^2 = 4$, we have $a = 1$ or -3. But only $a = 1$ satisfies $(a-5)^2 > 4$. Thus, $a = 1$ Also, when $a = 1$, Eq. (i) has solutions $x = 2 + \sqrt{3}$, and Eq. (ii) has solutions $x = -1, -1$. As $2 \pm \sqrt{3} > 0$ and $-1 < 0$, we see that $a = 1$ is indeed a solution.

Case II From $(a-5)^2 = 4$, we have $a = 3$ or 7 Both these values of a satisfy the inequality $(a+1)^2 > 4$. When $a = 3$, Eq. (i) has solutions $x = 1, 1$ and Eq. (ii) has the solutions $x = -2 \pm \sqrt{3}$. As $1 > 0$ and $-2 \pm \sqrt{3} < 0$, we see that $a = 3$ is in fact a solution.

When $a = 7$, Eq. (i) has solutions $x = -1, -1$, which are negative contradicting $x \geq 0$.

Thus, $a = 1, a = 3$ are the two desired values.

67. Since, α and β are the roots of the equation $x^2 + mx - 1 = 0$, we have $\alpha^2 + m\alpha - 1 = 0$, $\beta^2 + m\beta - 1 = 0$. Multiplying by $\alpha^{n-2}, \beta^{n-2}$ respectively we have $\alpha^n + m\alpha^{n-1} - \alpha^{n-2} = 0$ and $\beta^n + m\beta^{n-1} - \beta^{n-2} = 0$.

Adding we obtain

$$\alpha^n + \beta^n = -m(\alpha^{n-1} + \beta^{n-1}) + (\alpha^{n-2} + \beta^{n-2})$$

This gives a recurrence relation for $n \geq 2$

$$\lambda_n = -\lambda_{n-1} + \lambda_{n-2}, n \leq 2 \quad \ldots\text{(i)}$$

(a) Now, $\lambda_0 = 1 + 1 = 2$ and $\lambda_1 = \alpha + \beta = -m$. Thus, λ_0 and λ_1 are integers. By induction, in follows from Eq. (i) that λ_n is an integer for each $n \geq 0$.

(b) We again use Eq. (i) to prove by induction that GCD $(\lambda_n, \lambda_{n+1}) = 1$. This is clearly true for $n = 0$, as GCD $(2, -m) = 1$, by the given condition that m is odd.

Let GCD $(\lambda_{n-2}, \lambda_{n-1}) = 1$, $n \geq 2$. If it were to happen that GCD $(\lambda_{n-1}, \lambda_n) > 1$, take a prime p that divides both λ_{n-1} and λ_n. Then, from Eq. (i), we get that p divides λ_{n-2} also. Thus, p is a factor of GCD $(\lambda_{n-2}, \lambda_{n-1})$, a contradiction. So, GCD $(\lambda_{n-1}, \lambda_n) = 1$. Hence, we have GCD $(\lambda_n, \lambda_{n+1}) = 1, \forall n \geq 0$.

68. Consider the equation $x^2 + ax + b = 0$. It has two roots (not necessarily real), say α and β. Either $\alpha = \beta$ or $\alpha \neq \beta$.

Case I Suppose $\alpha = \beta$, so that α is a double root. Since, $\alpha^2 - 2$ is also a root, the only possibility is $\alpha = \alpha^2 - 2$. This reduces to $(\alpha + 1)(\alpha - 2) = 0$. Hence, $\alpha = -1$ or $\alpha = 2$. Observe that $a = -2\alpha$ and $b = \alpha^2$. Thus, $(a, b) = (2, 1)$ or $(-4, 4)$.

Case II Suppose $\alpha \neq \beta$. There are four possibilities;

(I) $\alpha = \alpha^2 - 2$ and $\beta = \beta^2 - 2$;

(II) $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$;

(III) $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$;

(IV) $\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$

(I) Here, $(\alpha, \beta) = (2, -1)$ or $(-1, 2)$. Hence, $(a, b) = (-(\alpha + \beta), \alpha\beta) = (-1, -2)$.

(II) Suppose $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$. Then,

$$\alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$$

Since, $\alpha \neq \beta$ we get $\beta + \alpha = -1$. However, we also have

$$\alpha + \beta = \beta^2 + \alpha^2 - 4 = (\alpha + \beta)^2 - 2\alpha\beta - 4$$

Thus, $-1 = 1 - 2\alpha\beta - 4$, which implies that $\alpha\beta = -1$.

Therefore, $(a, b) = (-(\alpha + \beta), \alpha\beta) = (1, -1)$

(III) If $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$, then $\alpha = -\beta$. Thus, $\alpha = 2, \beta = -2$ or $\alpha = -1, \beta = 1$. In this case $(a, b) = (0, -4)$ and $(0, -1)$.

(IV) Note that $\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$ is identical to (III), so that we get exactly same pairs (a, b).

Thus, we get 6 pairs;

$$(a, b) = (-4, 4), (2, 1), (-1, -2), (1, -1), (0, -4), (0, -1)$$

69. Let α, β, γ be the roots of the given equation. We have,

$$\alpha + \beta + \gamma = \frac{1}{a}, \alpha\beta + \beta\gamma + \gamma\alpha = \frac{b}{a}, \alpha\beta\gamma = \frac{1}{a}$$

It follows that a, b are positive. We thus obtain,

$$\frac{3b}{a} = 3(\alpha\beta + \beta\gamma + \gamma\alpha) \leq (\alpha + \beta + \gamma)^2 = \frac{1}{a^2},$$

which gives $0 < 3ab \leq 1$. Moreover,

$$\frac{b^2}{a^2} = (\alpha\beta + \beta\gamma + \gamma\alpha)^2$$

$$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + \frac{2}{a^2}$$

Thus,

$$\frac{b^2 - 2}{a^2} = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$$

$$\geq \frac{1}{3}(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \frac{b^2}{3a^2}$$

This implies that $3(b^2 - 2) \geq b^2$ or $b^2 \geq 3$. Hence, $b \geq \sqrt{3}$, the conclusion follows.

70. We have three relations

$$a\alpha^2 - b\alpha - c = \lambda,$$

$$b\alpha^2 - c\alpha - a = \lambda,$$

$$c\alpha^2 - a\alpha - b = \lambda,$$

where λ is the common value. Eliminating α^2 from these, taking these equations pairwise, we get three relations

$$(ca - b^2)\alpha - (bc - a^2) = \lambda(b - a),$$

$$(ab - c^2)\alpha - (ca - b^2) = \lambda(c - b),$$

$$(bc - a^2) - (ab - c^2) = \lambda(a - c)$$

Adding these three, we get

$$(ab + bc + ca - a^2 - b^2 - c^2)(\alpha - 1) = 0$$

(Alternatively, multiplying above relations respectively by $b - c$, $c - a$ and $a - b$ and adding also leads to this.) Thus, either $ab + bc + ca - a^2 - b^2 - c^2 = 0$ or $\alpha = 1$. In the first case

$$0 = ab + bc + ca - a^2 - b^2 - c^2$$

$$= \frac{1}{2}[(a - b^2) + (b - c)^2 + (c - a)^2]$$

Shows that $a = b = c$. If $\alpha = 1$, then we obtain

$$a - b - c = b - c - a = c - a - b,$$

and once again we obtain $a = b = c$.

Level 2

Level 2

1. Given that, $x + y + z = 1 = x^2 + y^2 + z^2$

$$\frac{a}{x}(x - p) = \frac{b}{y}(y - q) = \frac{c}{z}(z - r) = k \quad \text{(say)}$$

$$\frac{a}{x}(x - p) = k$$

or $\quad ax - ap = kx$

$$x(a - k) = ap \quad \therefore \; x = \frac{ap}{a - k}$$

Similarly, $\quad y = \dfrac{bq}{b - k}; \; z = \dfrac{cr}{c - k}$

Now, $\quad 1 = (x + y + z)$

$\therefore \quad 1 = (x + y + z)^2$

$$= x^2 + y^2 + z^2 + 2(yz + zx + xy)$$

But $\quad x^2 + y^2 + z^2 = 1$

$\therefore \quad 2(yz + zx + xy) = 0$

or $\quad yz + zx + xy = 0$

$$\frac{bq}{q - k} \times \frac{cr}{c - k} + \frac{cr}{c - k} \times \frac{ap}{a - k} + \frac{ap}{a - k} \times \frac{bq}{b - k} = 0$$

$$bcqr(a - k) + carp(b - k) + abpq(c - k) = 0$$

$$k(bcqr + carp + abpq) = abc(qr + rp + pq)$$

$$\therefore \quad k = \frac{abc(rp + pq + qr)}{bcqr + carp + abpq}$$

$$\therefore \quad k = \frac{ap}{a - k}$$

$$\therefore \quad x = \frac{ap}{a - \dfrac{abc(rp + pq + ar)}{bcqr + carp + abpq}}$$

$$\left(\because x = \frac{cr}{c - k}\right)$$

$$= \frac{ap(bcqr + carp + abpq)}{a^2p(cr + bq) - abc(rp + pq)}$$

$$= \frac{bcqr + carp + abpq}{a(bq + cr) - bc(q + r)}$$

Similarly, $\quad y = \dfrac{bcqr + carp + abpq}{b(cr + ap) - ca(r + p)}$

$$z = \frac{bcqr + carp + abpq}{c(ap + bq) - ab(p + q)}$$

Substituting these values in $x + y + z = 1$

$$\frac{1}{a(bq + cr) - bc(q + r)} + \frac{1}{b(cr + ap) - ca(r + p)} + \frac{1}{c(ap + bq) - ab(p + a)} = \frac{1}{bcqr + carp + abpq}$$

2. We have,

$$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$$

$$= x^3 + y^3 + \frac{3}{2}(x + y)[(x + y)^2 - (x^2 + y^2)]$$

And so $(x + y)^3 = 3(x + y)(x^2 + y^2) - 2(x^3 + y^3)$

But $\quad x^3 + y^3 = c$

Consequently the result of elimination

$$a^3 = 3ba - 2c$$

3. Let us find the equation whose roots are $\dfrac{1}{\alpha^2}$, $\dfrac{1}{\gamma^2}$ for which we put $y = \dfrac{1}{x^2}$ or $x = \dfrac{1}{\sqrt{y}}$ in the given equation,

$$x(2x^2 + 1) = -(x^2 + 1)$$

or $$\frac{1}{\sqrt{y}}\left(\frac{2}{y} + 1\right) = -\left(\frac{1}{y} + 1\right)$$

Squaring, we get

$$(2 + y)^2 = y(y + 1)^2$$

or $$4 + 4y + y^2 = y^3 + 2y^2 + y$$

or $$y^3 + y^2 - 3y - 4 = 0 \quad \text{...(i)}$$

Its roots are say A, B, C i.e., $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}, \dfrac{1}{\gamma^2}$.

Let $Z = \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} - \dfrac{1}{\alpha^2} = B + C - A = \Sigma A - 2A$

or $\quad Z = -1 - 2A; \quad \therefore \; A = -\dfrac{Z + 1}{2}$

But A is root of Eq. (i)

$$\therefore \quad \frac{-(Z + 1)^3}{8} + \frac{(Z + 1)^2}{4} + \frac{3(Z + 1)}{2} - 4 = 0$$

or $\quad (Z^3 + 3Z^2 + 3Z + 1) - 2(Z^2 + 2Z + 1) - 12(Z + 1) + 32 = 0$

or $\quad Z^3 + Z^2 - 13Z + 19 = 0$

4. Let $y = \dfrac{\alpha}{\beta}$

[There will be six such ratios as $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}, \dfrac{\beta}{\gamma}, \dfrac{\gamma}{\beta}, \dfrac{\gamma}{\alpha}, \dfrac{\alpha}{\gamma}$]

$$\therefore \quad \alpha = y\beta$$

Now, $f(\beta) = 0$ and $f(\alpha) = 0$

$\therefore$ $f(\beta y) = 0$

$$f(\beta) = \beta^3 + q\beta + r = 0$$

$$f(\beta y) = \beta^3 y^3 + q\beta y + r = 0$$

Eliminating β between these two

$$\frac{\beta^3}{qr(1-y)} = \frac{\beta}{r(y^3-1)} = \frac{1}{qy(1-y^2)}$$

$$\therefore \quad \beta = \frac{-r(1+y+y^2)}{qy(1+y)}$$

and $$\beta^3 = \frac{r}{y}\cdot\frac{1}{1+y}$$

Now, $$\beta^3 = (\beta)^3$$

$$\therefore \quad \frac{r}{y}\cdot\frac{1}{1+y} = \frac{-r^3(1+y+y^2)^3}{q^3y^3(1+y)^3}$$

or $$r^2(1+y+y^2)^3 + q^3y^2(1+y)^2 = 0$$

is the required equation of 6th degree.

5. Let us find the equation whose roots are squares of the roots of the given equation. Put

$$y = x^2,$$

$$\therefore \quad x(x^2+11) = 6(x^2+1)$$

or $$\sqrt{y}(y+11) = 6(y+1)$$

Squaring, we get

$$y(y^2+22y+121) = 36(y^2+2y+1)$$

or $$y^3 - 14y^2 + 49y - 36 = 0 \quad \text{...(i)}$$

Let its roots be A, B, C which stand for $\alpha^2, \beta^2, \gamma^2$.

Let $Z = \beta^2 + \gamma^2 = B + C = \Sigma A - A = 14 - A$

$\therefore A = -(Z-14)$. But A is a root of Eq. (i).

$$\therefore \quad -(Z-14)^3 - 14(Z-14)^2 - 49(Z-14) - 36 = 0$$

or $$Z^3 - 3Z^2(14) + 3Z(14)^2 - (14)^3 + 14(z^2 - 28z + 14^2) + 49Z - 49(14) + 36 = 0$$

or $$Z^3 - 28Z^2 + 14^2 Z\left(3 - 2 + \frac{1}{4}\right) - 686 + 36 = 0$$

or $$Z^3 - 28Z^2 + 245Z - 650 = 0.$$

6. $\because$ α, β, γ are the roots of

$$x^3 + px^2 + qx + r = 0 \quad \text{...(i)}$$

$$\therefore \quad \alpha+\beta+\gamma = -p, \alpha\beta+\beta\gamma+\gamma\alpha = q,$$
$$\alpha\beta\gamma = -4$$

If the transformed equation is in terms of y, then

$$y = \beta\gamma - \alpha^2 = \frac{\alpha\beta\gamma}{\alpha} - \alpha^2$$

$$= \frac{-r}{\alpha} - \alpha^2 = -\frac{r}{x} - x^2$$

or $$x^3 + xy + r = 0 \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we have

$$px^2 + (q-y)x = 0$$

$$px + (q-y) = 0 \quad (\because x \neq 0)$$

$$\therefore \quad x = \frac{y-q}{p}$$

Putting this value of x in Eq. (i), we get

$$\frac{(y-q)^3}{p^3} + p\cdot\frac{(y-q)^2}{p^2} + q\cdot\frac{y-q}{p} + r = 0$$

or $$(y-q)^3 + p^2(y-q)^2 + p^2q(y-q) + p^3r = 0$$

or $$y^3 + (y^2 - 3q)y^2 + (3q^2 - p^2q)y + p^3r - q^3 = 0$$

Which is required equation.

7. Roots of the equation

$$x^3 - ax^2 + bx - c = 0 \quad \text{...(i)}$$

are α, β, γ. If y is a root of the transformed equation, then

$$y = \alpha + \beta = (\alpha+\beta+\gamma) - \gamma = a - \gamma$$
$$[\because \alpha+\beta+\gamma = a]$$

$$\therefore \quad y = a - x \text{ or } x = a - y$$

Putting this value of x in Eq. (i), we have

$$(a-y)^3 - a(a-y)^2 + b(a-y) - c = 0$$

or $$y^3 - 2ay^2 + (a^2+b)y + (c-ab) = 0 \quad \text{...(ii)}$$

Which is required equation.

Its roots are $\alpha+\beta, \beta+\gamma, \gamma+\alpha$

Changing y into $\frac{1}{y}$ and multiplying by y^3, we get

$$(c-ab)y^3 + (a^2+b)y^2 - 2ay + 1 = 0 \quad \text{...(iii)}$$

Roots of this equation are the reciprocals of the roots of Eq. (ii).

$\therefore$ Roots of Eq. (iii) are $\frac{1}{\alpha+\beta}, \frac{1}{\beta+\gamma}, \frac{1}{\gamma+\alpha}$.

Now, $\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$ = sum of roots of Eq. (iii)

$$= \frac{-a^2+b}{c-ab} = \frac{a^2+b}{ab-c}$$

8. $$x^2 + 3x + 2 = 0 \quad \text{...(i)}$$

$\because$ Its roots are α, β, γ.

$\therefore$ $\alpha+\beta+\gamma = 0, \alpha\beta+\beta\gamma+\gamma\alpha = 3, \alpha\beta\gamma = -2$

Let y be a root of the transformed equation

$$\therefore \quad y = (\beta-\gamma)^2 = (\beta+\gamma)^2 - 4\beta\gamma$$

$$= (-\alpha)^2 - \frac{4\alpha\beta\gamma}{\alpha} \quad [\because \alpha+\beta+\gamma = 0]$$

$$= \alpha^2 + \frac{8}{\alpha} \quad [\because \alpha\beta\gamma = -2]$$

Replacing α by x,

$$\therefore \quad y = x^2 + \frac{8}{x}$$

or $$x^3 - xy + 8 = 0 \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i)

$$(3+y)x - 6 = 0$$

$$\therefore \quad x = \frac{6}{3+y}$$

Putting this value of x in Eq. (i), we get

$$\left(\frac{6}{3+y}\right)^3 + 3\cdot\frac{6}{3+y} + 2 = 0$$

$$216 + 18(3+y)^2 + 2(3+y)^3 = 0$$

$$y^3 + 18y^2 + 81y + 216 = 0$$

Which is the required equation.

Product of all its roots = – 216

$$\therefore \quad (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 = -216$$

RHS being -ve, one of the factors.

$\because$ That will make all the three roots imaginary which is not possible. Every odd degree equation with real coefficients has at least one real root on the LHS $(\alpha-\beta)^2$ is -ve.

$\therefore$ $\alpha - \beta$ is purely imaginary.

$\therefore$ α and β are conjugate complex roots

Hence, two roots of Eq. (i) are imaginary.

9. (a) $z^3 - (p^2-2q)z^2 - (p^4 - 4p^2q + 8pr)z$
$+ p^4 - 6p^4q + 8p^3r + 8p^2q^2$
$-16pqr + 8r^2 = 0$

(b) $z^3 - 2(p^2-2q)z^2 + (p^4 - 4p^2q + 5q^2 - 2pr)z$
$- (p^2q^2 - 2p^3r + 4pqr - 2q^3 - r^2) = 0$

10. (a) Here, $a_0 = 1, a_1 = -3, a_2 = 5, a_3 = -12, a_4 = 4$

$\therefore$ $a_0s_1 + a_1 = 0 \Rightarrow s_1 - 3 = 0$

$\therefore$ $s_1 = 3$

$a_0s_2 + a_1s_1 + 2a_2 = 0$

$\Rightarrow$ $s_2 - 3(3) + 10 = 0$

$\therefore$ $s_2 = -1$

$a_0s_3 + a_1s_2 + a_2s_1 + 3a_3 = 0$

$\Rightarrow$ $s_3 - 3(-1) + 5(3) - 36 = 0$

or $s_3 = -3 - 15 + 36 = 18$

If $\alpha, \beta, \gamma, \delta$ are the roots of the given equation, then putting $x = \alpha, \beta, \gamma, \delta$ in succession and adding.

$s_4 - 3s_3 + 5s_2 - 12s_1 + 16 = 0$

$\therefore$ $s_4 - 54 - 5 - 36 + 16 = 0$

$s_4 = 79$

Multiplying both sides of given equation by x

$$x^5 - 3x^4 + 5x^3 - 12x^2 + 4x = 0$$

Putting $x = \alpha, \beta, \gamma, \delta$ in succession and adding

$s_5 - 3s_4 + 5s_3 - 12s_2 + 4s_1 = 0$

or $s_5 - 237 + 90 + 12 + 12 = 0$

$\therefore$ $s_5 = 123$

(b) $S_5 = -1$

11. 45

12. Here, $a_0 = 1, a_1 = 0, a_2 = 2, a_3 = 6$

$a_0S_1 + a_1 = 0 \Rightarrow S_1 = 0$

$a_0S_2 + a_1S_1 + 2a_2 = 0 \Rightarrow S_2 = -4$

$a_0S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0$

$\Rightarrow$ $S_3 = -18$

Now, given equation can be written as

$$x^3 = -2x^2 - 6x \quad \text{...(i)}$$

Multiplying both sides by x, we get

$$x^4 = -2x^2 - 6x$$

Putting $x = \alpha, \beta, \gamma$ in sucession and adding

$$S_4 = -2S_2 - 6S_1 = 8$$

$\therefore \quad S_4 = 8$

Squaring Eq. (i), we get

$$x^6 = 4x^2 + 24x + 36 \quad \ldots(ii)$$

Putting $x = \alpha, \beta, \gamma$ in sucession and adding

$$S_6 = 4S_2 + 24S_1 + 108$$
$$= -16 + 108$$

$\therefore \quad S_6 = 92$

Multiplying both sides of Eq. (ii) by x

$$x^7 = 4x^3 + 24x^2 + 36x$$

Put $x = \alpha, \beta, \gamma$ and adding, we get

$$S_7 = 4S_3 + 24S_2 + 36S_1$$

$\therefore \quad S_7 = -72 - 96 = -168$

Also, $\quad 2(S_4 - S_6) = 2(8 - 92)$
$$= 2(-84) = -168$$
$$S_7 = 2(S_4 - S_6)$$

19. Here, $x > 0$ and $x \neq 1$

Let $\quad \log_2 x = p$ as $x \neq 1, p \neq 0$

Given inequality becomes $p + \frac{1}{p} + 2\cos y \leq 0$

i.e., $\quad \frac{p^2 + 1 + 2p\cos y}{p} \leq 0$

Case I When $p > 0$

$$p^2 + 1 + 2p\cos y \leq 0, \forall y \text{ and } p > 0$$
$$(p - 1)^2 + 2p(1 + \cos y) \leq 0$$

$\because \quad p > 0 \quad \ldots(i)$

$\therefore \quad 1 + \cos y \geq 0, (p - 1)^2 \geq 0$

$$(p - 1)^2 + 2p(1 + \cos y) \geq 0 \quad \ldots(ii)$$

Eqs. (i) and (ii) satisfied when

$$(p - 1)^2 + 2p(1 + \cos y) = 0$$

$\because (p - 1)^2 \geq 0$ and $2p(1 + \cos y) \geq 0$, we get

$$(p - 1)^2 = 0 \text{ and } 2p(1 + \cos y) = 0$$
$$p = 1 \text{ and } \cos y = -1$$
$$y = (2n + 1)\pi$$

Solution set is $x = 2, y = (2n + 1)\pi$

Case II When $p < 0$

$$p^2 + 1 + 2p\cos y \geq 0$$
$$(p + 1)^2 - 2p(1 - \cos y) \geq 0$$
$$(p + 1)^2 \geq 0 \text{ and } -p(1 - \cos y) \geq 0, \forall y$$

$\therefore$ Solution set is $0 < x < 1$ and all $y \in R$.

20. We have, $(a - b)^2 + (a - c)^2 = (b - c)^2$

$\Rightarrow \quad 2a^2 - 2ab - 2ac + 2bc = 0$

$\Rightarrow \quad a^2 - a(b + c) + bc = 0$

$\Rightarrow \quad (a - b)(a - c) = 0$

$\Rightarrow \quad a = b$ or $a = c$

The equation has no solution if a, b, c are all distinct.

21. We have, $(xy - 1)^2 = (x + 1)^2 + (y + 1)^2$

$\Rightarrow \quad (xy - 1)^2 - (x + 1)^2 = (y + 1)^2$

$\Rightarrow \quad (xy - x - 2)(xy + x) = (y + 1)^2$

$\Rightarrow \quad x(xy - x - 2)(y + 1) = (y + 1)^2 \quad \ldots(i)$

$\Rightarrow \quad (y + 1)[x(xy - x - 2) - (y + 1)] = 0 \quad \ldots(ii)$

If $y = -1$, then x takes all values from the set of integers.

Similarly, we also get

$$(x + 1)[y(xy - y - 2) - (x + 1)] = 0 \quad \ldots(iii)$$

If $x = -1$, then y takes all values from the set of integers.

If $x \neq -1$ and $y \neq -1$, then from Eq. (i)

$$x(xy - x - 2)(y + 1) = (y + 1)^2$$
$$x(xy - x - 2) = y + 1 \quad (\because y \neq -1)$$

$\Rightarrow \quad x^2y - x^2 - 2x - y - 1 = 0$

$\Rightarrow \quad y(x - 1)(x + 1) = (x + 1)^2$

Since, $x \neq -1$, we have

$$y(x - 1) = (x + 1)$$

$\Rightarrow \quad y = \frac{x + 1}{x - 1}$ is an integer.

If $x = 0$, then $y = -1$ and when $x = 2, y = 3$.

When $x = 3, y = 2$ for all other values of x, y is not an integer.

Hence, solution set is

$$(3, 2), (2, 3), (x, -1), (1, y)$$

22. If $x > 1$, then $y = x^3 + 3x(x^2 - 1)$

$$y > x^3 > x > 1$$
$$z = 4y^3 - 3y = y^3 + 3y(y^2 - 1) > y^3 > y > 1$$
$$x = 4z^3 - 3z = z^3 + 3z(z^2 - 1) > z^3 > z > 1$$

Thus, $x > y > z > x$ which is impossible and again $x < -1$ and lead to

$$x < z < y < x \text{ so } x < -1$$

so $\quad |x| \le 1, |y| \le 1, |z| \le 1$

We can write $x = \cos\theta$, where $0 \le \theta \le \pi$

Now, $\quad y = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$

$$z = 4y^3 - 3y = 4\cos^3 3\theta - 3\cos 3\theta$$

$$= \cos 3 \times 3\theta = \cos 9\theta$$

$$x = 4z^3 - 3z = 4\cos^3 9\theta - 3\cos 9\theta$$

$$= \cos 3 \times 9\theta = \cos 27\theta$$

Since, trigonometric functions are periodic, it is possible.

Thus, $\cos\theta = \cos 27\theta \Rightarrow \cos\theta - \cos 27\theta = 0$

$\Rightarrow \quad 2\sin 14\theta \sin 13\theta = 0$

$\Rightarrow \quad \sin 14\theta = 0$ or $\sin 13\theta = 0$

So, $\quad \theta = \frac{k\pi}{13}$, where $k = 0, 1, 2, \dots, 12$

$\theta = \frac{k\pi}{14}$, where $k = 0, 1, 2, \dots, 13$

Solution is $(x, y, z) = (\cos\theta, \cos 3\theta, \cos 9\theta)$

θ takes all above values.

23. We are given,

$$x_1^3x_3 + x_2^3x_1 + x_3^3x_2 = -8x_1x_2x_3$$

$$x_1 + x_2 + x_3 = 0; x_1x_2 + x_2x_3 + x_3x_1 = a;$$

$$x_1x_2x_3 = -a$$

and for $i = 1, 2, 3, \dots$

$$x_i^3 + ax_i + a = 0$$

Now, $\quad x_1^3 + ax_1 + a = 0$

$$x_2^3 + ax_2 + a = 0$$

$$x_3^3 + ax_3 + a = 0$$

$\Rightarrow \quad (x_1^3x_3 + x_2^3x_1 + x_3^3x_2) + a(x_1x_3 + x_2x_1 + x_3x_2) + a(x_3 + x_1 + x_2) = 0$

i.e., $\quad 8a + a^2 = 0 \Rightarrow a = -8$

So, given equation is $x^3 - 8x - 8 = 0$,

One root is -2, other roots are given by

$$x^2 - 2x - 4 = 0$$

i.e., $\quad x = 1 \pm \sqrt{5}$

So, $\quad \{x_1, x_2, x_3\} = \{-2, 1 - \sqrt{5}, 1 + \sqrt{5}\}$

24. If (x, y) is a solution, then $(-x, y)$ is also a solution

$\because$ There should be only one solution, $x = 0$

$\therefore \quad y + a = 1$ and $y^2 = 1$

so $\quad y = \pm 1$

$\Rightarrow a = 2$ or 0, when $a = 2$, $(1, 0)$ and $(-1, 0)$ are solutions.

$\therefore$ We have to consider only the possibility that $a = 0$, we have

$$2^{|x|} + |x| = y + x^2 \text{ and } x^2 + y^2 = 1$$

$\therefore \quad x^2, y^2 \le 1 \Rightarrow x^2 \le |x|$ and $y \le 1$

Now, $\quad 2^{|x|} + |x| \ge 1 + |x| \ge 1 + x^2 \ge y + x^2$

$\Rightarrow \quad y = 1$

$\therefore \quad x = 0$

Thus, given system has a unique solution if and only if $a = 0$.

25. Roots of the equation $p(x)q(x) = 0$

i.e., $\quad (ax^2 + bx + c)(-ax^2 + bx + c) = 0$

will be roots of the equations.

$$ax^2 + bx + c = 0 \quad \text{...(i)}$$

$$-ax^2 + bx + c = 0 \quad \text{...(ii)}$$

If D_1 and D_2 be discriminants of Eqs. (i) and (ii), then

$$D_1 = b^2 - 4ac \text{ and } D_2 = b^2 + 4ac$$

Now, $\quad D_1 + D_2 = 2b^2 \ge 0 \quad (\because b$ may zero)

i.e., $\quad D_1 + D_2 \ge 0$

At least one of D_1 and $D_2 \ge 0$.

i.e., atleast one of the Eqs. (i) and (ii) has real roots.

$\therefore p(x)q(x) = 0$ has at least two real roots.

26. We have,

$$x^2 + 2ax + \frac{1}{16} = -a + \sqrt{\left(a^2 + x - \frac{1}{16}\right)}$$

Let $\quad y = x^2 + 2ax + \frac{1}{16} \quad$ ...(i)

and $\quad y_1 = -a + \sqrt{\left(a^2 + x - \frac{1}{16}\right)} \quad$ ...(ii)

$\therefore \quad y = y_1 \quad$ ...(iii)

From Eq. (ii) we have

$$y_1 + a = \sqrt{\left(a^2 + x - \frac{1}{16}\right)}$$

$\Rightarrow \quad x = y_1^2 + 2ay_1 + \frac{1}{16}$

$x = y^2 + 2ay + \frac{1}{16} \quad$ ...(iv)

[from Eq. (iii)]

Eqs. (i) and (ii) represents parabolas. Both parabolas are symmetrical about the line

$$y = x \quad \text{...(v)}$$

From Eqs. (i) and (v), we get

$$x = x^2 + 2ax + \frac{1}{16}$$

$$\Rightarrow \quad x^2 - (1 - 2a)x + \frac{1}{16} = 0$$

$$x = \frac{(1 - 2a) \pm \sqrt{(1 - 2a)^2 - \frac{1}{4}}}{2}$$

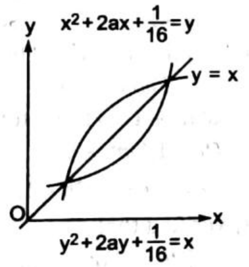
For real roots, $(1 - 2a)^2 - \frac{1}{4} \geq 0$

$$1 - 2a > \frac{1}{2} \Rightarrow a > \frac{1}{4}$$

Hence, $x_1 = \frac{(1 - 2a)}{2} + \sqrt{\left(\frac{1 - 2a}{2}\right)^2 - \frac{1}{16}}$

$$x_2 = \left(\frac{1 - 2a}{2}\right) - \sqrt{\left(\frac{1 - 2a}{2}\right)^2 - \frac{1}{16}}$$

are real roots and given fact satisfy original equation.



27. Given system of equation can be written as

$$ax_1^2 + (b - 1)x_1 + c = x_2 - x_1 = f(x_1) \quad \text{(say)}$$

$$ax_2^2 + (b - 1)x_2 + c = x_3 - x_2 = f(x_2) \quad \text{(say)}$$

$$\vdots$$

$$ax_{n-1}^2 + (b - 1)x_{n-1} + c = x_n - x_{n-1}$$

$$= f(x_{n-1}) \quad \text{(say)}$$

$$ax_n^2 + (b - 1)x_n + c = x_1 - x_n = f(x_n) \quad \text{(say)}$$

$$f(x_1) + f(x_2) + \dots + f(x_n) = 0 \quad \text{...(i)}$$

Case I When $(b - 1)^2 < 4ac$

Each roots of $ax_1^2 + (b - 1)x_1 + c = 0$ are imaginary.

If $a > 0$, then $f(x_1) + f(x_2) + \dots + f(x_n) > 0$

If $a < 0$, then $f(x_1) + f(x_2) + \dots + f(x_n) \neq 0$

$\therefore$ No solution.

Case II When $(b - 1)^2 = 4ac$

In case 1 and 2 all of

$$f(x_1), f(x_2), \dots, f(x_n) \geq 0$$

$$f(x_1), f(x_2), \dots, f(x_n) \leq 0$$

From Eq. (i),

$$f(x_1) + f(x_2) + \dots + f(x_n) = 0$$

$$f(x_1) = f(x_2) = \dots = f(x_n) = 0$$

But $f(x_i) = 0 \Rightarrow ax_i^2 + (b - 1)x_i + c = 0$

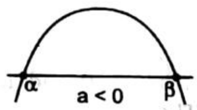
$$x_i = \frac{-(b - 1) \pm 0}{2a} \quad [\because (b - 1)^2 = 4ac]$$

$$= \frac{1 - b}{2a}$$

Hence, $x_1 = x_2 = \dots = x_n = \frac{1 - b}{2a}$

Case III When $(b - 1)^2 > 4ac$

Roots of $ax_i^2 + (b - 1)x_i + c = 0$ are real and unequal.



Let α and β be roots.

If $a < 0$, $\forall x_1 \in [\alpha, \beta]$

$$ax_1^2 + (b - 1)x_1 + c \geq 0$$

i.e., $f(x_1) \geq 0$

Similarly for all $x_i \in [\alpha, \beta]$, $(i = 1, 2, 3, \dots, n)$

i.e., $f(x_i) \geq 0$

But $f(x_1) + f(x_2) + \dots + f(x_n) = 0$

$$f(x_1) = f(x_2) = \dots = f(x_n) = 0$$

$\therefore$ $x_1 = x_2 = \dots = x_n$

then each $ax_i^2 + (b - 1)x_i + c = 0$

So, $x_1 = x_2 = \dots = x_n = \frac{-(b - 1) \pm \sqrt{(b - 1)^2 - 4ac}}{2a}$.

Also, $\forall x_1 \notin (\alpha, \beta) (i = 1, 2, 3, \dots, n)$

i.e., $f(x_i) \leq 0$

but $f(x_1) + f(x_2) + \dots + f(x_n) = 0$

So, $f(x_1) = f(x_2) = \dots = f(x_n)$

$\therefore \quad x_1 = x_2 = \dots = x_n$

then each $ax_i^2 + (b-1)x_i + c = 0$

So, $x_1 = x_2 = \dots = x_n$

$$= \frac{-(+b-1) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$$

when $a > 0$, we get

$$x_1 = x_2 = \dots = x_n$$

$$= \frac{(1-b) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$$

28. Let $p(x) \equiv (x-2)\,q(x) + p(2)$, $q(x)$ would have integer coefficients.

Let r be an integer such that $p(r) = 0$

Then, $p(r) = (r-2) \cdot q(r) + 13 = 0$

so $\quad r - \frac{2}{13}$

Thus, $r - 2$ can only equal $\pm$ or ± 13.

Leading to $r = 3, 1, 15$, or -11.

Let $\quad P(x) \equiv (x-10) \cdot F(x) + P(10)$

Leads to $r - \frac{10}{5}$. So, r can only be 11, 9, 15 or 5.

Thus, $r = 15$.

29. Suppose $a + b + c \neq 0$ and let the common value be λ. Then,

$$\lambda = \frac{xb + (1-x)c + xc + (1-x)a + xa + (1-x)b}{a+b+c}$$

$$= 1$$

We get two equations

$$-a + xb + (1-x)c = 0,$$

$$(1-x)a - b + xc = 0$$

(The other equation is a linear combination of these two.) Using these two equations, we get the relations

$$\frac{a}{1 - x + x^2} = \frac{b}{x^2 - x + 1} = \frac{c}{(1-x)^2 + x}$$

Since, $1 - x + x^2 \neq 0$, we get $a = b = c$

30. Eliminating z from the given set of equations, we get

$$x^3 + y^3 + \{1 - (x+y)\}^2 = 1$$

This factors to

$$(x+y)(x^2 - xy + y^2 + x + y - 2) = 0$$

Case I Suppose $x + y = 0$. Then, $z = 1$ and $(x, y, z) = (m, -m, 1)$, where m is an integer give one family of solutions.

Case II Suppose $x + y \neq 0$. Then, we must have

$$x^2 - xy + y^2 + x + y - 2 = 0$$

This can be written in the form

$$(2x - y + 1)^2 + 3(y+1)^2 = 12$$

Here, there are two possibilities

$$2x - y + 1 = 0, y + 1 = \pm 2;$$

$$2x - y + 1 = \pm 3, y + 1 = \pm 1$$

Analysing all these cases, we get

$$(x, y, z) = (0, 1, 0), (-2, -3, 6), (1, 0, 0), (0, -2, 3), (-2, 0, 3), (-3, -2, 6).$$

31. We seek solutions (x, y, z) which are in arithmetic progression. Let us put $y - x = z - y = d > 0$ so that the equation reduces to the form

$$3y^2 + 2d^2 = 2d^3.$$

Thus, we get $3y^2 = (d-1)d^2$. We conclude that $2(d-1)$ is 3 times a square. This is satisfied if $d - 1 = 6n^2$ for some n. Thus $d = 6n^2 + 1$ and $3y^2 = d^2 \cdot 2(6n^2)$ giving us $y^2 = 4d^2n^2$. Thus, we can take $y = 2dn = 2n(6n^2 + 1)$. From this we obtain $x = y - d = (2n-1)(6n^2+1)$,

$$z = y + d = (2n+1)(6n^2+1).$$

It is easily verified that

$$(x, y, z) = [(2n-1)(6n^2+1), 2n(6n^2+1), (2n+1)(6n^2+1)]$$

is indeed a solution for a fixed n and this gives an infinite set of solutions as n varies over natural numbers.

32. Substituting $\quad x = y + (1/2) \quad \dots$(i)

in the equation, we obtain the equation in y

$$8y^4 + 4y^2 + a - \frac{3}{2} = 0 \quad \dots\text{(ii)}$$

Using the transformation $z = y^2$, we get a quadratic equation in z

$$8z^2 + 4z + a - \frac{3}{2} = 0 \quad \dots\text{(iii)}$$

The discriminant of this equation is $32(2-a)$ which is non-negative if and only if $a \leq 2$. For $a \leq 2$, we obtain the roots

$$z_1 = \frac{-1 + \sqrt{2(2-a)}}{4}$$

$$z_2 = \frac{-1 - \sqrt{2(2-a)}}{4}$$

For getting real y we need $z \geq 0$. Obviously $z_2 < 0$ and hence it gives only non-real values of y. But $z_1 \geq 0$ if and only if $a \leq \frac{3}{2}$. In this case we obtain two real values for y and hence two real roots for the original Eq. (i). Thus, we conclude that there are two real roots and two non-real roots for $a \leq \frac{3}{2}$ and four non-real roots for $a > \frac{3}{2}$. Obviously the sum of all the roots of the equation is 2. For $a \leq \frac{3}{2}$, two real roots of Eq. (ii) are given by $y_1 = +\sqrt{z_1}$ and $y_2 = -\sqrt{z_1}$. Hence, the sum of real roots of Eq. (i) is given by $y_1 + \frac{1}{2} + y_2 + \frac{1}{2}$ which reduces to 1. It follows the sum of the non-real roots of Eq. (i) for $a \leq \frac{3}{2}$ is also 1. Thus,

$$\text{The sum of non-real roots} = \begin{cases} 1, & \text{for } a \leq \frac{3}{2} \\ 2, & \text{for } a > \frac{3}{2} \end{cases}$$

33. Suppose α is a real root of the given equation. Then,

$$\alpha^5 - \alpha^3 + \alpha - 2 = 0 \qquad \ldots(\text{i})$$

This gives $\alpha^5 - \alpha^3 + \alpha - 1 = 1$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) = 1$. Observe that $\alpha^4 + \alpha^3 + 1 \geq 2\alpha^2 + \alpha^3 = \alpha^2(\alpha + 2)$. If $-1 \leq \alpha < 0$, then $\alpha + 2 > 0$, giving $\alpha^2(\alpha + 2) > 0$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$.

If $\alpha < -1$, then $\alpha^4 + \alpha^3 = \alpha^3(\alpha + 1) > 0$ and hence $\alpha^4 + \alpha^3 + 1 > 0$.

This again gives $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$.

The above resoning shows that for $\alpha < 0$, we have $\alpha^5 - \alpha^3 + \alpha - 1 < 0$ and hence cannot be equal to 1. We conclude that a real root α of $x^5 - x^3 + x - 2 = 0$ is positive (obviously $\alpha \neq 0$).

Now, using $\alpha^5 - \alpha^3 + \alpha - 2 = 0$, we get

$$\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$$

The statement $[\alpha^6] = 3$ is equivalent to $3 \leq \alpha^6 < 4$.

Consider $\alpha^4 - \alpha^2 + 2\alpha < 4$. Since, $\alpha > 0$, this is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha$. Using the relation (i), we can write $2\alpha^2 - \alpha + 2 < 4\alpha$ or $2\alpha^2 - 5\alpha + 2 < 0$. Treating this as a quadratic, we get this is equivalent to $\frac{1}{2} < \alpha < 2$. Now, observe that if $\alpha \geq 2$, then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) \geq 25$ which is impossible. If $0 < \alpha \leq \frac{1}{2}$, then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$ which again is impossible. We conclude that $\frac{1}{2} < \alpha < 2$. Similarly, $\alpha^4 - \alpha^2 + 2\alpha \geq 3$ is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0$ which is equivalent to $2\alpha^2 - 4\alpha + 2 \geq 0$. But this is $2(\alpha - 1)^2 \geq 0$ which is valid. Hence, $3 \leq \alpha^6 < 4$ and we get $[\alpha^6] = 3$.

34. We write the equation in the form

$$a^2 + 2ap + p^2 + b(3a + 2p) = 0$$

Hence,

$$b = \frac{-(a+p)^2}{3a + 2p}$$

is an integer. This shows that $3a + 2p$ divides $(a + p)^2$ and hence also divides $(3a + 3p)^2$. But, we have

$$(3a + 3p)^2 = (3a + 2p + p)^2$$
$$= (3a + 2p)^2 + 2p(3a + 2p) + p^2$$

It follows that $3a + 2p$ divides p^2. Since, p is a prime, the only divisors of p^2 are ± 1, $\pm p$ and $\pm p^2$.

Since, $p > 3$, we also have $p = 3k + 1$ or $3k + 2$.

Case I Suppose $p = 2k + 1$. Obviously $3a + 2p = 1$ is not possible. Infact, we get $1 = 3a + 2p = 3a + 2(3k + 1) \Rightarrow 3a + 6k = -1$ which is impossible. On the other hand $3a + 2p = -1$ gives $3a = -2p - 1 = -6k - 3 \Rightarrow a = -2k - 1$ and $a + p = -2k - 1 + 3k + 1 = k$.

Thus, $b = \frac{-(a+p)^2}{(3k + 2p)} = k^2$. Thus, $(a, b) = (-2k - 1, k^2)$ when $p = 3k + 1$. Similarly, $3a + 2p = p \Rightarrow 3a = -p$ which is not possible. Considering $3a + 2p = -p$, we get $3a = -3p$ or $a = -p \Rightarrow b = 0$. Hence, $(a, b) = (-3k - 1, 0)$ where $p = 3k + 1$.

Let us consider $3a + 2p = p^2$. Hence, $3a = p^2 - 2p = p(p-2)$ and neither p nor $p-2$ is divisible by 3. If $3a + 2p = -p^2$, then $3a = -p(p+2) \Rightarrow a = -(3k+1)(k+1)$.
Hence, $a + p = (3k+1)(-k-1+1) = -(3k+1)k$
This gives $b = k^2$. Again $(a, b) = (-(k+1)(3k+1), k^2)$ when $p = 3k+1$.

Case II Suppose $p = 3k - 1$. If $3a + 2p = 1$, then $3a = -6k + 3$ or $a = -2k + 1$. We also get

$$b = \frac{-(a+p)^2}{1}$$
$$= \frac{-(-2k+1+3k-1)^2}{1} = -k^2$$

and we get the solution $(a, b) = (-2k+1, k^2)$. On the other hand $3a + 2p = -1$ does not have any solution integral solution for a. Similarly, there is no solution in the case $3a + 2p = p$. Taking $3a + 2p = -p$, we get $a = -p$ and hence $b = 0$. We get the solution $(a, b) = (-3k+1, 0)$. If $3a + 2p = p^2$, then $3a = p(p-2) = (3k-1)(3k-3)$ giving $a = (3k-1)(k-1)$ and hence $a + p = (3k-1)(1+k-1) = k(3k-1)$. This gives $b = -k^2$ and hence $(a, b) = (3k-1, -k^2)$. Finally $3a + 2p = -p^2$ does not have any solution.

35. Consider the discriminants of the three equations

$$px^2 + qx + r = 0 \qquad \text{...(i)}$$
$$qx^2 + rx + p = 0 \qquad \text{...(ii)}$$
$$rx^2 + px + q = 0 \qquad \text{...(iii)}$$

Let us denote them by D_1, D_2, D_3 respectively. Then, we have

$$D_1 = 4(q^2 - rp)$$
$$D_2 = 4(r^2 - pq)$$
$$D_3 = 4(p^2 - qr)$$

We observe that

$$D_1 + D_2 + D_3 = 4(p^2 + q^2 + r^2 - pq - qr - rp)$$
$$= 2\{(p-q)^2 + (q-r)^2 + (r-p)^2\} > 0$$

since p, q, r are not all equal. Hence, atleast one of D_1, D_2, D_3 must be positive. We may assume $D_1 > 0$.

Suppose $D_2 < 0$ and $D_3 < 0$. In this case both the Eqs. (ii) and (iii) have only non-real roots and Eq. (i) has only real roots. Hence, the common root α must be between Eqs. (ii) and (iii). But then $\bar{\alpha}$ is the other root of both Eqs. (ii) and (iii). Hence, it follows that Eqs. (ii) and (iii) have same set of roots. This implies that

$$\frac{q}{r} = \frac{r}{p} = \frac{p}{q}$$

Thus $p = q = r$ contradicting the given condition. Hence, both D_2 and D_3 cannot be negative. We may assume $D_2 \geq 0$. Thus, we have,

$$q^2 - rp > 0, r^2 - pq \geq 0.$$

These two give

$$q^2r^2 > p^2qr$$

since p, q, r are all positive. Hence, we obtain $qr > p^2$ or $D_3 < 0$. We conclude that the common root must be between Eqs. (i) and (ii). Thus,

$$p\alpha^2 + q\alpha + r = 0$$
$$q\alpha^2 + r\alpha + p = 0$$

Eliminating α^2, we obtain

$$2(q^2 - pr)\alpha = p^2 - qr$$

Since, $q^2 - pr > 0$ and $p^2 - qr < 0$, we conclude that $\alpha < 0$.

The condition $p^2 - qr < 0$ implies that the Eq. (iii) has only non-real roots.

Aliter One can argue as follows. Suppose α is a common root of two equations, say, Eqs. (i) and (ii). If α is non-real, then $\bar{\alpha}$ is also a root of both Eqs. (i) and (ii). Hence, the coefficients of Eqs. (i) and (ii) are proportional. This forces $p = q = r$, a contradiction. Hence, the common root between any two equations cannot be non-real. Looking at the coefficients, we conclude that the common root α must be negative. If Eqs. (i) and (ii) have common root α, then $q^2 \geq rp$ and $r^2 \geq pq$. Here, atleast one inequality is strict for $q^2 = pr$ and $r^2 = pq$ forces $p = q = r$. Hence, $q^2r^2 > p^2qr$. This gives $p^2 < qr$ and hence Eq. (iii) has non-real roots.

36. If α and β are both integers, then
$[m\alpha] + [m\beta] = m\alpha + m\beta = m(\alpha + \beta) = m^2$
This proves one implication.

Observe that $\alpha + \beta = m$ and $\alpha\beta = n$. We use the property of integer function
$x - 1 < [x] \leq x$ for any real number x. Thus,

$$m^2 - 2 = m(\alpha + \beta) - 2$$
$$= m\alpha - 1 + m\beta - 1 < [m\alpha] + [m\beta] \leq m(\alpha + \beta)$$
$$= m^2$$

Since, m and n are positive integers, both α and β must be positive. If $m \geq 2$, we observe that there is no square between $m^2 - 2$ and m^2. Hence, either $m = 1$ or $[m\alpha] + [m\beta] = m^2$. If $m = 1$, then $\alpha + \beta = 1$ implies that both α and β are positive reals smaller than 1. Hence, $n = \alpha\beta$ cannot be a positive integer. We conclude that $[m\alpha] + [m\beta] = m^2$. Putting $m = \alpha + \beta$ in this relation, we get

$$[\alpha^2 + n] + [\beta^2 + n] = (\alpha + \beta)^2$$

Using $[x + k] = [x] + k$ for any real number x and integer k, this reduces to

$$[\alpha^2] + [\beta^2] = \alpha^2 + \beta^2$$

This shows that α^2 and β^2 are both integers. On the other hand,

$$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = m(\alpha - \beta)$$

Thus, $$(\alpha - \beta) = \frac{\alpha^2 - \beta^2}{m},$$

is a rational number. Since, $\alpha + \beta = m$ is a rational number, it follows that both α and β are rational numbers. However, both α^2 and β^2 are integers. Hence, each of α and β is an integer.

37. We begin with the standard factorisation

$$y^4 + 4 = (y^2 - 2y + 2)(y^2 + 2y + 2)$$

Thus, we have $y^2 - 2y + 2 = p^m$ and $y^2 + 2y + 2 = p^n$ for some positive integers m and n such that $m + n = x$. Since, $y^2 - 2y + 2 < y^2 + 2y + 2$, we have $m < n$ so that p^m divides p^n. Thus, $y^2 - 2y + 2$ divides $y^2 + 2y + 2$. Writing

$$y^2 + 2y + 2 = y^2 - 2y + 2 + 4y,$$

we infer that $y^2 - 2y + 2$ divides $4y$ and hence $y^2 - 2y + 2$ divides $4y^2$. But

$$4y^2 = 4(y^2 - 2y + 2) + 8(y - 1)$$

Thus, $y^2 - 2y + 2$ divides $8(y - 1)$. Since, $y^2 - 2y + 2$ divides both $4y$ and $8(y - 1)$, we conclude that it also divides 8. This gives $y^2 - 2y + 2 = 1, 2, 4$ or 8.

If $y^2 - 2y + 2 = 1$, then $y = 1$ and $y^4 + 4 = 5$, giving $p = 5$ and $x = 1$. If $y^2 - 2y + 2 = 2$, then $y^2 - 2y = 0$ giving $y = 2$. But then $y^4 + 4 = 20$ is not the power of a prime. The equations $y^2 - 2y + 2 = 4$ and $y^2 - 2y + 2 = 8$ have no integer solutions. We conclude that $(p, x, y) = (5, 1, 1)$ is the only solution.

Aliter Using $y^2 - 2y + 2 = p^m$ and $y^2 + 2y + 2 = p^n$, we may get

$$4y = p^m (p^{n-m} - 1)$$

If $m > 0$, then p divides 4 or y. If p divides 4, then $p = 2$. If p divides y, then $y^2 - 2y + 2 = p^m$ shows that p divides 2 and hence $p = 2$. But then $2^x = y^4 + 4$, which shows that y is even. Taking $y = 2z$, we get $2^{x-2} = 4x^4 + 1$. This implies that $z = 0$ and hence $y = 0$, which is a contradiction. Thus, $m = 0$ and $y^2 - 2y + 2 = 1$. This gives $y = 1$ and hence $p = 5, x = 1$.

38. Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial with integer coefficients.

Case I We may write

$$P(x) = a_0 + a_2x^2 + a_4x^4 + \ldots$$
$$+ x(a_1 + a_3x^2 + a_0x^5 + \ldots).$$

Define

$$Q(x) = a_0 + a_2x^2 + a_4x^4 + \ldots$$
$$- x(a_1 + a_3x^2 + a_5x_5 + \ldots).$$

Then, $Q(x)$ is also a polynomial with integer coefficients and

$$P(x)Q(x) = (a_0 + a_2x^2 + a_4x^4 + \ldots)^2$$
$$- x^2(a_1 + a_3x^2 + a_5x^5 + \ldots)^2$$

is a polynomial in x^2.

Case II We write again

$$P(x) = A(x) + xB(x) + x^2C(x),$$

where

$$A(x) = a_0 + a_3x^3 + a_6x^6 + \ldots,$$
$$B(x) = a_1 + a_4x^3 + a_7x^6 + \ldots,$$
$$C(x) = a_2 + a_5x^3 + a_8x^6 + \ldots,$$

Note that $A(x)$, $B(x)$ and $C(x)$ are polynomials with integer coefficients and each of these is a polynomial in x^3. We may introduce

$$S(x) = A(x) + \omega xB(x) + \omega^2x^2C(x),$$
$$T(x) = A(x) + \omega^2xB(x) + \omega x^2C(x)$$

where ω is an imaginary cube root of unity.

Then,

$$S(x)T(x) = (A(x))^2 + x^2(B(x))^2 + x^4(C(x))^2 - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x)$$

Since, $\omega^3 = 1$ and $\omega + \omega^2 = -1$

Taking $R(x) = S(x)T(x)$, we obtain

$$P(x)R(x) = (A(x))^3 + x^3(B(x))^3 + x^6(C(x))^3 - 3x^3A(x)B(x)C(x),$$

which is a polynomial in x^3. This follows from the identity

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc$$

Alternately, $R(x)$ may be directly defined by

$$R(x) = (A(x))^2 + x^2(B(x))^2 + x^4(C(x))^2 - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x).$$

39. Adding 1 both sides, the equation reduces to

$$[(x+1)^2] = ([x+1])^2$$

We have, used $[x] + m = [x + m]$ for every integer m. Suppose $x + 1 \le 0$. Then, $[x+1] \le x + 1 \le 0$. Thus

$$([x+1])^2 \ge (x+1)^2 \ge [(x+1)^2] = ([x+1])^2$$

Thus equality holds everywhere. This gives $[x+1] = x + 1$ and thus $x + 1$ is an integer. Using $x + 1 \le 0$, we conclude that

$$x \in \{-1, -2, -3, \ldots\}.$$

Suppose $x + 1 > 0$. We have,

$$(x+1)^2 \ge [(x+1)^2] = ([x+1])^2$$

Moreover, we also have

$$(x+1)^2 \le 1 + [(x+1)^2] = 1 + ([x+1])^2$$

Thus, we obtain

$$[x] + 1 = [x+1] \le (x+1) < \sqrt{1 + ([x+1])^2} = \sqrt{1 + ([x]+1)^2}$$

This shows that

$$x \in [n, \sqrt{1 + (n+1)^2} - 1],$$

where $n \ge -1$ is an integer. Thus, the solution set is

$$\{-1, -2, -3, \ldots\} \cup \{\cup_{n=-1}^{\infty} [n, \sqrt{1 + (n+1)^2} - 1]\}$$

It is easy verify that all the real numbers in this set indeed satisfy the given equation.

40. Since, $xyz \ne 0$, we can divide the second relation by the first. Observe that

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),$$

holds for any x, y. Thus, we get

$$(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) = x^2y^2z^2$$

However, for any real numbers x, y, we have

$$x^2 - xy + y^2 \ge |xy|$$

Since, $x^2y^2z^2 = |xy||yz||zx|$, we get

$$|xy||yz||zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \ge |xy||yz||zx|$$

This is possible only if

$$x^2 - xy + y^2 = |xy|, y^2 - yz + z^2 = |yz|, z^2 - zx + x^2 = |zx|,$$

hold simultaneously. However $|xy| = \pm xy$. If $x^2 - xy + y^2 = -xy$, then $x^2 + y^2 = 0$ giving $x = y = 0$. Since, we are looking for non-zero x, y, z, we conclude that $x^2 - xy + y^2 = xy$ which is same as $x = y$. Using the other two relations, we also get $y = z$ and $z = x$. The first equation now gives $27x^6 = x^3$. This gives $x^3 = 1/27$ (since $x \ne 0$) or $x = 1/3$. We thus have, $x = y = z = 1/3$. These also satisfy the second relation, as may be verified.

41. Let $r = u/v$ where GCD $(u, v) = 1$. Then, we get

$$a_nu^n + a_{n-1}u^{n-1}v + \ldots + a_1uv^{n-1} + a_0v^n = 0,$$
$$b_nu^n + b_{n-1}u^{n-1}v + \ldots + b_1uv^{n-1} + b_0v^n = 0$$

Subtraction gives

$$(a_n - b_n)u^n + (a_{n-2} - b_{n-2})u^{n-2}v^2 + \ldots + (a_1 - b_1)uv^{n-1} + (a_0 - b_0)v^n = 0$$

Since $a_{n-1} = b_{n-1}$. This shows that v divides $(a_n - b_n)u^n$ and hence it divides $a_n - b_n$. Since $a_n - b_n$ is a prime, either $v = 1$ or $v = a_n - b_n$. Suppose the latter holds. The relation takes the form

$$u^n + (a_{n-2} - b_{n-2})u^{n-2}v + \ldots + (a_1 - b_1)uv^{n-2} + (a_0 - b_0)v^{n-1} = 0.$$

(Here we have divided throughout by v.) If $n > 1$, this forces $v | u$, which is impossible since GCD $(v, u) = 1 (v > 1$ since it is equal to the prime $(a_n - b_n)$. If $n = 1$, then we get two equations

$$a_1u + a_0v = 0, b_1u + b_0v = 0.$$

This forces $a_1b_0 - a_0b_1 = 0$ contradicting $a_nb_0 - a_0b_n \ne 0$.

Note The condition $a_nb_0 - a_0b_n \ne 0$ is extraneous. The condition $a_{n-1} = b_{n-1}$ forces that for $n = 1$, we have $a_0 = b_0$. Thus, we obtain, after subtraction $(a_1 - b_1)u = 0$. This implies that $u = 0$ and hence $r = 0$ is an integer).

Unit 3 Inequalities

Unit 3

Inequalities

Inequality

The relation between two unequal numbers (real numbers) is called an *inequality.*
A quantity x is said to be greater than the quantity y, if $x - y$ is +ve.
The quantity x is said to be less than quantity y, if $x - y$ is –ve.
The symbols > and < are used for greater than and less than respectively.

Some Important Properties

Property 1. If $a > b, b > c$, then $a > c$

We have $$a - c = (a - b) + (b - c) = + \text{ ve}$$

Since, $a > b$ and $b > c$ *i.e.*, $(a - b)$ and $(b - c)$ are +ve.

$$\therefore \quad a > c$$

In same way, we can prove that if, we are given $a < b, b < c$, then $a < c$.

Property 2. If $a > b$, then $a + c > b + c$

We have $$(a + c) - (b + c) = a - b = + \text{ ve} \qquad [\because a > b]$$

$$\therefore \quad (a + c) > (b + c)$$

Similarly, we can prove that

$$a - c > b - c, \frac{a}{c} > \frac{b}{c} \text{ and } a \times c > b \times c$$

provided c is +ve.

Similarly, we can also prove that, if $a < b$, then

$$a + c < b + c \text{ and } a - c < b - c$$

Hence, if both sides of an inequality are increased or diminished by the same number (+ve, –ve or zero), then the sign of inequality remains unaltered *i.e.*, remains the same.

Property 3. In an equality any term may be transposed from one side to the other provided its sign is changed.

If $$a + b > c + d \text{, then } a + b - c > d$$

or $$-c - d > - a - b \text{ etc.}$$

Property 4. If the sides of an inequality be changed the sign of inequality is reversed, if $a > b$, then $b < a$.

Property 5. If both sides of an inequality are multiplied or divided by same +ve number, the sign of inequality remains unaltered.

If $a > b$, then $a - b$ is +ve *i.e.*, $a - b > 0$ multiplying both sides by x where $x > 0$ we have $x(a - b) > 0$ or $xa - xb > 0$. Hence proved.

Again dividing both sides of $a - b > 0$ by x, where $x > 0$ *i.e.*, multiplying both sides of $a - b > 0$ by $\frac{1}{x}$, where $x > 0$, we get

$$\frac{1}{x}(a-b) > 0 \text{ or } \frac{a}{x} - \frac{b}{x} > 0$$

or
$$\frac{a}{x} > \frac{b}{x}$$

Hence proved.

Similarly, we can prove that if $a < b$ and $x > 0$, then $xa < xb$, $\frac{a}{x} < \frac{b}{x}$.

Property 6. If both sides of an inequality be multiplied by a -ve quantity, the sign of inequality is reversed.

If $a > b$, then $a - b$ is +ve multiplying both sides by $-x$, we get

$$-x(a-b) = \text{-ve} \quad \text{or} \quad -xa + xb = \text{-ve}$$

or
$$-xa + xb < 0 \text{ or } -xa < -xb$$

i.e.,
$$-x(a) < -x(b)$$

Hence proved.

Similarly, we can prove that if $a < b$ and $x > 0$,

then
$$-x(a) > -x(b)$$

Property 7. If a and b are two +ve numbers and $a > b$, then $\frac{1}{a} < \frac{1}{b}$.

$\because$
$$a > 0 \text{ and } b > 0\text{, so } ab > 0$$

Also, it is given that $a > b$

Dividing both sides of this inequality by ab, where $ab > 0$. So, by property 5

$$\frac{a}{ab} > \frac{b}{ab} \text{ or } \frac{1}{b} > \frac{1}{a}$$

or
$$\frac{1}{a} < \frac{1}{b}$$

Property 8. If $a_1 > b_1, a_2 > b_2, \ldots, a_n > b_n$ then $a_1 + a_2 + \ldots + a_n > b_1 + b_2 + \ldots + b_n$ for all +ve a's and +ve b's.

If
$$a_1 > b_1\text{, then } a_1 - b_1 > 0$$

Similarly,
$$a_2 - b_2 > 0, a_3 - b_3 > 0, \ldots,$$
$$a_n - b_n > 0$$

Adding these, we get

$$(a_1 - b_1) + (a_2 - b_2) + \ldots + (a_n - b_n) > 0$$

or
$$(a_1 + a_2 + \ldots + a_n) - (b_1 + b_2 + \ldots + b_n) > 0$$

or
$$a_1 + a_2 + \ldots + a_n > b_1 + b_2 + \ldots + b_n$$

Property 9. If $a_1 > b_1, a_2 > b_2, \ldots, a_n > b_n$, then $a_1a_2a_3 \ldots a_n > b_1b_2b_3 \ldots b_n$ for all +ve numbers a's and b's.

From property 5, we know that if $a_1 > b_1$, then $a_1x > b_1x$ where $x > 0$.

Substituting $a_2a_3 \ldots a_n$ for x, we get

$$a_1a_2a_3 \ldots a_n > b_1a_2a_3 \ldots a_n \quad \ldots\text{(i)}$$

Again as $a_2 > b_2$ so $a_2y > b_2y$ where $y > 0$

Substituting $b_1a_3 \ldots a_n$ for y, we get

$$a_2b_1a_3 \ldots a_n > b_2b_1a_3 \ldots a_n$$
$$b_1a_2a_3 \ldots a_n > b_1b_2\, a_3 \ldots a_n \quad \ldots\text{(ii)}$$

From (i) and (ii)

$$a_1a_2a_3 \dots a_n > b_1b_2a_3 \dots a_n$$

Applying this method successively, we get

$$a_1a_2a_3 \dots a_n > b_1b_2b_3 \dots b_n$$

Hence proved.

Property 10. If $a > b$ and n is a +ve integer, then $a^n > b^n$ and $a^{1/n} > b^{1/n}$ provided a and b are both +ve and only real +ve values of the nth roots are taken into account.

From property 9, if $a_1 > b_1, a_2 > b_2, \dots, a_n > b_n$ we have $a_1a_2a_3 \dots a_n > b_1b_2b_3 \dots b_n$

Put $$a_1 = a_2 = a_3 = \dots = a$$

and $$b_1 = b_2 = b_3 = \dots = b,$$

we get $a \cdot a \cdot a \dots n$ times $> b \cdot b \cdot b \dots n$ times or $a^n > b^n$

Similarly, we can prove second result.

Concept We know that given any 2 distinct real numbers a, b.

Either $$a < b \text{ or } a > b$$

But to decide whether it is the former inequality or the latter for any given pair of real number often requires use of certain facts in clever way.

For example $2^5, 3^3$

if $0 \le m < n$, we know that $2^m < 2^n$ or $3^m < 3^n$

Again for $$m > 0, 2^m < 3^m \quad (2 < 3)$$

But these inequalities as such do not help to compare 2^5 and 3^3.

On the other hand

$$3^3 = (2+1)^3 = 2^3 + 3 \times 2^2 + 3(2) + 1 < 2^3 + 4(2)^2 + 4 \times 2$$
$$= 2^3 + 2^4 + 2^3$$
$$= 2 \times 2^3 + 2^4 = 2 \cdot 2^4 = 2^5$$

of course $2^5 = 32$ and $3^3 = 27$ are easily computable and is evident that $2^5 > 3^3$.

If however the indices m, n of $2^m, 3^n$ are very large, it is necessary to think of smaller indices from which we could derive the inequality of larger indices.

e.g., We know that $2^5 > 3^3$

We immediately deduce

$$2^{5 \times 121} = 2^{605} > 3^{3 \times 121} = 3^{363}$$

Also, $$2^7 > 3^4$$

$\because$ $$2^2 \cdot 2^5 > 2^2 \cdot 3^3 = 4 \times 3^3 > 3 \times 3^3 = 3^4$$

Hence, $2^{700} > 3^{400}$ (raising power to 100).

Like that another simple deduction in the routine

$$9 = 3^2 > 2^3 = 8$$

$\Rightarrow$ $$3^{200} > 2^{300}$$

Example 1 *Which number is greater $(31)^{12}$ or $(17)^{17}$?*

Solution Now, $31 < 32$

Raising the power to 12

$$(31)^{12} < (32)^{12}$$

$\Rightarrow$ $$(31)^{12} < (2^5)^{12} = 2^{60} \quad \dots\text{(i)}$$

Now, $2^{60} < 2^{68} = 2^{4\times 17}$ ...(ii)

$\Rightarrow$ $(2^4)^{17} < (16)^{17}$

$\Rightarrow$ $(16)^{17} < (17)^{17}$ ...(iii)

From (i), (ii) and (iii), we get

$$(31)^{12} < 2^{60} < 2^{68} < (17)^{17}$$

$\Rightarrow$ $(31)^{12} < (17)^{17}$

$\therefore$ $(17)^{17} > (31)^{12}$

Example 2 *Which number is greater.* $(30)^{100}$ *or* 2^{567}*?*

Solution $(30)^{100} < (32)^{100}$

So, $(2^5)^{100} = (2^{10})^{50} = (1024)^{50}$

Now, $(1024)^{50} < (1024)^{54} = ((1024)^2)^{27} = (2^{20})^{27}$

Now, $2^{20} < 2^{21}$

$$(2^{20})^{27} < (2^{21})^{27} = 2^{567}$$

$\therefore$ $(30)^{100} < 2^{567}$

$\therefore$ $2^{567} > (30)^{100}$

Example 3 *Which is greater* 7^{92} *or* 8^{91}*?*

Solution If a, b are +ve and n is a natural number, then

$$(a+b)^n = a^n + na^{n-1}b + \ldots$$

(term involving powers of a and b).

Also, $(a+b)^n \geq a^n + na^{n-1}b$ (equality for $n = 1$)

Now, $(8)^{91} = (7+1)^{91} > 7^{91} + 91\cdot 7^{90}$

$$= 7^{90}(7+91) = 7^{90}(98)$$

Now, $7^{90}(98) > 7^{90}\cdot 49 = 7^{90}\cdot 7^2 = 7^{92}$

Hence, $(8)^{91} > 7^{92}$

Example 4 *Which is greater*

$$(150)^{300} \text{ or } (20000)^{100} \times (100)^{100}?$$

Solution We know $(150)^3 = 150 \times 150 \times 150$

$$= 30 \times 30 \times 30 \times 125 = 27000 \times 125$$

Now, $27000 \times 125 > 20000 \times 100$

Hence, $(150)^3 > 20000 \times 100$

Raising the power to 100

$$(150)^{300} > (20000)^{100} \times (100)^{100}$$

Example 5 *Show that* $\sqrt{2} < \dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2+\sqrt{3}}} < 2.$

Solution $\because$ $\sqrt{3} > \sqrt{2}$ and $\sqrt{3} < 2$

$\because$ $\sqrt{3} > \sqrt{2}$

$$\sqrt{2}+\sqrt{3} > 2\sqrt{2} \quad \text{...(i)}$$

$$2 > \sqrt{3}$$

$$2+2 > \sqrt{3}+2$$

$$\Rightarrow \quad \sqrt{4} > \sqrt{\sqrt{3}+2} \quad \text{...(ii)}$$

$$\Rightarrow \quad \sqrt{\sqrt{3}+2} < 2 \quad \text{...(iii)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{\sqrt{3}+2}} > \frac{1}{2} \quad \text{...(iv)}$$

Multiplying (i) and (iv), we get

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{\sqrt{3}+2}} > \sqrt{2} \quad \text{...(v)}$$

Again, $\sqrt{2} < \sqrt{3}$

$$\sqrt{2}+\sqrt{3} < 2\sqrt{3} \quad \text{...(vi)}$$

$$1 < \sqrt{3}$$

$$1+2 < \sqrt{3}+2$$

$$3 < \sqrt{3}+2$$

$$\sqrt{3} < \sqrt{\sqrt{3}+2}$$

or

$$\frac{1}{\sqrt{3}} > \frac{1}{\sqrt{\sqrt{3}+2}} \quad \text{...(vii)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{\sqrt{3}+2}} < \frac{1}{\sqrt{3}} \quad \text{...(viii)}$$

Multiplying (vi) and (viii), we get

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{\sqrt{3}+2}} < 2\sqrt{3} \times \frac{1}{\sqrt{3}} = 2 \quad \text{...(ix)}$$

From (ix) $\quad \sqrt{2} < \dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{\sqrt{3}+2}} < 2$

Hence proved.

Example 6 *Show that* $(1.01)^{1000} > 1000$.

Solution We can write 1.01 as $(1 + 0.01)$

Now, $\quad (1+0.01)^8 > 1+0.08 \qquad [\because (1+x)^n > 1+nx]$

$\Rightarrow \quad (1.01)^{1000} > (1+0.08)^{125} = ((1+0.08)^5)^{25} > (1+0.4)^{25}$

$\Rightarrow \quad (1+0.4)^{25} > (1+0.4)^{24} = ((1.4)^3)^8 \qquad [\because (1+0.08)^5 = (1+0.4)]$

Now, $\quad ((1.4)^3)^8 > (2.7)^8 > 7^4 = 2401 > 1000$

$\therefore \quad (1.01)^{1000} > 1000$

Author's Observation

Consider the numbers $\dfrac{101}{1001}, \dfrac{10001}{100001}$

Note that the pattern of the numbers in both the numerator and denominator of each fraction is same *i.e.*, 0 is flanked by 1. The only difference is that the denominator has one more zero than the numerator. If x is the numerator of any one of the two fractions, then the corresponding denominator is $10x - 9$.

For example if $x = 101$

$$10x - 9 = (101)10 - 9 = 1001$$

$\therefore$ Let $\dfrac{x}{10x-9}$ be denoted by a

then $$\frac{1}{a} = \frac{10x-9}{x}$$

which is equal to $10 - \dfrac{9}{x}$.

Thus, if x increases, $\dfrac{9}{x}$ decreases and so $\dfrac{-9}{x}$ increases. Hence, $\dfrac{1}{a}$ increases where 'x' increases or 'a' decreases, when x increases.

In other words $$\frac{101}{1001} > \frac{10001}{100001}$$

$$\frac{\overbrace{100\ldots\ldots\ldots\ldots01}^{K\text{ zero}}}{\underbrace{1000\ldots\ldots\ldots\ldots01}_{K+1\text{ zero}}} > \frac{\overbrace{10\ldots\ldots\ldots\ldots\ldots01}^{L\text{ zero}}}{\underbrace{100\ldots\ldots\ldots\ldots\ldots01}_{L+1\text{ zero}}}$$

$\therefore$ In general

If $L > K$

Concept If we have to prove $f(x) > g(x)$.

We consider difference of $f(x)$ and $g(x)$ say $f(x) - g(x)$.

By proving the difference $f(x) - g(x) > 0$ by some method we get the required result.

Example 1 *Prove that* $m^3 + 1 > m^2 + m,\ m \neq 1 \Rightarrow m > -1$.

Solution We consider difference of

$$\begin{aligned}(m^3+1)-(m^2+m) &= (m^3-m^2)-(m-1)\\ &= m^2(m-1)-(m-1) = (m-1)(m^2-1)\\ &= (m-1)(m-1)(m+1) = (m-1)^2(m+1)\end{aligned}$$

$\Rightarrow$ The expression is positive.

Hence, $$m^3 + 1 > m^2 + m$$

Example 2 *Prove that*

$$a^3b + ab^3 < a^4 + b^4.$$

Solution Consider difference of

$$\begin{aligned}(a^4+b^4)-(a^3b+ab^3) &= a^3(a-b)-b^3(a-b) = (a-b)(a^3-b^3)\\ &= (a-b)(a-b)(a^2+ab+b^2) = (a-b)^2(a^2+ab+b^2)\end{aligned}$$

$\because$ $(a-b)^2$ is +ve
and a^2+ab+b^2 is +ve
$\therefore$ $(a-b)^2(a^2+ab+b^2)$ is +ve.
$\therefore$ $a^3b+ab^3 < a^4+b^4$
Hence proved.

Example 3 *If $x, y > 0$, then prove $x^5+y^5 > x^4y + xy^4$ unless $x = y$.*

Solution Consider $x^5+y^5-x^4y-xy^4$

$$=(x^5-x^4y)+(y^5-xy^4)$$
$$=x^4(x-y)+y^4(y-x)=(x^4-y^4)(x-y)$$
$$=(x^2+y^2)(x^2-y^2)(x-y)$$
$$=(x^2+y^2)(x-y)^2(x+y)=+\text{ve, if } x\neq y$$

Example 4 *If $a > b$ and x is +ve, then prove $\frac{a+x}{b+x} < \frac{a}{b}$.*

Solution Consider the difference

$$\frac{a+x}{b+x}-\frac{a}{b}=\frac{b(a+x)-a(b+x)}{b(b+x)}=\frac{bx-ax}{b(b+x)}=\frac{-x(a-b)}{b(b+x)}=\frac{-x(a-b)}{b(b+x)} \text{ is } -\text{ve}.$$

$$\therefore \quad \frac{a+x}{b+x}<\frac{a}{b}$$

Hence proved.

Example 5 *If $x > 0$, then prove $x+\frac{1}{x}\geq 2$.*

Solution Consider difference of $x+\frac{1}{x}-2$

$$\Rightarrow \quad \frac{x^2-2x+1}{x} \Rightarrow \frac{(x-1)^2}{x}$$

$\because$ $x > 0$ given $(x-1)^2$ is > 0.

$$\therefore \quad x+\frac{1}{x}-2\geq 0 \text{ for } x>0 \Rightarrow x+\frac{1}{x}\geq 2$$

So, $\quad x+\frac{1}{x}=2$ for $x=1$

Note Similarly, if $x < 0$, then $x+\frac{1}{x}\leq -2$ combining the above inequalities, we get $\left|x+\frac{1}{x}\right|\geq 2 \ \forall x$.

Example 6 *Show that $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)$ cannot be –ve.*

Solution Whenever we have symmetrical expression in a, b, c then without loss of generality we can assume

$$a\geq b\geq c$$
$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)$$
$$=(a-b)[a(a-c)-b(b-c)]+c(c-a)(c-b)$$
$$=(a-b)[(a^2-b^2)+(bc-ac)]+c(c-a)(c-b)$$

$$= (a-b)[a^2 - b^2 - c(a-b)] + c(c-a)(c-b)$$
$$= (a-b)[(a-b)(a+b-c)] + c(c-a)(c-b)$$
$$= (a-b)^2(a+b-c) + c(c-a)(c-b)$$
$$= (a-b)^2(a+b-c) + c(a-c)(b-c) = \text{+ve}$$

As $a > b > c$

$\therefore$ $b - c$ is +ve

and $a - c$ is +ve

If $a = b = c$, then it is zero.

$\therefore$ $a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)$

It cannot be –ve.

Example 7 *For real a, b, c show that*

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Solution $a^2 + b^2 + c^2 \geq ab + bc + ca$

Multiply 2 on both sides, we get

$$2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca$$
$$\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \geq 0$$
$$\Rightarrow \quad (a^2 - 2ab + b^2) + (b^2 - bc + c^2) + (c^2 - ac + c^2) \geq 0$$
$$\Rightarrow \quad (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$$

Hence proved.

Note Egives us a new concept by use of which we can prove several inequalities *i.e.*, without using Arithmetic mean (AM) and Geometric mean (GM). Let us solve some examples by using above concept.

Example 8 *Prove that*

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2.$$

Solution We know that

$$a^2 + b^2 + c^2 \geq ab + bc + ca \quad \text{...(i)}$$

Now, we can write $a^4 + b^4 + c^4$ as

$$(a^2)^2 + (b^2)^2 + (c^2)^2$$

Let $a^2 = u$ ...(ii)

$b^2 = v$ ...(iii)

$c^2 = w$ ...(iv)

Using (i), we get

$$u^2 + v^2 + w^2 \geq uv + vw + uw$$

Substitute the values of u, v, w, we get

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + a^2c^2$$

Example 9 *Prove that*

$$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c).$$

Solution Let $ab = u$

$$bc = v$$
$$ca = w$$

Now, we have

$$u^2 + v^2 + w^2 \geq uv + vw + uw$$

Substitute the value, we get

$$a^2b^2 + b^2c^2 + c^2a^2 \geq (ab)(bc) + (bc)(ca) + (ab)(ca)$$

$\Rightarrow$ $$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$$

Hence proved.

Example 10 *Without using AM and GM, prove that*

$$\sqrt{\frac{ab}{c}} + \sqrt{\frac{bc}{a}} + \sqrt{\frac{ac}{b}} > \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

Solution Let $\sqrt{\frac{ab}{c}} = u^2$

$$\sqrt{\frac{bc}{a}} = v^2 \text{ and } \sqrt{\frac{ac}{b}} = w^2$$

As we know that

$$u^2 + v^2 + w^2 \geq uv + vw + uw$$

Now, $$u^2v^2 = \sqrt{\frac{ab}{c} \times \frac{bc}{a}}$$

$$u^2v^2 = \sqrt{b^2}$$

$$u^2v^2 = b \quad [\because b > 0]$$

$$uv = \sqrt{b}$$

Similarly, $$v^2w^2 = \sqrt{\frac{bc}{a} \times \frac{ac}{b}}$$

$$v^2w^2 = \sqrt{c^2}$$

$$vw = \sqrt{c} \quad [\because c > 0]$$

$\therefore$ Like wise $$uw = \sqrt{a} \quad [\therefore a > 0]$$

$\therefore$ We get

$$\sqrt{\frac{ab}{c}} + \sqrt{\frac{bc}{a}} + \sqrt{\frac{ac}{b}} > \sqrt{a} + \sqrt{b} + \sqrt{c}$$

Example 11 *For real $a, b, c > 0$. Without using AM or GM, prove that*

$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} > ab + bc + ca.$$

Solution $$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} > \left(\frac{ab}{c}\right)\left(\frac{bc}{a}\right) + \left(\frac{bc}{a}\right)\left(\frac{ca}{b}\right) + \left(\frac{ab}{c}\right)\left(\frac{ca}{b}\right)$$

$\therefore$ $$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} > b^2 + a^2 + c^2 \quad \text{...(i)}$$

Now, we know that

$$a^2 + b^2 + c^2 > ab + bc + ca \quad \text{...(ii)}$$

$\therefore$ From (i) and (ii)

$$\frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2} > ab+bc+ac$$

Concept

Maximization Principle

(a) Whenever we write $x \geq a$, this is analysed as 'a' is the minimum value of x.

(b) Whenever we write $x \leq a$, this is analysed as 'a' is maximum value of x.

(c) If we have $a \leq x \leq b$, this is analysed as 'a' is minimum value of x *and* 'b' is maximum value of x.

(I) Now, suppose we have

$$S = k - x$$

where k is constant.

Now, 'maximum value' of S

$$= k - (\text{minimum value of } x)$$

i.e., $\quad S \leq k - a$

$\therefore$ We read as $k - a$ is maximum value of S.

Now, 'minimum value' of $S = k -$ (maximum value of x)

i.e., $\quad S \geq (k - b)$

$\therefore$ We read as $k - b$ is minimum values of S.

(II) Now, suppose we have

$$S = k + x$$

Maximum value of $S = k +$ maximum (value of x)

$$= k + b$$

Minimum value of $S = k +$ minimum (value of x)

$$= k + a$$

Note Student should keep the above concept for proving various inequalities.

Example 1 *Let* $y = \dfrac{1}{x+\dfrac{1}{x}+5}$, $x \neq 0$.

Find maximum and minimum value of y.

Solution Now, $y_{max} = \dfrac{1}{\min\left(x+\dfrac{1}{x}+2\right)}$

Now, $\quad \min\left(x+\dfrac{1}{x}+2\right) = \min\left(x+\dfrac{1}{x}\right)+2 = 2+5 = 7 \quad \left(\because x+\dfrac{1}{x} \geq 2 \; \forall \; x > 0\right)$

$\therefore \quad y_{max} = \dfrac{1}{7}$ *i.e.*, $y \leq \dfrac{1}{7}$

Now, $\quad y_{min} = \dfrac{1}{\max\left(x+\dfrac{1}{x}+5\right)}$

Now, $\max\left(x+\frac{1}{x}+5\right) \Rightarrow \max\left(x+\frac{1}{x}\right)+5$

$\Rightarrow$ $-2+5=3$

$\therefore$ $x+\frac{1}{x} \le -2, \forall x<0$

$$y_{\min}=\frac{1}{3} \text{ i.e., } y \ge \frac{1}{3}$$

Example 2 *If $x+y=1$, $x>0$, $y>0$, then without using AM – GM prove*

(*i*) $0<xy\le\frac{1}{4}$ (*ii*) $x^2+y^2\ge\frac{1}{2}$

(*iii*) $x^4+y^4\ge\frac{1}{8}$ (*iv*) $\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2\ge\frac{25}{2}$.

Solution (i) Consider $(x-y)^2\ge 0$

$\Rightarrow$ $x^2+y^2-2xy\ge 0$

Add $4xy$ on both sides,

$\Rightarrow$ $x^2+y^2+2xy\ge 4xy \Rightarrow (x+y)^2\ge 4xy$

$\Rightarrow$ $1\ge 4xy$ $[\because x+y=1]$

$\Rightarrow$ $xy\le\frac{1}{4}$ ...(i)

Also, $x>0, y>0 \Rightarrow xy>0$ ...(ii)

From (i) and (ii)

$$0<xy\le\frac{1}{4} \quad \text{...(iii)}$$

(ii) Let $S=(x^2+y^2)$

$$S=(x+y)^2-2xy$$

$$S=1-2xy$$

$\because$ We have to find minimum of S.

$\therefore$ $\min(S)=1-2\max(xy)$

$\Rightarrow$ $\min(S)=1-2\left(\frac{1}{4}\right)$ $\left[\text{from (iii) } xy\le\frac{1}{4}\right]$

i.e., $S\ge 1-\frac{1}{2}=\frac{1}{2}$

$\Rightarrow$ $x^2+y^2\ge\frac{1}{2}$ ...(iv)

(iii) Let $S=x^4+y^4$

$$S=(x^2+y^2)^2-2x^2y^2$$

$$S=S_1-S_2=S_1+(-S_2)$$

Let $S_1=(x^2+y^2)^2$

We have, $x^2+y^2\ge\frac{1}{2}$

Squaring, $$(x^2 + y^2)^2 \geq \frac{1}{4} \quad \ldots(v)$$

$\therefore$ $$S_1 \geq \frac{1}{4} \quad \ldots(vi)$$

Let $$S_2 = -2x^2y^2$$

We have, $$xy \leq \frac{1}{4} \quad \text{[from (iii)]}$$

$$x^2y^2 \leq \frac{1}{16} - 2x^2y^2 \geq -\frac{1}{8}$$

$\therefore$ $$S_2 \geq -\frac{1}{8} \quad \ldots(vii)$$

$\because$ $$a_1 > b_1 \quad \text{and} \quad a_2 > b_2$$

$\Rightarrow$ $$a_1 + a_2 > b_1 + b_2$$

$\Rightarrow$ $$S_1 + S_2 \geq \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

(iv) Let $S = \left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 = x^2 + y^2 + \frac{1}{x^2} + \frac{1}{y^2} + 4$

$$= x^2 + y^2 + \frac{(x^2 + y^2)}{x^2y^2} + 4$$

$$S = S_1 + S_2 + S_3$$

Let $$S_1 = x^2 + y^2$$

$$S_1 \geq \frac{1}{2} \quad \text{[from (iv)]} \ldots(viii)$$

Let $$S_2 = \frac{x^2 + y^2}{x^2y^2}$$

$$x^2 + y^2 \geq \frac{1}{2} \quad \ldots(ix)$$

$$x^2y^2 \leq \frac{1}{16}$$

$$\frac{1}{x^2y^2} \geq 16 \quad \ldots(x)$$

Multiplying (ix) and (x)

$$\frac{x^2 + y^2}{x^2y^2} \geq 8 \quad \ldots(xi)$$

$\Rightarrow$ $$S_2 \geq 8 \quad \ldots(xii)$$

Adding (viii) and (xii)

$$S_1 + S_2 \geq \frac{1}{2} + 8$$

On adding S_3 on both sides

$\Rightarrow$ $$S_1 + S_2 + S_3 \geq \frac{1}{2} + 8 + S_3$$

$\Rightarrow$ $$S_1 + S_2 + S_3 \geq \frac{1}{2} + 8 + 4$$

$$\Rightarrow \quad S_1 + S_2 + S_3 \geq \frac{25}{2} \quad [\because S_3 = 4]$$

$$\Rightarrow \quad S \geq \frac{25}{2}$$

So, $$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq \frac{25}{2}$$

Example 3 *If $a > 0$ and $n \in N$, then prove*

$$\frac{a^n}{1 + a + a^2 + \ldots + a^{2n}} < \frac{1}{2n}.$$

Solution Let $S = \dfrac{a^n}{1 + a + a^2 + \ldots + a^{2n}}$

Dividing numerator and denominator by a^n

$$S = \frac{1}{\frac{1}{a^n} + \frac{1}{a^{n-1}} + \frac{1}{a^{n-2}} + \ldots + \frac{1}{a} + 1 + a + a^2 + \ldots + a^n}$$

$$S = \frac{1}{1 + \left(a + \frac{1}{a}\right) + \left(a^2 + \frac{1}{a^2}\right) + \ldots + \left(a^n + \frac{1}{a^n}\right)}$$

$\because$ We have to find maximum value of S.

$\therefore$ Maximum S

$$= \frac{1}{\min\left[1 + \left(a + \frac{1}{a}\right) + \left(a^2 + \frac{1}{a^2}\right) + \ldots + \left(a^n + \frac{1}{a^n}\right)\right]}$$

$$= \frac{1}{1 + (2 + 2 + \ldots n \text{ times})}$$

$$\because \quad a^k + \frac{1}{a^k} \geq 2, \ \forall\, k = 1, 2, 3, \ldots, n$$

$$\text{Max}\,(S) = \frac{1}{1 + 2n}$$

$$\therefore \quad S \leq \frac{1}{1 + 2n} \quad \ldots\text{(i)}$$

Now, $$1 + 2n > 2n, \ \forall\, n \in N$$

$$\Rightarrow \quad \frac{1}{1 + 2n} < \frac{1}{2n} \quad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$S \leq \frac{1}{1 + 2n} < \frac{1}{2n}$$

$$\Rightarrow \quad S < \frac{1}{2n}$$

$$\Rightarrow \quad \frac{a^n}{1 + a + a^2 + \ldots + a^{2n}} < \frac{1}{2n}$$

First Fundamental Special Concept

We know $(a-b)^2 \geq 0$

$$\Rightarrow \quad a^2 + b^2 - 2ab \geq 0$$

$$\Rightarrow \quad a^2 + b^2 \geq 2ab$$

$$\Rightarrow \quad 2a^2 + 2b^2 \geq a^2 + b^2 + 2ab$$

$$\Rightarrow \quad 2(a^2 + b^2) \geq (a+b)^2$$

$$\Rightarrow \quad (a^2 + b^2) \geq \frac{1}{2}(a+b)^2$$

$$\Rightarrow \quad \frac{a^2 + b^2}{a+b} \geq \frac{1}{2}(a+b)$$

Second Special Fundamental Concept

$$(a^2 - b^2)^2 \geq 0$$

$$a^4 + b^4 - 2a^2b^2 \geq 0$$

$$a^4 + b^4 \geq 2a^2b^2$$

$$2a^4 + 2b^4 \geq a^4 + b^4 + 2a^2b^2$$

$$2(a^4 + b^4) \geq (a^2 + b^2)^2$$

$$\frac{a^4 + b^4}{a^2 + b^2} \geq \frac{1}{2}(a^2 + b^2)$$

Third Special Fundamental Concept

$$\sqrt{x^2 + y^2} + \sqrt{a^2 + b^2} \geq \sqrt{(x+a)^2 + (y+b)^2} \qquad (\because \text{both sides are +ve})$$

$\because$ Squaring, we get

$$x^2 + y^2 + a^2 + b^2 + 2\sqrt{x^2 + y^2}\sqrt{a^2 + b^2} \geq (x+a)^2 + (y+b)^2$$

$$\Rightarrow \quad 2\sqrt{x^2 + y^2}\sqrt{a^2 + b^2} \geq 2ax + 2by$$

$$\Rightarrow \quad \sqrt{x^2 + y^2}\sqrt{a^2 + b^2} \geq ax + by \qquad \text{...(i)}$$

If $ax + by < 0$, then (i) is true and given inequality is true.

If $ax + by \geq 0$, then squaring (i)

$$(x^2 + y^2)(a^2 + b^2) \geq (ax + by)^2$$

which on arranging becomes

$$(ax - by)^2 \geq 0$$

which is true.

$\therefore$ Result is proved.

Fourth Special Fundamental Concept

We have, $(a-b)^2 \geq 0$, a, b are +ve

$$a^2 + b^2 - 2ab \geq 0$$

$$a^2 + b^2 \geq 2ab$$

Note $a^2 + b^2 - ab > ab$

Multiplying $(a + b)$ on both sides

$$(a + b)(a^2 - ab + b^2) > ab(a + b)$$

$$a^3 + b^3 > ab(a + b)$$

$$\frac{a^3 + b^3}{a + b} > ab$$

Fifth Special Fundamental Concept

We have proved earlier

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Now, we can write

$$(a + b + c)^2 - 2ab - 2bc - 2ac > ab + bc + ca$$

or $$(a + b + c)^2 > 3(ab + bc + ca)$$

or $$\left(\frac{a + b + c}{3}\right)^2 > \frac{1}{3}(ab + bc + ca)$$

Example 1 *Without using AM and GM, prove* $\dfrac{b^2 + c^2}{b + c} + \dfrac{c^2 + a^2}{c + a} + \dfrac{a^2 + b^2}{a + b} \geq a + b + c.$

Solution From above fundamental concept

We have, $$\frac{a^2 + b^2}{a + b} \geq \frac{1}{2}(a + b) \quad \text{...(i)}$$

$$\frac{b^2 + c^2}{b + c} \geq \frac{1}{2}(b + c) \quad \text{...(ii)}$$

and $$\frac{c^2 + a^2}{c + a} \geq \frac{1}{2}(c + a) \quad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$\frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} \geq a + b + c$$

Example 2 *Without using AM and GM, prove*

$$\frac{a^4 + b^4}{a^2 + b^2} + \frac{b^4 + c^4}{b^2 + c^2} + \frac{c^4 + a^4}{c^2 + a^2} \geq ab + bc + ca.$$

Solution From second fundamental concept

$$\frac{a^4 + b^4}{a^2 + b^2} \geq \frac{1}{2}(a^2 + b^2) \quad \text{...(i)}$$

$$\frac{b^4 + c^4}{b^2 + c^2} \geq \frac{1}{2}(b^2 + c^2) \quad \text{...(ii)}$$

and $$\frac{c^4 + a^4}{c^2 + a^2} \geq \frac{1}{2}(c^2 + a^2) \quad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$\frac{a^4+b^4}{a^2+b^2}+\frac{b^4+c^4}{b^2+c^2}+\frac{c^4+a^4}{c^2+a^2} \geq a^2+b^2+c^2 \quad \text{...(i)}$$

Also, $$a^2+b^2+c^2 \geq ab+bc+ca \quad \text{...(ii)}$$

$$\therefore \quad \frac{a^4+b^4}{a^2+b^2}+\frac{b^4+c^4}{b^2+c^2}+\frac{c^4+a^4}{c^2+a^2} \geq ab+bc+ca$$

Example 3 *If $A+B+C=\pi$, without using AM and GM, show that*

$$\sin^4 A+\sin^4 B+\sin^4 C \geq 32 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}.$$

Solution We have, $a^4+b^4+c^4 \geq abc(a+b+c)$

$$\therefore \quad \sin^4 A+\sin^4 B+\sin^4 C \geq \sin A \sin B \sin C(\sin A+\sin B+\sin C)$$

$$\Rightarrow \quad \sin^4 A+\sin^4 B+\sin^4 C \geq \sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}$$

$$\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{3}\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right) \quad [\because A+B+C=\pi]$$

$$\because \quad \sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\Rightarrow \quad \sin^4 A+\sin^4 B+\sin^4 C \geq 32\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos^4\frac{A}{2}\cos^4\frac{B}{2}\cos^4\frac{C}{2}$$

Example 4 *a, b, c are +ve, prove that*

$$a^3+b^3+c^3 \geq \frac{1}{2}[ab(a+b)+bc(b+c)+ca(c+a)]$$

Solution From fourth fundamental concept, we have

$$a^3+b^3 > ab(a+b) \quad \text{...(i)}$$

$$b^3+c^3 > bc(b+c) \quad \text{...(ii)}$$

$$c^3+a^3 > ca(c+a) \quad \text{...(iii)}$$

Adding (i), (ii), (iii), we get

$$\Rightarrow \quad 2(a^3+b^3+c^3) > ab(a+b)+bc(b+c)+ca(c+a)$$

$$a^3+b^3+c^3 \geq \frac{1}{2}[ab(a+b)+bc(b+c)+ca(c+a)]$$

Example 5 *Without using AM – GM, prove that*

$\tan^2 A\tan^2 B\tan^2 C > 3(\tan A\tan B+\tan B\tan C+\tan A\tan C)$.

Solution From fifth fundamental concept

$$\left(\frac{a+b+c}{3}\right)^2 > \frac{1}{3}(ab+bc+ca)$$

Let $$a=\tan A$$

$$b=\tan B$$

$$c=\tan C$$

We have,

$$\left(\frac{\tan A + \tan B + \tan C}{3}\right)^2 > \frac{1}{3}(\tan A \tan B + \tan B \tan C + \tan C \tan A)$$

If $\quad A + B + C = \pi$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow \quad \tan^2 A \tan^2 B \tan^2 C > 3(\tan A \tan B + \tan B \tan C + \tan A \tan C)$

Arithmetic Mean (AM)

AM of a set of n +ve real numbers $a_1, a_2, \ldots, a_n$ is defined to be the average value of the n numbers.

i.e.,
$$\frac{a_1 + a_2 + \ldots + a_n}{n}$$

Geometric Mean (GM)

GM of the same set of numbers is defined to be $\sqrt[n]{a_1 \times a_2 \times \ldots \times a_n}$.

If $a, b > 0$, then $a, \frac{a+b}{2}, b$ are such that the difference between consecutive terms, *viz.*, $\frac{a+b}{2} - a, b - \frac{a+b}{2}$ are each $\frac{b-a}{2}$ *i.e.*, the same. The three terms form an AP $\frac{a+b}{2}$ is thus, an AM inserted between a and b to get an AP. Again $a, \sqrt{ab}, b$ are such that the quotient of consecutive terms $\frac{\sqrt{ab}}{a}, \frac{b}{\sqrt{ab}}$ are each $\sqrt{\frac{b}{a}}$ *i.e.*, the same $a, \sqrt{ab}, b$ is said to form a GP and $\sqrt{ab}$ has been inserted (as a GM) between a and b.

$$\sqrt{ab} \le \frac{a+b}{2} \quad \ldots\text{(i)}$$

$$\text{GM} \le \text{AM}$$

Now, (i) is equivalent to $2\sqrt{ab} \le a + b$

(Multiplying an inequality by a +ve real number does not alter the inequality).

i.e.,
$$0 \le a + b - 2\sqrt{ab}$$

Subtracting the same real number from both sides of an inequality does not alter the inequality.

i.e.,
$$a + b - 2\sqrt{ab} = (\sqrt{a} - \sqrt{b})^2 \ge 0$$

Note The proof shows that $\sqrt{ab} = \frac{a+b}{2}$, if and only if $\sqrt{a} = \sqrt{b}$ or $a = b$.

To sum up $\sqrt{ab} \le \frac{a+b}{2}$ *i.e.*, GM$\le$ AM equality holds, if and only if both the quantities are equal.

The GM – AM inequality is true in the general form

$$\sqrt[n]{a_1 \ldots a_n} \le \frac{a_1 + \ldots + a_n}{n}$$

If equality holds and only if $a_1 = a_2 = \ldots = a_n$

It is easy to deduce it for a set of 4 +ve numbers from the case of 2 numbers for

$$\begin{aligned}\sqrt[4]{a_1 a_2 a_3 a_4} &= \sqrt[4]{a_1 a_2} \cdot \sqrt[4]{a_3 a_4} \\ &= \sqrt{\sqrt{a_1 a_2} \cdot \sqrt{a_3 a_4}} \\ &\le \frac{1}{2}(\sqrt{a_1 a_2} + \sqrt{a_3 a_4})\end{aligned}$$

$$\le \frac{1}{2}\left[\frac{1}{2}(a_1 + a_2) + \frac{1}{2}(a_3 + a_4)\right]$$

$$= \frac{a_1 + a_2 + a_3 + a_4}{4}$$

This method can be extended to the case $n = 2^k$ for some $k = 2, 3, 4, \ldots$

Now, we have to prove

$$\sqrt[3]{a_1a_2a_3} \le \frac{a_1 + a_2 + a_3}{3} \quad \ldots\text{(ii)}$$

for $a_i > 0, i = 1, 2, 3$

Let $a_1 = x_1^3$

$a_2 = x_2^3$

$a_3 = x_3^3$

$x_i > 0, i = 1, 2, 3$

Now, (ii) becomes

$$3x_1x_2x_3 \le x_1^3 + x_2^3 + x_3^3$$

or

$$x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 \ge 0 \quad \ldots\text{(iii)}$$

Rewrite LHS of (iii) as

$$(x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1)$$

$$= (x_1 + x_2 + x_3)\frac{[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2]}{2}$$

$\because$ The sum of three squares is always non-negative.

The equality can occur if and only if

$$x_1 = x_2 = x_3 \text{ or } a_1 = a_2 = a_3.$$

Geometrical Interpretation of the AM–GM Inequality

We have segments of length a, b on a straight line putting them side by side along PR and RQ

O is the mid-point of PQ.

Draw a perpendicular to PQ at R to cut the semicircle on PQ at A and A'.

By the secant theorem

$$PR \cdot RQ = a \cdot b = A'R \cdot AR = AR^2$$

or

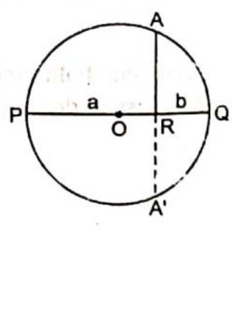
$$AR = \sqrt{a \cdot b}$$

AR represents the GM of a and b.

AM being PO = radius of circle

$\because$ In any circle, half of a chord is less than the radius.

$$\sqrt{ab} \le \frac{(a + b)}{2}$$



equality hold if and only if

Cauchy's Inequality or Cauchy Schwarz Inequality

If a, b, x, y are real, then

$$|ax + by| \le \sqrt{a^2 + b^2}\,\sqrt{x^2 + y^2}$$

Proof To prove

$$(ax + by)^2 \le (a^2 + b^2)(x^2 + y^2) \quad \text{...(A)}$$

i.e., $$a^2x^2 + 2axby + b^2y^2 \le a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2$$

$\Rightarrow$ $$2axby \le a^2y^2 + b^2x^2$$

$\Rightarrow$ $$a^2y^2 - 2axby + b^2y^2 \ge 0$$

$\Rightarrow$ $$(ay - bx)^2 \ge 0$$

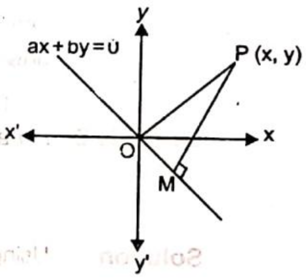
which is true.

Similarly, if $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$ are all real.

Then $|a_1b_1 + a_2b_2 + \dots + a_nb_n| \le \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \dots + b_n^2}$

Note Let $P(x, y)$ be a point in the plane having origin O.
Let $ax + by = 0$ be a given straight line through the origin.

Then $\dfrac{|ax + by|}{\sqrt{a^2 + b^2}}$ is the perpendicular distance PM of $P(x, y)$ from the given straight line $ax + by = 0$, $\sqrt{x^2 + y^2}$ is the distance of x, y from origin.



The slope of the line is $\dfrac{-a}{b}$ and of the line joining origin (0, 0) to (x, y) is $\dfrac{y}{x}$.

If these two lines are perpendicular and only then, $\dfrac{y}{x}\left(\dfrac{-a}{b}\right) = -1$ or $\dfrac{a}{x} = \dfrac{b}{y}$

This is the only case in which equality occur.

Harmonic Mean (HM)

The +ve number

$$\left[\frac{\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)}{n}\right]^{-1}$$

is called the HM of $a_1, a_2, \dots a_n$ and is denoted by H.

Example 1 *If $a > 0, b > 0, a + b = 1$, show that*

$$\sqrt{8 + \frac{1}{a^2}} + \sqrt{8 + \frac{1}{b^2}} \ge 24.$$

Solution $\because$ $$AM \ge GM$$

$\Rightarrow$ $$\frac{a + b}{2} \ge \sqrt{ab}$$

$$a + b \ge 2\sqrt{ab}$$

$$1 \ge 2\sqrt{ab}$$

$$4ab \le 1$$

$$ab \le \frac{1}{4} \quad \text{...(i)}$$

Now, $$\frac{\sqrt{8+\frac{1}{a^2}}+\sqrt{8+\frac{1}{b^2}}}{2} \geq \sqrt{8+\frac{1}{a^2}}\sqrt{8+\frac{1}{b^2}} \qquad [\because \text{AM} \geq \text{GM}]$$

$$\Rightarrow \quad \sqrt{8+\frac{1}{a^2}}+\sqrt{8+\frac{1}{b^2}} \geq 2\sqrt{64+8\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{a^2b^2}}$$

$$\geq 2\sqrt{64+8\left(\frac{b^2+a^2}{a^2b^2}\right)+\frac{1}{a^2b^2}}$$

$$\geq 2\sqrt{64+\frac{16}{ab}+\frac{1}{a^2b^2}}$$

$$\geq 2\sqrt{64+16\times 4+16} \qquad \left[\begin{array}{l} \because a^2+b^2 \geq 2ab \\ 8(a^2+b^2) \geq 16ab \\ 8\dfrac{(a^2+b^2)}{a^2b^2} \geq \dfrac{16}{ab} \end{array}\right]$$

$$\geq 2\sqrt{144}$$

$$\geq 2\times 12 \geq 24$$

Example 2 *For any +ve a, b. Prove that*

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq 8.$$

Solution Using AM-GM inequality

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq 2\sqrt{\left(a+\frac{1}{a}\right)^2\left(b+\frac{1}{b}\right)^2}$$

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq 2\left(ab+\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}\right)$$

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq 2(2+2) \qquad \left[\begin{array}{l} \because ab+\dfrac{1}{ab} \geq 2 \\ \dfrac{a}{b}+\dfrac{b}{a} \geq 2 \end{array}\right]$$

Hence, $$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq 8$$

Example 3 *If $a_1, a_2, a_3, \ldots, a_n$ are non –ve and $a_1a_2a_3 \ldots a_n = 1$, then show that*

$$(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n) \geq 2^n.$$

Solution By AM-GM

$$\left(\frac{1+a_i}{2}\right) \geq \sqrt{a_i}, \; i = 1, 2, \ldots, n$$

Multiplying the inequalities,

$$(1+a_1)(1+a_2)\ldots(1+a_n) \geq 2^n\sqrt{a_1a_2\ldots a_n}$$

$$(1+a_1)(1+a_2)\ldots(1+a_n) \geq 2^n \qquad [\because a_1a_2\ldots a_n = 1]$$

Hence proved.

Example 4 *If $a, b, c > 0$, then show that*

$$a^2(b+c)+b^2(c+a)+c^2(a+b) \geq 6abc.$$

Solution By applying AM ≥ GM for 6 numbers

$$a^2b, a^2c, b^2c, b^2a, c^2a, c^2b$$

We get,
$$\frac{a^2b+a^2c+b^2c+b^2a+c^2a+c^2b}{6} \geq [a^2b \cdot a^2c \cdot b^2c \cdot b^2a \cdot c^2a \cdot c^2b]^{\frac{1}{6}}$$

$$\Rightarrow \quad a^2(b+c)+b^2(c+a)+c^2(a+b) \geq 6abc$$

Example 5 *If a, b, c are greater than zero. Prove that*

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} > 6$$

or
$$bc(b+c)+ca(c+a)+ab(a+b) > 6abc.$$

Solution From example 4, we have

$$a^2(b+c)+b^2(c+a)+c^2(a+b) \geq 6abc$$

$$\Rightarrow \quad (a^2b+b^2a)+(a^2c+c^2a)+(b^2c+c^2b) \geq 6abc$$

$$\Rightarrow \quad ab(a+b)+ac(a+c)+bc(b+c) \geq 6abc$$

$$\Rightarrow \quad \left(\frac{a+b}{c}\right)+\left(\frac{a+c}{b}\right)+\left(\frac{b+c}{a}\right) \geq 6$$

Aliter

$$\frac{\frac{a}{b}+\frac{b}{a}}{2} > \sqrt{\frac{a}{b} \times \frac{b}{a}}$$

$$\Rightarrow \quad \frac{a}{b}+\frac{b}{a} > 2 \qquad \text{...(i)}$$

Similarly,
$$\frac{b}{c}+\frac{c}{b} > 2 \qquad \text{...(ii)}$$

and
$$\frac{a}{c}+\frac{c}{a} > 2 \qquad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right) > 6$$

or
$$\left(\frac{a}{b}+\frac{c}{b}\right)+\left(\frac{b}{a}+\frac{c}{a}\right)+\left(\frac{a}{c}+\frac{b}{c}\right) > 6$$

$$\Rightarrow \quad \left(\frac{a+c}{b}\right)+\left(\frac{b+c}{a}\right)+\left(\frac{a+b}{c}\right) > 6$$

Hence proved.

Example 6 *If $a_i > 0$, $\forall\, i = 1, 2, \ldots, n$. Prove that*

$$(a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) > n^2$$

Solution $\because$ AM > GM, we have

$$\frac{a_1+a_2+\ldots+a_n}{n} > (a_1a_2\ldots a_n)^{1/n}$$

and $\dfrac{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}}{n} > \left(\dfrac{1}{a_1}\cdot\dfrac{1}{a_2}\ldots\dfrac{1}{a_n}\right)^{1/n}$

$\Rightarrow$ $(a_1 + a_2 + \ldots + a_n) > n(a_1a_2 \ldots a_n)^{1/n}$

and $\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}\right) > n\left(\dfrac{1}{a_1a_2\ldots a_n}\right)^{1/n}$

$\Rightarrow$ $(a_1 + a_2 \ldots + a_n)\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}\right) > n[a_1a_2 \ldots a_n]^{(1/n)}\left[\dfrac{n}{(a_1a_2\ldots a_n)^{1/n}}\right]$

$\Rightarrow$ $(a_1 + a_2 + \ldots + a_n)\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}\right) > n^2$

Example 7 *Show that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ac}}$$

Solution $\because$ AM > GM

$\therefore$ $\dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}\right) > \sqrt{\dfrac{1}{a}\cdot\dfrac{1}{b}}$

$$\frac{1}{a}+\frac{1}{b} > 2\left(\frac{1}{\sqrt{ab}}\right)$$

Similarly, $\dfrac{1}{2}\left(\dfrac{1}{b}+\dfrac{1}{c}\right) > \sqrt{\left(\dfrac{1}{bc}\right)}$

i.e., $\dfrac{1}{b}+\dfrac{1}{c} > 2\left(\dfrac{1}{\sqrt{bc}}\right)$

and $\dfrac{1}{2}\left(\dfrac{1}{c}+\dfrac{1}{a}\right) > \sqrt{\dfrac{1}{ca}}$ *i.e.*, $\dfrac{1}{c}+\dfrac{1}{a} > 2\left(\dfrac{1}{\sqrt{ca}}\right)$

Adding these three results

$$\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{c}+\frac{1}{a}\right) > \frac{2}{\sqrt{ab}}+\frac{2}{\sqrt{bc}}+\frac{2}{\sqrt{ca}}$$

or $2\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right) > 2\left(\dfrac{1}{\sqrt{ab}}+\dfrac{1}{\sqrt{bc}}+\dfrac{1}{\sqrt{ca}}\right)$

or $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} > \dfrac{1}{\sqrt{ab}}+\dfrac{1}{\sqrt{bc}}+\dfrac{1}{\sqrt{ca}}$

Hence proved.

Example 8 *If x, y, z are three +ve integers. Prove that*

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) > 9 \qquad \text{(RMO 1994)}$$

Solution $\because$ AM > GM

$\therefore$ $\dfrac{x+y+z}{3} > (xyz)^{1/3}$

or $$x + y + z > 3(xyz)^{1/3} \quad \text{...(i)}$$

Similarly, $$\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) > 3\left(\frac{1}{x} \cdot \frac{1}{y} \cdot \frac{1}{z}\right)^{1/3} \quad \text{...(ii)}$$

Multiplying (i) and (ii), we get

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) > 9(xyz)^{1/3}\left(\frac{1}{xyz}\right)^{1/3}$$

$$\Rightarrow \quad (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) > 9$$

Hence proved.

Example 9 *Prove that*

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} > n \qquad \textbf{(RMO Punjab 1993)}$$

Solution $\because$ AM > GM

$$\therefore \quad \frac{1}{n}\left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}\right) > \left(\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \frac{a_3}{a_4} \dots \frac{a_{n-1}}{a_n} \cdot \frac{a_n}{a_1}\right)$$

or $$\left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}\right) > n(1)^n$$

or $$\left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \dots + \frac{a_{n-1}}{a_n}\right) > n$$

Hence proved.

Example 10 *If $a, b, c > 0$, show that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \qquad \textbf{(IMP)}$$

Solution

$$\frac{a}{b+c} = \frac{a+b+c}{b+c} - 1$$

$$\frac{b}{c+a} = \frac{a+b+c}{c+a} - 1$$

and $$\frac{c}{a+b} = \frac{a+b+c}{a+b} - 1$$

Now,

$$\text{LHS} = (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) - 3$$

But $$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq 3\sqrt[3]{\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}}$$

Also, $$2(a+b+c) = (b+c) + (c+a) + (a+b) \geq 3\,(b+c)(c+a)(a+b)$$

So, $$(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq \frac{3 \times 3}{2}$$

$$\frac{a}{a+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{9}{2} - 3 = \frac{3}{2}$$

Example 11 *Find the minimum value of* $\sqrt{a^2+b^2}$, *when* $3a+4b=15$.

Solution
$$|3a+4b| \le \sqrt{a^2+b^2}\,\sqrt{3^2+4^2}$$
$$= 5\sqrt{a^2+b^2} \qquad \text{[By Cauchy's inequality]}$$

So, under given condition
$$\sqrt{a^2+b^2} \ge \frac{15}{5} = 3$$

Weighted AM-GM Inequality

If $a_1, a_2, \dots, a_n$ are n +ve real numbers and $m_1, m_2, \dots, m_n$ are n +ve rational numbers, then
$$\frac{m_1a_1 + m_2a_2 + \dots + m_na_n}{m_1 + m_2 + \dots + m_n} > (a_1^{m_1} \cdot a_2^{m_2} \dots a_n^{m_n})^{\frac{1}{m_1 + m_2 + \dots + m_n}}$$

Some Important Inequalities

(A) If $a_1, a_2, \dots, a_n$ are n +ve distinct real numbers, then

(i) $$\frac{a_1^m + a_2^m + \dots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^m$$

if $\qquad m < 0$ or $m > 1$

(ii) $$\frac{a_1^m + a_2^m + \dots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^m$$

if $\qquad 0 < m < 1$

i.e., the AM of mth powers of n +ve quantities is greater than the mth power of their AM except when
m is a +ve proper fraction.

(iii) If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are rational numbers and m is a rational number, then
$$\frac{b_1a_1^m + b_2a_2^m + \dots + b_na_n^m}{b_1 + b_2 + \dots + b_n} > \left(\frac{b_1a_1 + b_2a_2 + \dots + b_na_n}{b_1 + b_2 + \dots + b_n}\right)^m$$

if $m < 0$ or $m > 1$

and
$$\frac{b_1a_1^m + b_2a_2^m + \dots + b_na_n^m}{b_1 + b_2 + \dots + b_n} < \left(\frac{b_1a_1 + b_2a_2 + \dots + b_na_n}{b_1 + b_2 + \dots + b_n}\right)^m$$

if $\qquad 0 < m < 1$

(B) If $a_1, a_2, a_3, \dots, a_n$ are distinct +ve real numbers and p, q, r are natural numbers, then
$$\frac{a_1^{p+q+r} + a_2^{p+q+r} + \dots + a_n^{p+q+r}}{n} > \frac{a_1^p + a_2^p + \dots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \dots + a_n^q}{n} \cdot \frac{a_1^r + a_2^r + \dots + a_n^r}{n}$$

(C) Weierstrass Inequality

(i) If $a_1, a_2, \dots, a_n$ are n +ve real numbers, then for $n \ge 2$
$$(1 + a_1)(1 + a_2)\dots(1 + a_n) > 1 + a_1 + a_2 + \dots + a_n$$

(ii) If $a_1, a_2, \dots, a_n$ are +ve real numbers less than unity, then
$$(1 - a_1)(1 - a_2)\dots(1 - a_n) > 1 - a_1 - a_2 - \dots - a_n$$

(D) Tchebychef's Inequality

If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are real numbers such that

$$a_1 \le a_2 \le a_3 \le \dots \le a_n$$

and
$$b_1 \le b_2 \le b_3 \le \dots \le b_n$$

Then
$$n(a_1b_1 + a_2b_2 + \dots + a_nb_n) \ge (a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)$$

or
$$\frac{a_1b_1 + a_2b_2 + \dots + a_nb_n}{n} \ge \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \dots + b_n}{n}\right)$$

(E) Holder's Inequality

$$(a_1b_1 + a_2b_2 + \dots a_nb_n)^{pq} \le (a_1^p + a_2^p + \dots + a_n^p)^q (b_1^q + b_2^q + \dots + b_n{}^q)^p$$

where $\frac{1}{p} + \frac{1}{q} = 1$, a_i and b_i are non-negative real numbers.

Example 1 *If a, b are +ve real numbers. Prove that*

(i) $\left(\frac{a^2 + b^2}{a + b}\right)^{a+b} > a^ab^b$ (ii) $a^bb^a < \left(\frac{a + b}{2}\right)^{a+b} < a^ab^b$

Solution (i) $\because$ Weighted AM > Weighted GM

$$\therefore \quad \left(\frac{a \cdot a + b \cdot b}{a + b}\right) > (a^a \cdot b^b)^{\frac{1}{a+b}}$$

$$\Rightarrow \quad \left(\frac{a^2 + b^2}{a + b}\right)^{a+b} > a^a\, b^b$$

(ii) $\because$ Weighted AM > Weighted GM

$$\frac{b \cdot a + a \cdot b}{b + a} > (a^b \cdot b^a)^{\frac{1}{a+b}}$$

$$\Rightarrow \quad \frac{2ab}{a + b} > (a^b \cdot b^a)^{\frac{1}{a+b}}$$

$$\Rightarrow \quad \frac{a + b}{2} > \frac{2ab}{a + b} > (a^b \cdot b^a)^{\frac{1}{a+b}} \qquad \left[\because \text{AM} > \text{HM}, \therefore \frac{a + b}{2} > \frac{2ab}{a + b}\right]$$

$$\Rightarrow \quad \frac{a + b}{2} > (a^b \cdot b^a)^{\frac{1}{a+b}}$$

$$\Rightarrow \quad \left(\frac{a + b}{2}\right)^{a+b} > a^bb^a \qquad \dots(i)$$

Again, $\because$ weighted AM > weighted GM

$$\Rightarrow \quad \frac{a \cdot \frac{1}{a} + b \cdot \frac{1}{b}}{a + b} > \left[\left(\frac{1}{a}\right)^a \cdot \left(\frac{1}{b}\right)^b\right]^{\frac{1}{a+b}}$$

$$\Rightarrow \quad \frac{2}{a+b} > \left(\frac{1}{a^a b^b}\right)^{\frac{1}{a+b}}$$

$$\Rightarrow \quad \left(\frac{2}{a+b}\right)^{a+b} > \frac{1}{a^a b^b}$$

$$\Rightarrow \quad a^a b^b > \left(\frac{a+b}{2}\right)^{a+b} \quad \text{...(ii)}$$

From (i) and (ii), we have

$$a^b \cdot b^a < \left(\frac{a+b}{2}\right)^{a+b} < a^a \cdot b^b$$

Example 2 *Prove that*

$$\frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} + \frac{x^2+y^2}{x+y} > x+y+z.$$

Solution AM of 2nd power > 2nd power of AM

$$\frac{y^2+z^2}{2} > \left(\frac{y+z}{2}\right)^2$$

$$\Rightarrow \quad y^2+z^2 > 2\left(\frac{y+z}{2}\right)^2$$

$$\Rightarrow \quad \frac{y^2+z^2}{y+z} > \frac{y+z}{2} \quad \text{...(i)}$$

Similarly, $$\frac{x^2+z^2}{x+z} > \frac{z+x}{2} \quad \text{...(ii)}$$

and $$\frac{x^2+y^2}{x+y} > \frac{x+y}{2} \quad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$\frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} + \frac{x^2+y^2}{x+y} > x+y+z$$

Example 3 *If $a, b > 0$ such that $a^3 + b^3 = 8$, then show that $a + b \le 2$.*

Solution AM of $\left(\frac{1}{3}\right)$ th power $\le \left(\frac{1}{3}\right)$ th power of AM

$$\therefore \quad \frac{(a^3)^{1/3} + (b^3)^{1/3}}{2} \le \left(\frac{a^3+b^3}{2}\right)^{1/3}$$

$$\Rightarrow \quad \frac{a+b}{2} \le 1$$

$$\Rightarrow \quad a+b \le 2$$

Example 4 *If $m > 1, n \in N$ show that*

$$1^m + 2^m + 2^{2m} + 2^{3m} + \ldots + 2^{nm-m} > n^{1-m}(2^n - 1)^m.$$

Solution $\because$ $m > 0$

AM of mth power > mth power of AM

$$\frac{1^m + 2^m + 4^m + 8^m + \ldots + (2^{n-1})^m}{n} > \left(\frac{1+2+4+\ldots+2^{n-1}}{n}\right)^m$$

$$\Rightarrow \quad 1^m + 2^m + 4^m + \ldots + 2^{(n-1)m} > n\left(\frac{2^n - 1}{n}\right)^m > n^{1-m}(2^n - 1)^m$$

Hence proved.

Example 5 *If $m > 1$, then show that $2^m + 4^m + 6^m + \ldots + (2n)^m > n(n+1)^m$*

Solution AM of mth power > mth power of AM

$$\frac{2^m + 4^m + 6^m + \ldots + (2n)^m}{n} > \left[\frac{(2+4+6+\ldots+2n)}{n}\right]^m$$

$$\Rightarrow \quad 2^m + 4^m + 6^m + \ldots + (2n)^m > n\left[\frac{n(n+1)}{n}\right]^m$$

$$\Rightarrow \quad 2^m + 4^m + 6^m + \ldots + (2n)^m > n(n+1)^m$$

Hence proved.

Example 6 *If n is a +ve integer > 1, show that*

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} < n\sqrt{\frac{n+1}{2}}$$

Solution AM of $\left(\frac{1}{2}\right)$ th power < $\left(\frac{1}{2}\right)$ th power of AM

Since, $0 < m < 1$

$$\therefore \quad \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}}{n} < \left(\frac{1+2+3\ldots+n}{n}\right)^{1/2}$$

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} < n\left(\frac{\sqrt{n+1}}{2}\right)$$

Example 7 *Show that* $C_0^4 + C_1^4 + C_2^4 + \ldots + C_n^4 > \dfrac{2^{4n}}{n^3}$,

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Solution AM of 4th power > 4th power of AM

$$\frac{C_0^4 + C_1^4 + \ldots + C_n^4}{n} > \left(\frac{C_0 + C_1 + C_2 + \ldots + C_n}{n}\right)^4$$

$$\Rightarrow \quad C_0^4 + C_1^4 + C_2^4 + \ldots + C_n^4 > n\left(\frac{2^n}{n}\right)^4$$

$$\Rightarrow \quad C_0^4 + C_1^4 + C_2^4 + \ldots + C_n^4 > \frac{2^{4n}}{n^3}$$

Example 8 *If $a_1, a_2, \dots, a_n$ are +ve real numbers less than unity and $S_n = a_1 + a_2 + \dots + a_n$, then*

$$(i)\ 1 - S_n < (1-a_1)(1-a_2)\dots(1-a_n) < \frac{1}{1+S_n}$$

$$(ii)\ 1 + S_n < (1+a_1)(1+a_2)\dots(1+a_n) < \frac{1}{1-S_n}$$

provided $S_n < 1$.

Solution We have

$$0 < a_i < 1\ \forall\ i = 1, 2, \dots, n$$

$$\Rightarrow\quad 0 < 1 - a_i^2 < 1\ \forall\ i = 1, 2, \dots, n$$

$$\Rightarrow\quad 0 < (1-a_i)(1+a_i) < 1\ \forall\ i = 1, 2, \dots, n$$

$$\Rightarrow\quad 0 < 1 - a_i < \frac{1}{1+a_i} \text{ and } 0 < 1 + a_i < \frac{1}{1-a_i}\ \forall\ i = 1, 2, \dots, n$$

$$\Rightarrow\quad 0 < (1+a_1)(1+a_2)\dots(1+a_n) < \frac{1}{(1-a_1)(1+a_2)\dots(1+a_n)} \qquad \dots(i)$$

By Weierstrass inequality

$$(1+a_1)(1+a_2)\dots(1+a_n) > 1 + (a_1 + a_2 + \dots + a_n)$$

and

$$(1-a_1)(1-a_2)\dots(1-a_n) > 1 - (a_1 + a_2 + \dots + a_n)$$

$$\Rightarrow\quad (1+a_1)(1+a_2)\dots(1+a_n) > 1 + S_n \qquad \dots(ii)$$

and

$$(1-a_1)(1-a_2)\dots(1-a_n) > 1 - S_n$$

$$\Rightarrow\quad \frac{1}{(1+a_1)(1+a_2)\dots(1+a_n)} < \frac{1}{1+S_n} \qquad \dots(iii)$$

and

$$\frac{1}{(1-a_1)(1-a_2)\dots(1-a_n)} < \frac{1}{1-S_n}$$

From (i), (ii) and (iii), we get

$$1 - S_n < (1-a_1)(1-a_2)\dots(1-a_n) < \frac{1}{1+S_n}$$

and

$$1 + S_n < (1+a_1)(1+a_2)\dots(1+a_n) < \frac{1}{1-S_n}$$

provided $S_n < 1$

Example 9 *If $a_1, a_2, \dots, a_n$ be +ve real numbers. Prove that*

$$(a_1 + a_2 + \dots + a_n)^3 \le n^2(a_1^3 + a_2^3 + \dots + a_n^3)$$

Also, show that equality sign holds iff

$$a_1 = a_2 = \dots = a_n.$$

Solution Let two sets of numbers are

$$a_1^{3/2}, a_2^{3/2}, \dots, a_n^{3/2} \text{ and } a_1^{1/2}, a_2^{1/2}, \dots, a_n^{1/2}$$

By Cauchy schwartz inequality, we have

$$(a_1^{3/2}a_1^{1/2} + a_2^{3/2}a_2^{1/2} + \dots + a_n^{3/2}a_n^{1/2})^2 \le (a_1^3 + a_2^3 + \dots + a_n^3) \times (a_1 + a_2 + \dots + a_n)$$

or

$$(a_1^2 + a_2^2 + \dots + a_n^2)^2 \le (a_1^3 + a_2^3 + \dots + a_n^3)(a_1 + a_2 + \dots + a_n) \qquad \dots(i)$$

Again taking two sets of numbers $a_1, a_2, \dots, a_n$ and $1, 1, \dots, 1$.

Applying Cauchy Schwartz inequality

$$(a_1 \cdot 1 + a_2 \cdot 1 + \ldots + a_n \cdot 1)^2 \le (a_1^2 + a_2^2 + \ldots + a_n^2)(1^2 + 1^2 + 1^2 + \ldots + 1^2)$$

or $\quad (a_1 + a_2 + \ldots + a_n)^2 \le n(a_1^2 + a_2^2 + \ldots + a_n^2)^2 \quad \ldots$(ii)

From (i) and (ii), we get

$$(a_1 + a_2 + \ldots + a_n)^4 \le n^2(a_1^3 + a_2^3 + \ldots + a_n^3)(a_1 + a_2 + \ldots + a_n)$$

or $\quad (a_1 + a_2 + \ldots + a_n)^3 \le n^2(a_1^3 + a_2^3 + \ldots + a_n^3)$

Equality sign hold if and only if

$$\frac{a_1^{3/2}}{a_1^{1/2}} = \frac{a_2^{3/2}}{a_2^{1/2}} = \ldots = \frac{a_n^{3/2}}{a_n^{1/2}}$$

i.e., if and only if $a_1 = a_2 = \ldots = a_n$

Example 10 *Prove that*

$$\frac{1}{n} + \frac{2}{n-1} + \frac{3}{n-2} + \ldots + \frac{n}{1} \ge \frac{n+1}{2}\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{1}\right)$$

where n is any +ve integer.

Solution Let n is a +ve integer such that $1 < 2 < 3 < \ldots < (n-2) < (n-1) < n$

$$\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \ldots > \frac{1}{n-2} > \frac{1}{n-1} > \frac{1}{n}$$

or $$\frac{1}{n} < \frac{1}{n-1} < \frac{1}{n-2} < \ldots < \frac{1}{3} < \frac{1}{2} < \frac{1}{1}$$

For two sets of numbers 1, 2, 3,..., n

and $$\frac{1}{n}, \frac{1}{n-1}, \frac{1}{n-2}, \ldots \frac{1}{2}, \frac{1}{1}$$

By Tchebychef's inequality, we have

$$n\left(1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n-1} + 3 \cdot \frac{1}{n-2} + \ldots + n \cdot \frac{1}{1}\right)$$

$$\ge (1 + 2 + 3 + \ldots + n) \times \left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{1}\right)$$

$$\Rightarrow \quad n\left(\frac{1}{n} + \frac{2}{n-1} + \frac{3}{n-2} + \ldots + \frac{n}{1}\right) \ge \frac{n(n+1)}{2}\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{1}\right)$$

$$\therefore \quad \left(\frac{1}{n} + \frac{2}{n-1} + \frac{3}{n-2} + \ldots + \frac{n}{1}\right) \ge \frac{n+1}{2}\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{1}\right)$$

Example 11 *If a, b, c, d are +ve real numbers. Prove that*

$$(a^5 + b^5 + c^5 + d^5) \ge abcd(a + b + c + d).$$

Solution Assume $a < b < c < d$

Then $\quad a^4 < b^4 < c^4 < d^4$

For two sets *i.e.*, a, b, c, d

and $\quad a^4, b^4, c^4, d^4$

Apply Tchebychef's inequality, we have

$$(a+b+c+d)(a^4+b^4+c^4+d^4) \le 4(a\cdot a^4 + b\cdot b^4 + c\cdot c^4 + d\cdot d^4)$$

or $$(a^5+b^5+c^5+d^5) \ge (a+b+c+d)\frac{(a^4+b^4+c^4+d^4)}{4} \quad \ldots(i)$$

$\therefore$ $$\frac{a^4+b^4+c^4+d^4}{4} \ge (a^4b^4c^4d^4)^{1/4}$$

$\Rightarrow$ $$\frac{a^4+b^4+c^4+d^4}{4} \ge abcd$$

or $$(a+b+c+d)\frac{(a^4+b^4+c^4+d^4)}{4} \ge abcd\,(a+b+c+d)$$

Thus, $$(a^5+b^5+c^5+d^5) \ge abcd\,(a+b+c+d)$$

Jensen's Inequality

Suppose $f(x)$ is a twice differentiable function on an interval $[a, b]$ and $f''(x) < 0\ \forall\, a < x < b$.
Then for every +ve integer m and for all points.
$x_1, x_2, x_3, \ldots, x_m$ in $[a, b]$, we have

$$f\left(\frac{x_1+x_2+x_3+\ldots+x_m}{m}\right) \ge \frac{f(x_1)+f(x_2)+\ldots+f(x_m)}{m}$$

Moreover equality holds if and only if $x_1 = x_2 = x_3 = \ldots = x_n$

Note In this case, graph of $f(x)$ is concave down.

Corollary *Let $f'(x) > 0$ and $f''(x) < 0$ and let $x_1, x_2 \in [a, b]$, then*

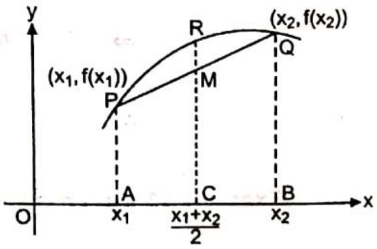
$$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$$

Coordinates of M are $\left\{\frac{x_1+x_2}{2}, \frac{f(x_1)+f(x_2)}{2}\right\}$

Thus, $$(a^5+b^5+c^5+d^5) \ge abcd\,(a+b+c+d)$$

Coordinates of R are $\left\{\frac{x_1+x_2}{2}, f\left(\frac{x_1+x_2}{2}\right)\right\}$

Now, from the figure



$$RC > MC$$

$$RC = f\left(\frac{x_1+x_2}{2}\right)$$

$$MC = \frac{f(x_1)+f(x_2)}{2}$$

$\therefore$ $$f\left(\frac{x_1+x_2}{2}\right) \ge \frac{f(x_1)+f(x_2)}{2}$$

Concept If $f''(x) > 0,\ \forall x \in\ ]a, b[$, then

$$f\left(\frac{x_1+x_2+\ldots+x_m}{m}\right) \le \frac{f(x_1)+f(x_2)+\ldots+f(x_m)}{m}$$

Note In this case, graph of $f(x)$ is concave up.

It is easy to derive the AM-GM inequality as a special case of the Jensen's inequality.

Suppose $y_1, y_2, \dots, y_m$ are +ve real numbers.

Let
$$x_i = \log y_i \quad \text{for } i = 1, 2, \dots, m$$
i.e., logarithm w.r.t. the base e (so that for $i = 1, 2, \dots, m$, we have $y_i = e^{x_i}$).

Take $[a, b]$ to be any interval containing all the numbers $x_1, x_2, \dots, x_m$

The function $h(x) = e^x$ satisfies the condition in the inequality.

$\because$
$$h''(x) = e^x \text{ is +ve.}$$
$\therefore$ By Jensen's inequality, we have
$$e^{\frac{x_1 + x_2 + \dots + x_m}{m}} \le \frac{e^{x_1} + e^{x_2} + \dots + e^{x_m}}{m}$$
$\because$
$$e^{x_i} = y_i \quad \text{for } i = 1, 2, \dots, m$$
i.e.,
$$(y_1 y_2 \dots y_m)^{\frac{1}{m}} \le \frac{y_1 + y_2 + \dots + y_m}{m}$$
which is truly by AM-GM inequality. Also, equality holds if and only if all x_i' are equal which is same thing as saying that all y_i's are equal.

Example 1 *ABC is an acute angled triangle, show that*
$$\cos A + \cos B + \cos C \le 3/2.$$

Solution $\because$ *ABC* is an acute angled triangle.

$\therefore$ A, B, C would lie in the interval $[0, \pi/2]$

Let
$$h(x) = \cos x$$
$$h'(x) = -\sin x$$
$$h''(x) = -\cos x < 0 \quad \forall x \in \left]0, \frac{\pi}{2}\right[$$
By using Jensen's inequality
$$f\left(\frac{x_1 + x_2 + \dots + x_m}{m}\right) \ge \frac{f(x_1) + f(x_2) \dots f(x_m)}{m}$$
$$\Rightarrow \quad \cos\left(\frac{A+B+C}{3}\right) \ge \frac{\cos A + \cos B + \cos C}{3}$$
Now,
$$A + B + C = \pi$$
$$\Rightarrow \quad \cos\frac{\pi}{3} \ge \frac{\cos A + \cos B + \cos C}{3}$$
$$\Rightarrow \quad \cos A + \cos B + \cos C \le 3/2$$
Equality holds if and only if $A = B = C$

Example 2 *In a* ΔABC; $A, B, C \in (0, \pi)$, *show that*
$$\cos A + \cos B + \cos C \le 3/2.$$

Solution Here, it is not given that the ΔABC is acute angled triangle.

So, we need to modify the solution a little.

For this, we first note that in any ΔABC (whether it is acute angled or not).

$$\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2} \text{ are always +ve.}$$

Hence,

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$= 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) \le 2\sin\frac{C}{2}$$

Similarly, $\cos B + \cos C \le 2\sin\frac{A}{2}$

$$\cos C + \cos A \le 2\sin\frac{B}{2}$$

Adding these we have

$$\cos A + \cos B + \cos C \le \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \qquad \text{...(i)}$$

Instead of applying Jensen's inequality to the function $f(x) = \cos x$ over $[0, \pi]$, as we do not know about second derivative of $\cos x$ which is neither +ve throughout nor –ve throughout in $0 < x < \pi$.

$\therefore$ By applying Jensen's inequality

for $$g(x) = \sin\left(\frac{x}{2}\right)$$

$$g'(x) = \frac{1}{2}\cos\frac{x}{2}$$

$$g''(x) = -\frac{1}{4}\sin\frac{x}{2} < 0 \;\forall\; x \text{ in }]\,0, \pi\,[$$

$\therefore$ Applying the inequality

$$g\left(\frac{A+B+C}{3}\right) \ge \frac{g(A)+g(B)+g(C)}{3}$$

$$\Rightarrow \quad \frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \le \sin\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\cos A + \cos B + \cos C \le \frac{3}{2}$$

Example 3 *Show that*

$$\tan A + \tan B + \tan C \ge 3\sqrt{3},\; A, B, C \in \left[0, \frac{\pi}{2}\right]$$

Solution Let

$$f(x) = \tan x$$

$$f'(x) = \sec^2 x$$

$$f''(x) = 2\sec^2 x \tan x > 0 \;\forall\; x \in [0, \pi/2[$$

$$f\left(\frac{A+B+C}{3}\right) \le \frac{f(A)+f(B)+f(C)}{3}$$

$$\tan A + \tan B + \tan C \geq 3 \tan\left(\frac{A+B+C}{3}\right)$$

$$\geq 3 \tan \frac{\pi}{3}$$

$$\therefore \quad \tan A + \tan B + \tan C \geq 3\sqrt{3}$$

Example 4 *If* $0 < A_i < \pi$, $\forall\, i = 1, 2, \ldots, n$, *show that*

$$\sin A_1 + \sin A_2 + \ldots + \sin A_n \leq n \sin\left(\frac{A_1 + A_2 + \ldots + A_n}{n}\right)$$

Solution Let

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x < 0,\ \forall\, x \in\ ]0, \pi[$$

$\therefore$ Using Jensen's inequality

$$f\left(\frac{A_1 + A_2 + \ldots + A_n}{n}\right) \geq \frac{f(A_1) + f(A_2) + \ldots + f(A_n)}{n}$$

$$\Rightarrow \quad \sin\left(\frac{A_1 + A_2 + \ldots + A_n}{n}\right) \geq \frac{\sin A_1 + \sin A_2 + \ldots + \sin A_n}{n}$$

$$\Rightarrow \quad \sin A_1 + \sin A_2 + \ldots + \sin A_n \leq n \sin \frac{(A_1 + A_2 + \ldots + A_n)}{n}$$

Finding the greatest value of $a^m b^n c^p \ldots$, when $m, n, p, \ldots$ being +ve integers, $a + b + c$ is constant.

Let Z denotes the product $a^m b^n c^p \ldots$

Then $\quad Z = a^m b^n c^p$

$$= \left[m^m\left(\frac{a}{m}\right)^m\right]\left[n^n\left(\frac{b}{n}\right)^n\right]\left[p^p\left(\frac{c}{p}\right)^p\right]\ldots$$

$$= m^m \cdot n^n \cdot p^p \ldots \left(\frac{a}{m}\right)^m\left(\frac{b}{n}\right)^n\left(\frac{c}{p}\right)^p \qquad \ldots\text{(i)}$$

$\because$ m, n, p are constants.

m^m, n^n, p^p are also constants.

Hence, Z will be maximum when

$$\left(\frac{a}{m}\right)^m\left(\frac{b}{n}\right)^n\left(\frac{c}{p}\right)^p \ldots \text{ is maximum.}$$

But $\left(\frac{a}{m}\right)^m\left(\frac{b}{n}\right)^n\left(\frac{c}{p}\right)^p \ldots$ is the product of m factors such that each equal to $\left(\frac{a}{m}\right)$, n factors each equal to $\left(\frac{b}{n}\right)$, p factors each equal to $\left(\frac{c}{p}\right)$ etc.

The sum of these factors is equal to $m\left(\frac{a}{m}\right) + n\left(\frac{b}{n}\right) + p\left(\frac{c}{p}\right) + \ldots$ or $a + b + c \ldots$, which is given to be constant. Hence, the product $\left(\frac{a}{m}\right)^m\left(\frac{b}{n}\right)^n\left(\frac{c}{p}\right)^p \ldots$

will be maximum when all the factors $\frac{a}{m}, \frac{b}{n}, \frac{c}{p}$, ... are equal *i.e.,* when

$$\frac{a}{m}=\frac{b}{n}=\frac{c}{p}=\dots=\frac{a+b+c+\dots}{m+n+p+\dots}$$

Thus, the greatest value of the product Z from Eq. (i)

$$= m^m n^n p^p \dots \left(\frac{a+b+c+\dots}{m+n+p+\dots}\right)^m$$

$$\left(\frac{a+b+c+\dots}{m+n+p+\dots}\right)^n = m^m n^n p^p \dots \left(\frac{a+b+c+\dots}{m+n+p+\dots}\right)^{m+n+p+\dots}$$

Corollary Let $x_1, x_2, x_3, \dots, x_n$ are n +ve variables and Z is constant, then

If $x_1 + x_2 + x_3 + \dots + x_n = Z$ (constant)

The value of $x_1 x_2 x_3 \dots x_n$ is greatest when $x_1 = x_2 = x_3 = \dots = x_n$ and is given by $\left(\frac{Z}{n}\right)^n$.

Example 1 *Find the greatest value of $a^2b^3c^4$ subject to the condition $a + b + c = 18$.*

Solution Let $Z = a^2b^3c^4$

$$Z = 2^2 3^3 4^4 \left(\frac{a}{2}\right)^2 \left(\frac{b}{3}\right)^3 \left(\frac{c}{4}\right)^4 \quad \dots(i)$$

$\therefore$ Z will have maximum value when

$$\left(\frac{a}{2}\right)^2 \left(\frac{b}{3}\right)^3 \left(\frac{c}{4}\right)^4 \text{ is maximum.}$$

But $\left(\frac{a}{2}\right)^2 \left(\frac{b}{3}\right)^3 \left(\frac{c}{4}\right)^4$ is product of 2 + 3 + 4

i.e., 9 factors whose sum

$$= 2\left(\frac{a}{2}\right) + 3\left(\frac{b}{3}\right) + 4\left(\frac{c}{4}\right)$$

$$= a + b + c = 18 \quad \text{(constant)}$$

$\therefore$ $\left(\frac{a}{2}\right)^2 \left(\frac{b}{3}\right)^3 \left(\frac{c}{4}\right)^4$ will be maximum if all the factors are equal.

i.e., If $\frac{a}{2}=\frac{b}{3}=\frac{c}{4}=\frac{a+b+c}{2+3+4}=\frac{18}{9}=2$

$\therefore$ Maximum value of Z from Eq. (i)

$$= 2^2 \cdot 3^3 \cdot 4^4 (2)^2 (2)^3 (2)^4 \quad \left[\because \frac{a}{2}=\frac{b}{3}=\frac{c}{4}=2\right]$$

$$= 2^{19} \times 3^3$$

Example 2 *If $2x + 3y = 7$ and $x \geq 0$, $y \geq 0$, then find the greatest value of x^3y^4.*

Solution Let $Z = x^3y^4 = \left(\frac{3}{2}\right)^3 \left(\frac{4}{3}\right)^4 \left(\frac{2x}{3}\right)^3 \left(\frac{3y}{4}\right) \quad \dots(i)$

$\therefore$ Z will have maximum value when $\left(\frac{2x}{3}\right)^3 \left(\frac{3y}{4}\right)^4$ is maximum.

But $\left(\frac{2x}{3}\right)^3\left(\frac{3y}{4}\right)^4$ is the product of $3 + 4 = 7$ factors, the sum of which

$$= 3\left(\frac{2x}{3}\right) + 4\left(\frac{3y}{4}\right) = 2x + 3y = 7 \text{(constant)}$$

$\therefore \left(\frac{2x}{3}\right)^3\left(\frac{3y}{4}\right)^4$ will be maximum if all the factors are equal.

i.e., If $\frac{2x}{3} = \frac{3y}{4} = \frac{2x + 3y}{3 + 4} = \frac{7}{7} = 1$ $[\because 2x + 3y = 7]$

$\therefore$ From Eq. (i) the maximum value of Z *i.e.*, x^3y^4

$$= \left(\frac{3}{2}\right)^3\left(\frac{4}{3}\right)^4(1)^3(1)^4 \quad \left[\because \frac{2x}{3} = \frac{3y}{4} = 1\right]$$

$$= \frac{27}{8} \times \frac{256}{81} = \frac{32}{3}$$

Example 3 *Find the greatest value of $x^2y^3z^4$. If $x^2 + y^2 + z^2 = 1$, where x, y, z are +ve.*

Solution Let $A = x^2y^3z^4$

then $A^2 = x^4y^6z^8$

$$= 2^2 \cdot 3^3 \cdot 4^4\left(\frac{x^2}{2}\right)^2\left(\frac{y^2}{3}\right)^3\left(\frac{z^2}{4}\right)^4 \quad \text{...(i)}$$

$\therefore A$ will have maximum value when A^2 is maximum *i.e.*, when $\left(\frac{x^2}{2}\right)^2\left(\frac{y^2}{3}\right)^3\left(\frac{z^2}{4}\right)^4$ is maximum.

But $\left(\frac{x^2}{2}\right)^2\left(\frac{y^2}{3}\right)^3\left(\frac{z^2}{4}\right)^4$ is the product of $2 + 3 + 4$ *i.e.*, 9 factors.

The sum of which $= 2\left(\frac{x^2}{2}\right) + 3\left(\frac{y^2}{3}\right) + 4\left(\frac{z^2}{4}\right)$

$$= x^2 + y^2 + z^2 = 1 \text{ (constant)}$$

$\therefore \left(\frac{x^2}{2}\right)^2\left(\frac{y^2}{3}\right)^3\left(\frac{z^2}{4}\right)^4$ will be maximum if all the factors are equal *i.e.*,

if $\frac{x^2}{2} = \frac{y^2}{3} = \frac{z^2}{4} = \frac{x^2 + y^2 + z^2}{2 + 3 + 4} = \frac{1}{9}$ $[\because x^2 + y^2 + z^2 = 1]$

From Eq. (i) maximum value of A^2 is

$$2^2 \cdot 3^3 \cdot 4^4\left(\frac{1}{9}\right)^2\left(\frac{1}{9}\right)^3\left(\frac{1}{9}\right)^4 = \frac{2^2 \times 3^3 \times 4^4}{9^9}$$

$$= \frac{2^2 \times 3^3 \times 2^8}{3^{18}}$$

$$= \frac{2^{10}}{3^{15}}$$

$\therefore$ Maximum value of $x^2y^3z^4$ is $2^5 \times 3^{-15/2}$.

Example 4 *If $yz + zx + xy = 12$, where x, y, z are +ve values, find the greatest value of xyz.*

Solution Let $A = xyz$

$$A^2 = x^2y^2z^2 = (xy)(yz)(zx) \quad \ldots(i)$$

$\therefore$ A will have maximum value when A^2 is maximum *i.e.*, when $(xy)(yz)(zx)$ is maximum.

But $(xy)(yz)(zx)$ is product of 3 factors. The sum of which $xy + yz + zx = 12$ (constant)

$\therefore$ $(xy)(yz)(zx)$ will be maximum if all the factors are equal *i.e.*, if $\frac{xy}{1} = \frac{yz}{1} = \frac{zx}{1}$

$$= \frac{xy + yz + zx}{1+1+1} = \frac{12}{3} = 4$$

From Eq. (i), maximum value of $A^2 = 64$

$\therefore$ Maximum value of xyz is 8.

Example 5 *Find the maximum value of xyz when $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.*

Solution Let $A = xyz$

$$\therefore \quad A^2 = x^2y^2z^2 = a^2b^2c^2\left(\frac{x^2}{a^2}\right)\left(\frac{y^2}{b^2}\right)\left(\frac{z^2}{c^2}\right) \quad \ldots(i)$$

$\therefore$ A will be maximum when A^2 is maximum *i.e.*, when $\left(\frac{x^2}{a^2}\right)\left(\frac{y^2}{b^2}\right)\left(\frac{z^2}{c^2}\right)$ is maximum.

But $\left(\frac{x^2}{a^2}\right)\left(\frac{y^2}{b^2}\right)\left(\frac{z^2}{c^2}\right)$ is the product of 3 factors.

The sum of which $= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (constant).

$\therefore$ $\left(\frac{x^2}{a^2}\right)\left(\frac{y^2}{b^2}\right)\left(\frac{z^2}{c^2}\right)$ will be maximum if all the factors are equal.

i.e., If
$$\frac{\frac{x^2}{a^2}}{1} = \frac{\frac{y^2}{b^2}}{1} = \frac{\frac{z^2}{c^2}}{1}$$

$$= \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}{1+1+1} = \frac{1}{3} \quad \left[\because \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right]$$

From Eq. (i) maximum value of $A^2 = a^2 \cdot b^2 \cdot c^2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$

$\therefore$ Maximum value of $A = \frac{abc}{\sqrt{27}}$

Example 6 *Prove that the greatest value of xy is $c^3/\sqrt{2ab}$, if $a^2x^4 + b^2y^4 = c^6$.*

Solution Let $z = xy$

$$z^4 = (xy)^4 = \frac{1}{(ab)^2}(a^2x^4)(b^2y^4) \quad \ldots(i)$$

$\therefore$ z will maximum when z^4 is maximum *i.e.*, when $(a^2x^4)(b^2y^4)$ is maximum.
But $(a^2x^4)(b^2y^4)$ is the product of 2 factors whose sum is $a^2x^4 + b^2y^4 = c^6$
$\therefore$ $(a^2x^4)(b^2y^4)$ will be maximum if both the factors are equal.

i.e., If
$$\frac{a^2x^4}{1} = \frac{b^2y^4}{1}$$
$$= \frac{a^2x^4 + b^2y^4}{1+1} = \frac{c^6}{2}$$

$$\therefore \text{Maximum value of } z^4 = \frac{1}{(ab)^2}\left(\frac{c^6}{2}\right)\left(\frac{c^6}{2}\right)$$

Maximum value of z is $\dfrac{c^3}{\sqrt{2ab}}$.

Hence proved.

Example 7 *If $x^2 + y^2 = c^2$, find the least value of $\dfrac{1}{x^2} + \dfrac{1}{y^2}$.*

Solution Let $z' = \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{y^2 + x^2}{x^2y^2} = \dfrac{c^2}{x^2y^2}$

$\therefore z'$ will be minimum when $\dfrac{x^2y^2}{c^2}$ will be maximum.

Now, Let $z = \dfrac{x^2y^2}{c^2} = \dfrac{1}{c^2}(x^2)(y^2)$ ...(i)

$\therefore z$ will maximum when x^2y^2 is maximum but $(x^2)(y^2)$ is the product of two factors whose sum is $x^2 + y^2 = c^2$

$\therefore$ x^2y^2 will be maximum when both these factors are equal *i.e.*, when
$$\frac{x^2}{1} = \frac{y^2}{1} = \frac{x^2 + y^2}{1} = \frac{c^2}{1}$$

From Eq. (i) maximum value of $z = \dfrac{c^2}{4}$

$\therefore$ Least value of $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{4}{c^2}$

Example 8 *Find the greatest value of $(a + x)^3(a - x)^4$ for any real value of x numerically less than a.*

Solution Let
$$z = (a + x)^3(a - x)^4$$
$$= 3^3 \cdot 4^4 \left(\frac{a + x}{3}\right)^3\left(\frac{a - x}{4}\right)^4 \quad \text{...(i)}$$

z will be maximum, when $\left(\dfrac{a + x}{3}\right)^3\left(\dfrac{a - x}{4}\right)^4$ is maximum but $\left(\dfrac{a + x}{3}\right)^3\left(\dfrac{a - x}{4}\right)^4$ is product of $3 + 4 = 7$ factors.

The sum of which
$$= 3\left(\frac{a + x}{3}\right) + 4\left(\frac{a - x}{4}\right) = (a + x) + (a - x) = 2a$$

$\therefore \left(\frac{a+x}{3}\right)^3\left(\frac{a-x}{4}\right)^4$ will be maximum if all the factors are equal *i.e.*, if $\frac{a+x}{3}=\frac{a-x}{4}$

or $\quad 4a+4x=3a-3x$ or $x=-\frac{a}{7}$

So, from Eq. (i) maximum value of z

$$=3^3\cdot 4^4\left[\frac{a-(a/7)}{3}\right]^3\left[\frac{a+(a/7)}{4}\right]^4$$

$$=3^3\cdot 4^4\cdot\left(\frac{6a}{3\times 7}\right)^3\left(\frac{8a}{7\times 4}\right)^4$$

$$=\frac{6^3\cdot 8^4}{7^7}a^7$$

Example 9 *Find the greatest value of $(a-x)(b-y)(c-z)(ax+by+cz)$, where a, b, c are +ve quantities and $(a-x),(b-y),(c-z)$ are also +ve.*

Solution Let $\quad z=(a-x)(b-y)(c-z)(ax+by+cz)$

$$=\frac{1}{abc}(a^2-ax)(b^2-by)(c^2-cz)(ax+by+cz) \quad \text{...(i)}$$

$\therefore$ z will be maximum when $(a^2-ax)(b^2-by)(c^2-cz)(ax+by+cz)$ is maximum.

But $(a^2-ax)(b^2-by)(c^2-cz)(ax+by+cz)$ is the product of 4 factors, the sum of which

$$=(a^2-ax)+(b^2-by)+(c^2-cz)+(ax+by+cz)$$
$$=a^2+b^2+c^2 \text{ (constant)}$$

$\therefore (a^2-ax)(b^2-by)(c^2-cz)(ax+by+cz)$ will be maximum, if all factors are equal, *i.e.*, if

$$\frac{a^2-ax}{1}=\frac{b^2-by}{1}=\frac{c^2-cz}{1}$$
$$=\frac{ax+by+cz}{1}$$
$$=\frac{(a^2-ax)+(b^2-by)+(c^2-cz)+(ax+by+cz)}{1+1+1+1}$$
$$=\frac{a^2+b^2+c^2}{4}$$

$\therefore$ From Eq. (i), the maximum value of $z=\frac{1}{abc}\left(\frac{a^2+b^2+c^2}{4}\right)^4$

Example 10 Find the value of $\frac{(a+x)(b+x)}{(c+x)}$.

Solution Let $c+x=y$ or $x=y-c$

Then, the given expression

$$=\frac{(a+y-c)(b+y-c)}{y}=\frac{(a-c+y)(b-c+y)}{y}$$

$$= \frac{(a-c)(b-c)+y(b-c)+y(a-c)+y^2}{y}$$

$$= \frac{(a-c)(b-c)}{y}+(b-c)+(a-c)+y$$

$$= \left[\frac{\sqrt{(a-c)(b-c)}}{\sqrt{y}}-\sqrt{y}\right]^2+(b-c)+(a-c)+2\sqrt{(a-c)(b-c)}$$

We find that as the value of the given expression varies as y varies. It is minimum when

$$\left[\frac{\sqrt{(a-c)(b-c)}}{y}-\sqrt{y}\right]^2 \text{ is zero.}$$

i.e., $$\frac{\sqrt{(a-c)(b-c)}}{\sqrt{y}}-\sqrt{y}=0$$

or $$\sqrt{(a-c)(b-c)}=y$$

$\therefore$ The minimum value of the given expression

$$=(b-c)+(a-c)+2\sqrt{(a-c)(b-c)}$$

Concept If the product of any number of +ve quantities is given, then their sum will be minimum if the quantities are all equal.

Proof Let us consider first two quantities x and y. Their sum is denoted by Y' and product by Z.

$\therefore$ $$Y' = x+y \text{ and } Z = xy$$

We know that $$(x+y)^2-(x-y)^2=4xy$$

or $$(Y')^2-(x-y)^2=4Z$$

or $$(Y')^2=4Z+(x-y)^2$$

It is evident that $(Y')^2$ will be minimum and Z remaining constant *i.e.*, when $(x-y)^2$ is zero or when $x-y=0$ or $x=y$.

Let us suppose that there are more than two quantities. If we replace any two quantities x and y by two equal quantities $\sqrt{xy}$ and $\sqrt{xy}$, their product xy remains unaltered. Their sum is $\sqrt{xy}+\sqrt{xy}$ *i.e.*, $2\sqrt{xy}$ which is less than $x+y$. [$\because$ AM> GM]

So, the sum of any two quantities can be diminished by making the quantities equal whereas their product remains unchanged.

$\therefore$ When all the quantities are equal their sum will have minimum value.

Corollary

1. If $x_1x_2x_3 \ldots x_n = z$ (constant), then the value of $x_1+x_2+x_3+\ldots+x_n$ is least when $x_1=x_2=\ldots=x_n$, then $n(z)^{1/n}$ is least value of $x_1+x_2+x_3+\ldots+x_n$.
2. If $x_1+x_2+x_3+\ldots+x_n=z$ (constant) and if r may or may not lie between 0 and 1, then the least or the greatest value of $x_1^r+x_2^r+x_3^r+\ldots+x_n^r$ occurs when $x_1=x_2=x_3=\ldots=x_n$ and is given by $n^{1-r}\cdot z^r$.
3. If $x_1^r+x_2^r+x_3^r+\ldots+x_n^r$ and if r may or may not lie between 0 and 1, then the least or the greatest value of $x_1+x_2+x_3+\ldots+x_n$ occurs when $x_1=x_2=x_3=\ldots=x_n$ and is given by $n^{1-\frac{1}{r}}\cdot z^{\frac{1}{r}}$.

Example 1 *If $x^2y^3 = 6$, find the least value of $3x + 4y$ for +ve value of x and y.*

Solution We have $x^2y^3 = 6$

or $$\left(\frac{3x}{2}\right)^2\left(\frac{4y}{3}\right)^3 = 6 \times \left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3$$

$$\Rightarrow \quad \left(\frac{3x}{2}\right)^2\left(\frac{4y}{3}\right)^3 = 32 \qquad \text{...(i)}$$

Now, $\left(\frac{3x}{2}\right)^2\left(\frac{4y}{3}\right)^3$ is product of 5 (3 + 2) factors and product = 32, which is constant. Hence, the sum of these factors will be minimum when all of them are equal.

i.e., $$\frac{4y}{3} = \frac{3x}{2} \qquad \text{...(ii)}$$

$$\therefore \quad \left(\frac{3x}{2}\right)^2\left(\frac{3x}{2}\right)^3 = 32 \text{ or } \left(\frac{3x}{2}\right)^5 = 32 = 2^5 \qquad \text{[from Eq. (i)]}$$

or $$\frac{3x}{2} = 2$$

∴ From Eq. (ii), we have

$$\frac{3x}{2} = \frac{4y}{3} = 2$$

∴ Minimum value of the sum of the factors of

$$\left(\frac{3x}{2}\right)^2\left(\frac{4y}{3}\right)^3 = 2\left(\frac{3x}{2}\right) + 3\left(\frac{4y}{3}\right) = 2(2) + 3(2) = 10$$

Example 2 *Find the minimum value of $bcx + cay + abz$ when $xyz = abc$.*

Solution We have $abc = xyz$

or $$(bcx)(cay)(abz) = (abc)^3 \qquad \text{...(i)}$$

Now, $(bcx)(cay)(abz)$ is product of 3 factors. Its product is constant.

∴ Sum of these factors will be minimum when all of them are equal.

i.e., $$abz = cay = bcx \qquad \text{...(ii)}$$

Now, $$(bcx)(bcx)(bcx) = (abc)^3$$

or $$(bcx)^3 = (abc)^3$$

Hence, $$abc = bcx$$

From Eq. (ii), we get $bcx = acy = abc = abz$

∴ Required minimum value is $3abc$.

Example 3 *Prove that the cube is the rectangular parallelopiped of maximum volume for given surface and of minimum surface for given volume.*

Solution We have a, b, c as the edges of rectangular parallelopiped.

Let V be its volume and S be its surface.

Then $$S = 2(ab + bc + ca) \quad \text{and} \quad V = abc$$

(i) Surface constant

Now, we have to find a, b, c when V is maximum.

i.e., $$V = abc$$

$$V^2 = (abc)^2 = \frac{1}{8}(2ab)(2bc)(2ca)$$

Now, V^2 is maximum when $(2ab)(2bc)(2ca)$ is maximum.

But $(2ab)(2bc)(2ca)$ is the product of 3 factors sum of which $2ab + 2bc + 2ca$ is constant.

$\therefore$ The product $(2ab)(2bc)(2ca)$ will be maximum when all factors are equal *i.e.,* when

$$2ab = 2bc = 2ca$$

i.e., when $$\frac{1}{a} = \frac{1}{b} = \frac{1}{c} \qquad \text{[Dividing by } 2abc\text{]}$$

$$a = b = c$$

$\therefore$ When all the edges of the parallelopiped are equal *i.e.,* when it is cube.

(ii) Volume constant

$$abc = \text{constant}$$

$\therefore$ $$8(abc)^2 = \text{constant}$$

i.e., $$(2ab)(2bc)(2ca) = \text{constant}$$

Product of these factors $(2ab)$, $(2bc)$ and $(2ca)$ being constant, their sum $(2ab + 2bc + 2ca)$ will be minimum when all of these factors are equal.

i.e., when $$2ab = 2bc = 2ca$$

i.e., when $$\frac{1}{c} = \frac{1}{b} = \frac{1}{a}$$

i.e., $$c = b = a$$

i.e., when parallelopiped is a cube.

Example 4 *Prove that the equilateral triangle has maximum area for given perimeter and minimum perimeter for given area.*

Solution A and P are the area and the perimeter of a triangle respectively.

$\therefore$ $$A = \sqrt{s(s-a)(s-b)(s-c)} \text{ and } P = 2s$$

where $$2s = a + b + c$$

Let $$s - a = x$$
$$s - b = y$$
$$s - c = z$$

Then $$A = \sqrt{(sxyz)} \text{ or } A^2 = sxyz$$

$$x + y + z = (s-a) + (s-b) + (s-c)$$
$$= 3s - (a + b + c) = 3s - 2s = s$$

(i) If perimeter of triangle is constant.

$$P = 2s = 2(x + y + z) = \text{constant}$$
$$A^2 = (sxyz)$$

A^2 will be maximum, if xyz is maximum.

Now, xyz is the product of 3 factors whose sum is $x + y + z$ (a constant)

$\therefore$ xyz will be maximum when all the factors are equal

i.e., when $x = y = z$ *i.e.*,

when $\quad (s-a) = (s-b) = (s-c)$

i.e., when $a = b = c$ *i.e.*, when triangle is equilateral.

(ii) When area is constant.

$$A^2 = s(xyz) = (x + y + z)(xyz) = x^2yz + y^2zx + z^2xy$$

Let $\quad x^2yz = D,\ y^2zx = E$, and $z^2xy = F$

Then $\quad A^2 = D + E + F = \text{constant}$

$$P = 2s = 2(x + y + z) = 2\left(\frac{D}{xyz} + \frac{E}{xyz} + \frac{F}{xyz}\right)$$

$$= \frac{2(D + E + F)}{xyz} = \frac{2A^2}{xyz} = \frac{2A^2}{(DEF)^{1/4}} \qquad [\because DEF = x^4y^4z^4]$$

or $\quad P = \dfrac{2A^2}{(DEF)^{1/4}}$

So, perimeter P will be minimum when $\dfrac{2A^2}{(DEF)^{1/4}}$ is minimum *i.e.*, when $\dfrac{(DEF)^{1/4}}{2A^2}$ is maximum or $(DEF)^{1/4}$ is maximum or DEF is maximum.

But $D + E + F = \text{constant}$ (given)

$\therefore$ DEF is maximum only when all the factors are equal *i.e.*, when $D = E = F$

i.e., $\quad x^2yz = y^2zx = z^2xy$

or $\quad x = y = z$ or $(s-a) = (s-b) = (s-c)$

or $\quad a = b = c$ *i.e.*, triangle is equilateral.

Example 5 *If $x^2y^3 = 6$, find least value of $3x + 4y$.*

Solution We have $x^2y^3 = 6$

Now, $\quad (x)(x)(y)(y)(y) = 6$

$$\therefore \quad 3x + 4y = 2\left(\frac{3x}{2}\right) + 3\left(\frac{4y}{3}\right) = \frac{3x}{2} + \frac{3x}{2} + \frac{4y}{3} + \frac{4y}{3} + \frac{4y}{3}$$

(Multiply and divide coefficient of x and y by 2 and 3 respectively).

$$\therefore \quad \left(\frac{3x}{2}\right)\left(\frac{3x}{2}\right)\left(\frac{4y}{3}\right)\left(\frac{4y}{3}\right)\left(\frac{4y}{3}\right) = \frac{3^2}{2^2} \times \frac{4^3}{3^3} \times 6 = 32$$

Here, $\quad n = 5$

$z = 32 \ \therefore\ 5(32)^{1/5} = 10$

Example 6 *Find the minimum value of $bcx + cay + abz$ when $xyz = abc$*

Solution We have $xyz = abc$

$$(bcx)(cay)(abz) = a^3b^3c^3 = z$$

$$n = 3$$

Minimum value of $bcx + cay + abz$

$$= n(z)^{1/n} = 3(a^3b^3c^3)^{1/3} = 3abc$$

Example 7 *Find the minimum value of* $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ for possible values of x, y, z which satisfy the condition $x + y + z = 9$.

Solution We have $r = -1$

$$n = 3$$

$\therefore$ Minimum value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is $3^{1-(-1)} \cdot 9^{-1}$

$$= 9/9 = 1$$

Example 8 *Find the greatest value of* $2(x)^{1/2} + 3(y)^{1/2} + 4(z)^{1/2} + 5(a)^{1/2}$ *for possible values of* x, y, z, a *which satisfy the condition*

$$4x + 9y + 16z + 25a = 720.$$

Solution

$$z = 720$$

Now,
$$2(x)^{1/2} + 3(y)^{1/2} + 4(z)^{1/2} + 5(a)^{1/2}$$
$$= \sqrt{4x} + \sqrt{9y} + \sqrt{16z} + \sqrt{25a}$$

Now, we have $r = \frac{1}{2}$ and $n = 4$

Hence, greatest value is $4^{1-\frac{1}{2}}(720)^{1/2}$

$$= 24\sqrt{5}$$

Example 9 *Find the maximum value of* $a + b + c$ *for possible values of* a, b, c *satisfying condition* $a^3 + b^3 + c^3 = 1$.

Solution Here, $r = 3$ and $n = 3$

$\therefore$ Maximum value of $x + y + z = 3^{1-\frac{1}{3}}1^{1/3}$

$$= 3^{2/3}$$

Concept 1. If a and b are +ve quantities and $a > b$. If x be a +ve quantity, then

$$\left(1+\frac{x}{a}\right)^a > \left(1+\frac{x}{b}\right)^b$$

Concept 2. If a and b are +ve proper fractions and $a > b$, then

$$\left(\frac{1+a}{1-a}\right)^{1/a} > \left(\frac{1+b}{1-b}\right)^{1/b}$$

Example 1 *If a, b, c are descending order of magnitude, show that* $\left(\frac{a+c}{a-c}\right)^a < \left(\frac{b+c}{b-c}\right)^b$

Solution $\because$ $a > b > c$

$\therefore$ c/a and c/b are +ve fraction.

Also, $c/a < c/b$

Now, we have to prove that $\left(\frac{a+c}{a-c}\right)^a < \left(\frac{b+c}{b-c}\right)^b$

or $$\left(\frac{1+\frac{c}{a}}{1-\frac{c}{a}}\right)^a < \left(\frac{1+\frac{c}{b}}{1-\frac{c}{b}}\right)^b \text{ or } \left(\frac{1+\frac{c}{a}}{1-\frac{c}{a}}\right)^{\frac{a}{c}} < \left(\frac{1+\frac{c}{b}}{1-\frac{c}{b}}\right)^{\frac{b}{c}}$$

or $$\left(\frac{1+y}{1-y}\right)^{\frac{1}{y}} < \left(\frac{1+x}{1-x}\right)^{\frac{1}{x}}$$

where $$x=\frac{c}{b}, y=\frac{c}{a} \text{ or } \left(\frac{1+x}{1-x}\right)^{\frac{1}{x}} > \left(\frac{1+y}{1-y}\right)^{\frac{1}{y}}$$

$$x > y \qquad [\because c/b > c/a]$$

$\therefore$ $$\left(\frac{a+c}{a-c}\right)^a < \left(\frac{b+c}{b-c}\right)^b$$

Example 2 *If x is a +ve proper fraction. Prove that*

$$(1+x)^{1-x}(1-x)^{1+x} < 1.$$

Also, show $$a^b b^a < \left(\frac{a+b}{2}\right)^{a+b}$$

Solution Let $z=(1+x)^{1-x}(1-x)^{1+x}$

$\therefore$ $$\log z = (1-x)\log(1+x)+(1+x)\log(1-x)$$

$$=[\log(1+x)+\log(1-x)]-x[\log(1+x)-\log(1-x)]$$

$$=\left[\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)+\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\right)\right]$$

$$-x\left[\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\dots\right)\right]$$

$$=-2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots\right)-2x\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right)$$

$$=-2\left[x^2\left(\frac{1}{2}+1\right)+x^4\left(\frac{1}{4}+\frac{1}{3}\right)+x^6\left(\frac{1}{6}+\frac{1}{5}\right)+\dots\right]$$

$$=-2\left(3\frac{x^2}{1\cdot 2}+\frac{7x^4}{3\cdot 4}+\frac{11x^6}{5\cdot 6}+\dots\right)$$

$= -$ve as x is a +ve proper fraction.

$\therefore$ $\log z = -$ve or $z<1$

or $$(1+x)^{1-x}(1-x)^{1+x} < 1 \qquad \dots(i)$$

Hence proved.

Let $$x=\frac{a-b}{a+b}, \text{ then } 1+x=\frac{2a}{a+b}$$

and $$1-x=\frac{2b}{a+b}$$

Substituting these values in (i), we get

$$\left(\frac{2a}{a+b}\right)^{\frac{2b}{a+b}}\left(\frac{2b}{a+b}\right)^{\frac{2a}{a+b}}<1 \quad \text{or} \quad \left(\frac{2a}{a+b}\right)^{b}\left(\frac{2b}{a+b}\right)^{a}<1$$

or $$\left(\frac{2}{a+b}\right)^{b}a^{b}\left(\frac{2}{a+b}\right)^{a}b^{a}<1$$

or $$a^{b}b^{a}\left(\frac{2}{a+b}\right)^{a+b}<1$$

Hence, $$a^{b}b^{a}<\left(\frac{a+b}{2}\right)^{a+b}$$

Example 3 *If $x<1$, prove that $(1+x)^{1+x}(1-x)^{1-x}<1$. Also, show that*

$$a^{a}b^{b}>\left(\frac{a+b}{2}\right)^{a+b}$$

Solution Let $z=(1+x)^{1+x}(1-x)^{1-x}$, then

$$\begin{aligned}
\log z &= (1+x)\log(1+x)+(1-x)\log(1-x)\\
&= [\log(1+x)+\log(1-x)]+x[\log(1+x)-\log(1-x)]\\
&= \left[\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)+\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\right)\right]\\
&\quad +x\left[\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\right)\right] \qquad [\because x<1]\\
&= -2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots\right)+2x\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right)\\
&= -2x^2\left(\frac{1}{2}-1\right)-2x^4\left(\frac{1}{4}-\frac{1}{3}\right)-2x^6\left(\frac{1}{6}-\frac{1}{5}\right)+\dots\\
&= x^2+\frac{1}{6}x^4+\frac{1}{15}x^6+\dots=+\text{ve}
\end{aligned}$$

$\therefore$ $$\log z>0 \text{ or } z>1$$

or $$(1+x)^{1+x}(1-x)^{1-x}>1 \qquad \dots\text{(i)}$$

Hence proved.

Now, let $$x=\frac{a-b}{a+b}$$

$$1+x=\frac{2a}{a+b} \text{ and } 1-x=\frac{2b}{a+b}$$

Substituting these values in (i), we get

$$\left(\frac{2a}{a+b}\right)^{\frac{2a}{a+b}}\left(\frac{2b}{a+b}\right)^{\frac{2b}{a+b}}>1$$

Raising both sides to power $\frac{a+b}{2}$, we get

$$a^a b^b (2/a+b)^{a+b} > 1$$

or $$a^a b^b > \left(\frac{a+b}{2}\right)^{a+b}$$

Hence proved.

Example 4 *If x is +ve, then show that* $\log(1+x) < x$ *and* $\log(1+x) > \frac{x}{1+x}$. *Also, show that*

$$\log(1+n) < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}.$$

Solution If $\log(1+x) < x$, then $(1+x) < e^x$

or $$1 + x < 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ which is true.} \qquad [\because x \text{ is +ve}]$$

$\therefore$ $$\log(1+x) < x \qquad \ldots\text{(i)}$$

If $\log(1+x) > \frac{x}{1+x}$, then $(1+x) > e^{\frac{x}{1+x}}$

or $$\frac{1}{1-\frac{x}{1+x}} > e^{\frac{x}{1+x}} \text{ or } \left(1 - \frac{x}{1+x}\right)^{-1} > e^{\frac{x}{1+x}}$$

or $$1 + \frac{x}{1+x} + \left(\frac{x}{1+x}\right)^2 + \left(\frac{x}{1+x}\right)^3 + \ldots$$
$$> 1 + \left(\frac{x}{1+x}\right) + \frac{1}{2!}\left(\frac{x}{1+x}\right)^2 + \frac{1}{3!}\left(\frac{x}{1+x}\right)^3 + \ldots$$

It is clear that above relation is true.

$\therefore$ $$\log(1+x) > \frac{x}{1+x}$$

Hence proved.

Now, $$\log(1+x) < x$$

$\therefore$ $$\log\left(1 + \frac{1}{y}\right) < \frac{1}{y} \qquad \left[\text{Put } x = \frac{1}{y}\right]$$

or $$\log\left(\frac{1+y}{y}\right) < \frac{1}{y} \quad \text{or} \quad \log(1+y) - \log y < \frac{1}{y}$$

Put $y = 1, 2, 3, \ldots, n,$ we get

$$\log 2 - \log 1 < \frac{1}{1}$$

$$\log 3 - \log 2 < \frac{1}{2}$$

$$\log 4 - \log 3 < \frac{1}{3}$$

$$\log 5 - \log 4 < \frac{1}{4}$$

....

....

....

$$\log(n+1) - \log n < \frac{1}{n}$$

Adding the above terms, we have

$$\therefore \quad \log(n+1) - \log 1 < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

or $$\log(1+n) < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \qquad [\because \log 1 = 0]$$

Example 5 *Show that* $e^n > \frac{(n+1)^n}{n!}$, *n being an integer.*

Solution $$e^n = 1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \ldots + \frac{n^n}{n!} + \ldots$$

$$e^n > 1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \ldots + \frac{n^n}{n!}$$

$$e^n > \frac{n^n}{n!} + \frac{n^{n-1}}{(n-1)!} + \frac{n^{n-2}}{(n-2)!} + \ldots + \frac{n^3}{3!} + \frac{n^2}{2!} + n + 1$$

or $$e^n > n^n \left[\frac{1}{n!} + \frac{1}{n(n-1)!} + \frac{1}{n^2(n-2)!} + \frac{1}{n^3(n-2)!} + \ldots \frac{1}{n^{n-1}} + \frac{1}{n^n}\right]$$

i.e., $$e^n > \frac{n^n}{n!}\left[1 + \frac{1}{n}\cdot n + \frac{1}{n^2}\cdot n(n-1) + \frac{1}{n^3}\cdot n(n-1)(n-2) + \ldots + \frac{n!}{n^n}\right]$$

$$e^n > \frac{n^n}{n!}\left[1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \ldots + \frac{1}{n^n}\right] \quad \left[\begin{array}{l} \because n(n-1) > \frac{n(n-1)}{2!} \\ n(n-1)(n-2) > \frac{n(n-1)(n-2)}{3!} \\ n! > 1 \qquad (\because n \text{ is an integer}) \end{array}\right]$$

$$\therefore \quad e^n > \frac{n^n}{n!}\left(1 + \frac{1}{n}\right)$$

i.e., $$e^n > \frac{n^n}{n!}\cdot\frac{(n+1)^n}{n^n}$$

$$e^n > \frac{(n+1)^n}{n!}$$

Hence proved.

Additional Solved Examples

Additional Solved Examples

Example 1. *If $a_1, a_2, a_3, \ldots, a_n$ be non-negative real numbers such that $a_1 + a_2 + a_3 + \ldots + a_n = m$, then prove that $\sum_{i<j} a_i a_j \le \frac{m^2}{2}$.*

Solution Now,

$$m = a_1 + a_2 + a_3 + \ldots + a_n$$

$\therefore$ $$m^2 = (a_1 + a_2 + a_3 + \ldots + a_n)^2$$

$$m^2 = a_1^2 + a_2^2 + \ldots + a_n^2 + 2\sum_{i<j} a_i a_j$$

$$m^2 - 2\Sigma a_i a_j = a_1^2 + a_2^2 + \ldots + a_n^2$$

$\because$ $$a_1^2 + a_2^2 + \ldots + a_n^2 \ge 0$$

$\therefore$ $$m^2 - 2\Sigma a_i a_j \ge 0$$

$\Rightarrow$ $$\sum_{i<j} a_i a_j \le \frac{m^2}{2}$$

Hence proved.

Example 2. *Prove $\sum_{i<j} i \cdot j\, C_i^2 C_j^2 < \frac{n^2}{2} ({}^{2n-1}C_n)^2$, where C_i, C_j are binomial coefficient.*

Solution From Binomial theorem, we know that

$$1C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2 = n\,{}^{2n-1}C_n$$

Squaring both sides, we get

$$(1C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2)^2 = n^2 ({}^{2n-1}C_n)^2$$

$$1^2C_1^4 + 2^2C_2^4 + 3^2C_3^4 + \ldots + n^2C_n^4 + 2\Sigma i \cdot j \cdot C_i^2 C_j^2$$

$$= n^2 ({}^{2n-1}C_n)^2$$

$\because$ $$1^2C_1^4 + 2^2C_2^4 + \ldots + n^2C_n^4 > 0$$

$\therefore$ $$n^2 ({}^{2n-1}C_n)^2 - 2\Sigma\, i \cdot j\, C_i^2 C_j^2 > 0$$

$\Rightarrow$ $$\sum_{i<j} i \cdot j\, C_i^2 C_j^2 < \frac{n^2}{2} ({}^{2n-1}C_n)^2$$

Hence proved.

Example 3. *Given $n^4 < 10^n$ for $n \ge 2$. Prove that $(n+1)^4 < 10^{n+1}$.*

Solution Consider $\left(\frac{n+1}{n}\right)^4 = \left(1 + \frac{1}{n}\right)^4$

We are given $n \ge 2$

$\Rightarrow$ $$\frac{1}{n} \le \frac{1}{2}$$

so $$1 + \frac{1}{n} \le 1 + \frac{1}{2}$$

or $$\left(1 + \frac{1}{n}\right)^4 \le \left(1 + \frac{1}{2}\right)^4$$

or $$\left(1+\frac{1}{n}\right)^4 \le \left(\frac{3}{2}\right)^4 \Rightarrow \left(\frac{n+1}{n}\right)^4 < 10$$

Since, the logic is that if $n \le 2$ it is also less than 3, 6, 7, 8 etc. So, we can consider any one.

$$n^4 \cdot 10 > (n+1)^4 \text{ or } (n+1)^4 < n^4 \cdot 10 \qquad \ldots\text{(i)}$$

So, $$n^4 < 10^n$$

$\therefore$ $$10n^4 < 10^{n+1} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$(n+1)^4 < 10^{n+1}$$

Hence proved.

Example 4. *If $\theta_i \in (-\pi, \pi)$ for $i = 2, 3, \ldots, n$. Prove* $\dfrac{1}{2^{\sin^2\theta_2}} \cdot \dfrac{1}{3^{\sin^2\theta_3}} \cdots \dfrac{1}{n^{\sin^2\theta_n}} \ge n!$

or $$2^{\text{cosec}^2\theta_2} \cdot 3^{\text{cosec}^2\theta_3} \ldots n^{\text{cosec}^2\theta_n} \ge n!$$

Solution $$\forall\, i = 2, 3, \ldots, n$$

$$\sin^2\theta_i \le 1$$

$\Rightarrow$ $$\frac{1}{\sin^2\theta_i} \ge 1 \Rightarrow \frac{1}{i^{\sin^2\theta_i}} \ge i^1$$

$$i = 2, 3, \ldots, n$$

$\Rightarrow$ $$\frac{1}{2^{\sin^2\theta_2}} \ge 2, \frac{1}{3^{\sin^2\theta_3}} \ge 3, \ldots, \frac{1}{n^{\sin^2\theta_n}} \ge n$$

Therefore,

$$\frac{1}{2^{\sin^2\theta_2}} \cdot \frac{1}{3^{\sin^2\theta_3}} \cdots \frac{1}{n^{\sin^2\theta_n}} \ge 2 \cdot 3 \ldots n \ge n!$$

or $$2^{\text{cosec}^2\theta_2} \cdot 3^{\text{cosec}^2\theta_3} \ldots n^{\text{cosec}^2\theta_n} \ge n!$$

Hence proved.

Example 5. *If* $^nC_r = \dfrac{n!}{r!(n-r)!}$, *then prove*

$$\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n} \le 2^{n-1} + \frac{n-1}{2}$$

Solution Consider

$$(\sqrt{C_1} - 1)^2 + (\sqrt{C_2} - 1)^2 + (\sqrt{C_3} - 1)^2 + \ldots + (\sqrt{C_n} - 1)^2 \ge 0$$

$\Rightarrow$ $$(C_1 + C_2 + C_3 + \ldots + C_n) - 2(\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n}) + n \ge 0$$

$\Rightarrow$ $$2^n - 1 + n \ge 2(\sqrt{C_1} + \sqrt{C_2} \ldots + \sqrt{C_n}) \qquad (\because\ C_0 + C_1 + C_2 + \ldots + C_n = 2^n)$$

$\Rightarrow$ $$(\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n}) \le \frac{2^n + n - 1}{2}$$

$$(\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n}) \le 2^{n-1} + \frac{(n-1)}{2}$$

Hence proved.

Example 6. *If $a > b > 0$, then without using AM-GM prove*

$$\frac{1+a+a^2+\dots+a^{n-1}}{1+a+a^2+\dots+a^n} < \frac{1+b+b^2+\dots+b^{n-1}}{1+b+b^2+\dots+b^n}.$$

Solution Let
$$A = \frac{1+a+a^2+\dots+a^{n-1}}{1+a+a^2+\dots+a^{n-1}+a^n}$$

and
$$B = \frac{1+b+b^2+\dots+b^{n-1}}{1+b+b^2+\dots+b^{n-1}+b^n}$$

$\therefore$
$$\frac{1}{A} = \frac{1+a+a^2+\dots+a^{n-1}+a^n}{1+a+a^2+\dots+a^{n-1}}$$

$$\frac{1}{A} = 1 + \frac{a^n}{1+a+a^2+\dots+a^{n-1}} \quad \dots\text{(i)}$$

$$\frac{1}{A} = 1 + \frac{1}{\frac{1}{a^n}+\frac{1}{a^{n-1}}+\dots+\frac{1}{a}} \quad \dots\text{(ii)}$$

Similarly,
$$\frac{1}{B} = 1 + \frac{b^n}{1+b+b^2+\dots+b^{n-1}} \quad \dots\text{(iii)}$$

$$\frac{1}{B} = 1 + \frac{1}{\frac{1}{b^n}+\frac{1}{b^{n-1}}+\dots+\frac{1}{b}} \quad \dots\text{(iv)}$$

It is given
$$a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b} \Rightarrow \frac{1}{a^n} < \frac{1}{b^n}, \forall n \in N$$

$\Rightarrow$
$$\left(\frac{1}{a}+\frac{1}{a^2}+\dots+\frac{1}{a^n}\right) < \left(\frac{1}{b}+\frac{1}{b^2}+\dots+\frac{1}{b^n}\right)$$

$\Rightarrow$
$$\frac{1}{\left(\frac{1}{a}+\frac{1}{a^2}+\dots+\frac{1}{a^n}\right)} > \frac{1}{\left(\frac{1}{b}+\frac{1}{b^2}+\dots+\frac{1}{b^n}\right)}$$

Using Eqs. (ii) and (iii), we get
$$\frac{1}{A} - 1 > \frac{1}{B} - 1 \Rightarrow \frac{1}{A} > \frac{1}{B} \Rightarrow A < B$$

$\Rightarrow$
$$\frac{1+a+a^2+\dots+a^{n-1}}{1+a+a^2+\dots+a^n} < \frac{1+b+b^2+\dots+b^{n-1}}{1+b+b^2+\dots+b^n}$$

Hence proved.

Example 7. *If $a_1, a_2, a_3, \dots, a_n$ and $b_1, b_2, b_3, \dots, b_n$ be +ve such that k the largest of fraction $\frac{a_i}{b_i}$, $i = 1, 2, \dots, n$. Prove*

$$\frac{a_1 + a_2^2 + a_3^3 + \dots + a_n^n}{b_1 + kb_2^2 + k^2b_3^3 + \dots + k^{n-1}b_n^n} \le k.$$

Solution $\because$ k is maximum $\frac{a_i}{b_i}$, $\forall\ i = 1, 2, \dots, n$

$\Rightarrow$
$$\frac{a_i}{b_i} \le k, \forall\, i = 1, 2, \dots, n$$

$$a_1 \le kb_1, a_2^2 \le k^2b_2^2, a_3^3 = k^3b_3^3, \dots, a_n^n \le k^n b_n^n$$

Adding these, we get

$$a_1 + a_2^2 + a_3^3 + \ldots + a_n^n \le kb_1 + k^2b_2^2 + k^3b_3^3 + \ldots + k^n b_n^n$$

$$\Rightarrow \quad a_1 + a_2^2 + a_3^3 + \ldots + a_n^n \le k(b_1 + b_2^2k + k^3b_3^3 + \ldots + k^{n-1}b_n^n)$$

$$\Rightarrow \quad \frac{a_1 + a_2^2 + a_3^3 + \ldots + a_n^n}{b_1 + kb_2^2 + k^2b_3^3 + \ldots + k^{n-1}b_n^n} \le k.$$

Example 8. *If $a_1, a_2, a_3, \ldots, a_n$ are unequal real number, then prove that*

$$\frac{(1 + a_1 + a_1^2)(1 + a_2 + a_2^2)\ldots(1 + a_n + a_n^2)}{a_1a_2a_3\ldots a_n} \ge 3^n.$$

Solution For any $a > 0$, we have

$$\frac{1 + a + a^2}{a} = a + \frac{1}{a} + 1$$

Now, let

$$a + \frac{1}{a} = S$$

$$\because \quad S \ge 2$$

$$S + 1 \ge 3$$

$$\therefore \quad a + \frac{1}{a} + 1 \ge 3$$

Thus,

$$\frac{1 + a_i + a_i^2}{a_i} \ge 3, \forall\, i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad \left(\frac{1 + a_1 + a_1^2}{a_1}\right)\left(\frac{1 + a_2 + a_2^2}{a_2}\right)\left(\frac{1 + a_3 + a_3^2}{a_3}\right)\cdots\left(\frac{1 + a_n + a_n^2}{a_n}\right) \ge 3 \times 3 \ldots n \text{ times} = 3^n$$

$$\therefore \quad \left(\frac{1 + a_1 + a_1^2}{a_1}\right)\left(\frac{1 + a_2 + a_2^2}{a_2}\right)\cdots\left(\frac{1 + a_n + a_n^2}{a_n}\right) \ge 3^n$$

Example 9. *If a, b, c are +ve real numbers, then prove that the expression*

$$\frac{1}{2}(a + b + c) - \frac{bc}{b + c} - \frac{ca}{c + a} - \frac{ab}{a + b}$$

is always non -ve. Find the condition that this expression is zero. **(RMO 1993)**

Solution Let

$$S = \frac{1}{2}(a + b + c) - \frac{bc}{b + c} - \frac{ca}{c + a} - \frac{ab}{a + b}$$

$$= \frac{1}{4}(2a + 2b + 2c) - \frac{4}{4}\left(\frac{bc}{b + c} + \frac{ca}{c + a} + \frac{ab}{a + b}\right)$$

$$= \frac{1}{4}\left[(b + c) - \frac{4bc}{b + c} + c + a - \frac{4ca}{c + a} + a + b - \frac{4ab}{a + b}\right]$$

$$= \frac{1}{4}\left[\frac{(b - c)^2}{b + c} + \frac{(c - a)^2}{c + a} + \frac{(a - b)^2}{a + b}\right] \ge 0 \qquad [\because a, b, c \text{ are all +ve}].$$

Now, $S = 0$ iff each of the three terms in the expression are zero *i.e.*,

$a = b = c$

Example 10. *If a, b, c, d, e, f are real numbers, then show that*

$$\frac{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}+\sqrt{e^2+f^2}}{\sqrt{(a+c)^2+(b+d)^2}+\sqrt{(c+e)^2+(d+f)^2}+\sqrt{(e+a)^2+(f+b)^2}} \geq 1/2.$$

Solution From Third fundamental concept

$$\sqrt{a^2+b^2}+\sqrt{c^2+d^2} \geq \sqrt{(a+c)^2+(b+d)^2} \quad \text{...(i)}$$

$$\sqrt{c^2+d^2}+\sqrt{e^2+f^2} \geq \sqrt{(c+e)^2+(d+f)^2} \quad \text{...(ii)}$$

$$\sqrt{e^2+f^2}+\sqrt{a^2+b^2} \geq \sqrt{(e+a)^2+(f+b)^2} \quad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$2(\sqrt{a^2+b^2}+\sqrt{c^2+d^2}+\sqrt{e^2+f^2})$$

$$\geq \sqrt{(a+c)^2+(b+d)^2}+\sqrt{(c+e)^2+(d+f)^2}+\sqrt{(e+a)^2+(f+b)^2}$$

$$\therefore \quad \frac{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}+\sqrt{e^2+f^2}}{\sqrt{(a+c)^2+(b+d)^2}+\sqrt{(c+e)^2+(d+f)^2}+\sqrt{(e+a)^2+(f+b)^2}} \geq \frac{1}{2}$$

Hence proved.

Example 11. *If a, b, c are real +ve, then prove that*

$$a^6+b^6+c^6 > \frac{1}{2}[b^3ac(a+e)+c^3ab(a+b)+a^3bc(b+c).$$

Without using AM-GM.

Solution We have, $(a^2-b^2)^2 \geq 0$

$$a^4+b^4-2a^2b^2 \geq 0$$

$$a^4+b^4-a^2b^2 > a^2b^2$$

$$(a^2+b^2)(a^4-a^2b^2+b^4) > a^2b^2(a^2+b^2)$$

$$a^6+b^6 > a^2b^2(a^2+b^2) \quad \text{...(i)}$$

Also,

$$a^2+b^2 > 2ab$$

$$a^2b^2(a^2+b^2) > 2a^3b^3 \quad \text{...(ii)}$$

Using (i) and (ii)

$\Rightarrow$
$$a^6+b^6 > 2a^3b^3 \quad \text{...(iii)}$$

Similarly,
$$b^6+c^6 > 2b^3c^3 \quad \text{...(iv)}$$

and
$$c^6+a^6 > 2c^3a^3 \quad \text{...(v)}$$

Adding (iii), (iv), (v), we get

$$2(a^6+b^6+c^6) > 2(a^3b^3+b^3c^3+c^3a^3)$$

$$a^6+b^6+c^6 > (a^3b^3+b^3c^3+c^3a^3) \quad \text{...(vi)}$$

Now, we know that

$$x^3+y^3+z^3 > \frac{1}{2}[xy(x+y)+yz(y+z)+xz(x+z)]$$

$$\therefore \quad a^3b^3 + b^3c^3 + c^3a^3 > \frac{1}{2}[b^3ac(a + c) + c^3ab(a + b) + a^3bc(b + c)] \quad \text{...(vii)}$$

From (vi) and (vii), we get

$$a^6 + b^6 + c^6 > \frac{1}{2}[b^3ac(a + c) + c^3ab(a + b) + a^3bc(b + c)]$$

Hence proved.

Example 12. *If a, b, c are +ve real numbers, then show that*

$$a^2b + ab^2 + c^2a + ca^2 + b^2c + bc^2 + 2abc \geq 8abc$$ (RMO 1992)

Solution Let $S = a^2b + ab^2 + c^2a + ca^2 + b^2c + bc^2 + 2abc$

Factorizing, we get

$$S = (b + c)(c + a)(a + b)$$

We have $a + b \geq 2\sqrt{ab}$

$$b + c \geq 2\sqrt{bc} \quad \text{and} \quad a + c \geq 2\sqrt{ac}$$

Multiplying these, we get

$$(a + b)(b + c)(a + c) \geq 8abc$$

$$S \geq 8abc$$

Hence proved.

Example 13. *Show that in any triangle with sides a, b, c, we have*

$$(a + b + c)^2 < 4(ab + bc + ca).$$ (RMO 1992)

Solution Without loss of generality, let us assume $a > b > c > 0$, since the sum of any two sides of a triangle is always greater than the third side. So, the difference between any two sides can never exceeds the third.

$$\therefore \quad 0 < b - c < a$$

$$0 < c - a < b$$

$$0 < a - b < c$$

Squaring and adding, we get

$$(a - b)^2 + (b - c)^2 + (c - a)^2 < a^2 + b^2 + c^2$$

On simplification, we have

$$a^2 + b^2 + c^2 > 2(bc + ca + ab)$$

$$\Rightarrow \quad (a + b + c)^2 - 2ab - 2bc - 2ac < 2(ab + bc + ca)$$

$$\Rightarrow \quad (a + b + c)^2 < 4(ab + bc + ca)$$

Example 14. *If $a < c$, $(b - a)(b - c) < 0$. Show that*

$$3\left(\frac{a}{c} + \frac{c}{a} + 1\right) > (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Solution $\because (b - a)(b - c) < 0$

$\Rightarrow b$ lies between a and c.

$$\therefore \quad a < b < c \quad [\because a < c]$$

$$\Rightarrow \quad \frac{1}{c} < \frac{1}{b} < \frac{1}{a}$$

Now, $$(a-b)\left(\frac{1}{c}-\frac{1}{b}\right)>0$$

$$(b-c)\left(\frac{1}{b}-\frac{1}{a}\right)>0,\ (a-c)\left(\frac{1}{c}-\frac{1}{a}\right)>0$$

On adding

$$(a-b)\left(\frac{1}{c}-\frac{1}{b}\right)+(b-c)\left(\frac{1}{b}-\frac{1}{a}\right)+(a-c)\left(\frac{1}{c}-\frac{1}{a}\right)>0$$

Simplifying, we get

$$3\left(\frac{a}{c}+1+\frac{c}{a}\right)>(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Hence proved.

Example 15. *If $a, b, c > 0$ are such that $a+b+c=1$, prove that $ab+bc+ca \le 1/3$.* **(RMO 1988)**

Solution For all real numbers a, b, c

$$(b-c)^2+(c-a)^2+(a-b)^2 \ge 0$$

so that $$a^2+b^2+c^2 \ge ab+bc+ca$$

$\Leftrightarrow$ $$(a+b+c)^2 \ge 3(ab+bc+ca)$$

$\because$ $a+b+c=1$, we get $1 \ge 3(ab+bc+ca)$

i.e., $$ab+bc+ca \le \frac{1}{3}$$

Hence proved.

Note If $a_1, a_2, a_3, \ldots, a_n$ are +ve real numbers whose sum is A, then

$$a_1a_2+a_2a_3+a_3a_4+\ldots+a_{n-1}a_n \le \frac{1}{4}A^2$$

To prove it, we put $a_1+a_2+a_3+\ldots+a_n=A$

$$a_1-a_2+a_3-a_4+\ldots=B$$

so that $$2(a_1+a_3+\ldots)=A+B,$$

$$2(a_2+a_4+a_6+\ldots)=A-B$$

Multiplying corresponding sides of the above relations

We have

$$4(a_1+a_3+\ldots)(a_2+a_4+\ldots)=A^2-B^2 \le A^2 \quad (\because B^2 \ge 0)$$

Now, $$a_1a_2+a_2a_3+a_3a_4+\ldots \le (a_1+a_3+\ldots)(a_2+a_4+\ldots) \le \frac{1}{4}A^2$$

Example 16. *If a, b, c, d are four non -ve real numbers and $a+b+c+d=1$, show that $ab+bc+cd \le 1/4$.* **(INMO 1993)**

Solution Let $a+b+c+d=A$

$$a-b+c-d=B$$

Then, $2(a+c)=A+B, 2(b+d)=A-B$, so that

$$4(a+c)(b+d)=A^2-B^2 \le A^2 \quad (\because B^2 \ge 0)$$

Now, we have $A=1$

So, $$(a+c)(b+d) \le \frac{1}{4}$$

Now, $$ab+bc+cd \le (a+c)(b+d) \le 1/4$$

Example 17. *If* $x, y, z > 0$ *and* $x + y + z = 1$.

Prove $$x^2 + y^2 + z^2 \geq 1/3.$$

Solution $\forall x, y, z$ we have

$$x^2 + y^2 + z^2 \geq xy + yz + zx \quad \text{...(i)}$$

Now, $$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$\Rightarrow \quad xy + yz + zx = \frac{(x + y + z)^2 - (x^2 + y^2 + z^2)}{2}$$

$$\Rightarrow \quad xy + yz + zx = \frac{1 - (x^2 + y^2 + z^2)}{2}$$

Substitute the value of $xy + yz + zx$ in (i), we get

$$x^2 + y^2 + z^2 \geq \frac{1 - (x^2 + y^2 + z^2)}{2}$$

$$\Rightarrow \quad x^2 + y^2 + z^2 \geq 1/3$$

Example 18. *For given real numbers,* $x_1, x_2, y_1, y_2, z_1, z_2$ *satisfying* $x_1 > 0$, $x_2 > 0$, $x_1y_1 - z_1^2 > 0$ *and* $x_2y_2 - z_2^2 > 0$.

Prove that $\dfrac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \leq \dfrac{1}{x_1y_1 - z_1^2} + \dfrac{1}{x_2y_2 - z_2^2}$, *give necessary and sufficient conditions for equality.*

Solution We have to prove

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2} \quad \text{...(i)}$$

Let us denote the denominators appearing in (i) by A, A_1, A_2 respectively.

$$A = A_1 + A_2 + x_1y_2 + x_2y_1 - 2z_1z_2 \quad \text{...(ii)}$$

Applying $$x + y \geq 2\sqrt{xy} \ (x > 0, y > 0)$$

$$\Rightarrow \quad A = A_1 + A_2 + \frac{x_1}{x_2}(A_2 + z_2^2) + \frac{x_2}{x_1}(A_1 + z_1^2) - 2z_1z_2$$

$$A = A_1 + A_2 + \frac{x_1}{x_2}A_2 + \frac{x_2}{x_1}A_1 + \left(z_1\sqrt{\frac{x_2}{x_1}} - z_2\sqrt{\frac{x_1}{x_2}}\right)^2$$

$$\therefore \quad A > 0 \quad \text{...(iii)}$$

$$\because \quad x + y \geq 2\sqrt{xy}$$

$$\therefore \quad A \geq A_1 + A_2 + \frac{x_1}{x_2}A_2 + \frac{x_2}{x_1}A_1 \geq A_1 + A_2 + 2\sqrt{A_1A_2}$$

$$= (\sqrt{A_1} + \sqrt{A_2})^2 \quad \text{...(iv)}$$

Equality holds only in case

$$z_1\sqrt{\frac{x_2}{x_1}} = z_2\sqrt{\frac{x_1}{x_2}} \text{ and } \frac{x_1}{x_2}A_2 = \frac{x_2}{x_1}A_1 \quad \text{...(v)}$$

Now, (i) can be written as

$$\frac{8}{A} \le \frac{1}{A_1} + \frac{1}{A_2} \text{ or } A \ge \frac{8A_1A_2}{A_1 + A_2}$$

Now, it is enough to show that

$$(\sqrt{A_1} + \sqrt{A_2})^2 \ge 8A_1A_2$$

i.e.,

$$\left(\frac{\sqrt{A_1} + \sqrt{A_2}}{2}\right)^2 \cdot \frac{A_1 + A_2}{2} \ge A_1A_2$$

Based on inequality between AM and GM, we have

$$\left(\frac{\sqrt{A_1} + \sqrt{A_2}}{2}\right)^2 \ge \sqrt{A_1A_2}$$

and

$$\frac{A_1 + A_2}{2} \ge \sqrt{A_1A_2}$$

Equality holds, if $A_1 = A_2$

Together with (v) it gives $x_1^2 = x_2^2$

Consequently $x_1 = x_2$ is necessary for equality. Based on (v) these conditions imply $z_1 = z_2$

Now, $A_1 = A_2$ forces $y_1 = y_2$

But $x_1 = x_2, y_1 = y_2, z_1 = z_2$ obviously imply equality in (i).

Example 19. *Let a, b, c be real numbers such that $a + b + c = 1$. Prove that*

$$a^2 + b^2 + c^2 \ge 4(ab + bc + ca) - 1$$

when does equality hold?

Solution $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 1 - 2(ab + bc + ca)$

So, it is enough to prove that

$$1 - 2(ab - bc - ca) \ge 4(ab + bc + ca) - 1$$

i.e.,

$$6(ab + bc + ca) \le 2 \text{ or } 3(ab + bc + ca) \le 1$$

It happens only if

$$0 \le 1 - 3(ab + bc + ca)$$

$$= (a + b + c)^2 - 3(ab + bc + ca)$$

$$= \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2}$$

which is always true.

Clearly, equality holds when $a = b = c = 1/3$

Example 20. *Let a, b, c be real numbers such that*

$0 < a < 1, 0 < b < 1, 0 < c < 1$. *Prove that*

(i) $\sqrt{ab} + \sqrt{(1 - a)(1 - b)} \le 1$ (ii) $\sqrt{abc} + \sqrt{(1 - a)(1 - b)(1 - c)} \le 1$

Solution Taking $a = \sin^2 \theta$

$$b = \sin^2 \phi \qquad [\because 0 < a < 1 \text{ and } 0 < b < 1]$$

Then, $$\sqrt{ab} + \sqrt{(1-a)(1-b)}$$

$$= \sin\theta \sin\phi + \cos\theta\cos\phi$$

$$= \cos(\theta - \phi) \le 1$$

$\because$ $$0 < \sqrt{c} < 1, 0 < \sqrt{1-c} < 1$$

We have

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} \le \sqrt{ab} + \sqrt{(1-a)(1-b)} \le 1$$

Hence proved.

Example 21. *Let a, b, c be +ve real numbers such that $abc = 1$. Prove that*

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \le 1$$

when does equality hold?

Solution Now, $$a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$$

$$= (a+b)[(a-b)^2(a^2 + ab + b^2) + a^2b^2] \ge (a+b)a^2b^2$$

if and only if $a = b$

So, $$\frac{ab}{a^5 + b^5 + ab} \le \frac{ab}{a^2b^2(a+b) + ab}$$

$$= \frac{1}{ab(a+b)+1} = \frac{1}{ab(a+b+c)}$$

$$= \frac{c}{a+b+c}$$

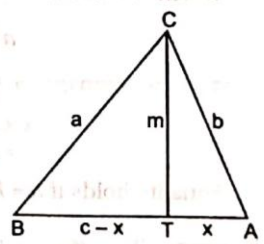
Now, writing Z for the sum $\Sigma\, ab/(a^5 + b^5 + ab)$, we get

$$Z \le \frac{c}{a+b+c} + \frac{a}{a+b+c} + \frac{b}{a+b+c} = 1$$

Equality holds, if $a = b = c = 1$.

Example 22. *Let a, b, c be sides of triangle while t is its area, show that $a^2 + b^2 + c^2 \ge 4t\sqrt{3}$, when does equality holds?*

Solution Suppose that the largest angle of the $\Delta\, ABC$ is at C. The foot of the altitude m at C is T which is an inner point of the interval AB. Let x denote AT.



Now, apply formula $cm = 2t$ and express a^2 and b^2 using the Pythagoras theorem.

$$a^2 + b^2 + c^2 - 4t\sqrt{3}$$

$$= [m^2 + (c-x)^2] + (m^2 + x^2) + c^2 - 2\sqrt{3}\,cm$$

$$= 2c^2 + 2m^2 + 2x^2 - 2cx - 2\sqrt{3}\,cm$$

$$= \frac{1}{2}[(c-2x)^2 + (c\sqrt{3} - 2m)^2] \ge 0$$

Equality holds if $x = \dfrac{c}{2}$ and $m = \dfrac{c\sqrt{3}}{2}$ *i.e.*, triangle is equilateral.

Aliter

Now, $$a^2 + b^2 + c^2 \geq 4t\sqrt{3} \qquad \ldots\text{(i)}$$

(i) is equivalent to

$$(a^2 + b^2 + c^2)^2 \geq 48t^2 = 3 \cdot 16t^2$$

Apply an appropriate version of formula of Heron

$$16t^2 = -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2$$

We only need to verify that

$$a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \geq -3a^4 - 3b^4 - 3c^4 + 6a^2b^2 + 6b^2c^2 + 6c^2a^2$$

which is equivalent to

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0$$

equality holds if and only if $a = b = c$.

Aliter

Apply Hero's formula again by AM-GM inequality, we have

$$(s - a)(s - b)(s - c) \leq \left[\frac{(s - a) + (s - b) + (s - c)}{3}\right]^3 = \frac{s^3}{27}$$

Hence, $$t = \sqrt{s\,(s - a)\,(s - b)\,(s - c)} \leq \frac{\sqrt{s^4}}{27} = \frac{s^2}{3\sqrt{3}}$$

Equality holds only for equilateral triangle.

This last inequality implies

$$4t\sqrt{3} \leq 3\left(\frac{2s}{3}\right)^2 = 3\left(\frac{a + b + c}{3}\right)^2$$

$$\leq 3\,\frac{a^2 + b^2 + c^2}{3} = a^2 + b^2 + c^2$$

Equality holds only if $a = b = c$.

Aliter

Apply area formula

It is equal to $ab \sin\theta$ and $c^2 = a^2 + b^2 - 2ab\cos\theta$

which is a direct consequence of the law of cosines

$$a^2 + b^2 + c^2 - 4t\sqrt{3} = 2a^2 + 2b^2 - 4ab\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)$$

$$= 2a^2 + 2b^2 - 4ab\sin(\theta + 30^\circ) \geq 2a^2 + 2b^2 - 4ab$$

$$= 2(a - b)^2 \geq 0$$

Equality holds if $a = b, \theta = 60^\circ$.

Example 23. *If $a > 0, b > 0$, then prove that for any x and y the following inequality holds true.*

$$a \cdot 2^x + b \cdot 3^y + 1 \leq \sqrt{(4^x + 9^y + 1)} \cdot \sqrt{a^2 + b^2 + 1}$$

Solution By hypothesis both sides of this inequality is +ve.

On squaring, we get

$$\Rightarrow \qquad (a \cdot 2^x + b \cdot 3^y + 1)^2 \leq (4^x + 9^y + 1)(a^2 + b^2 + 1)$$

$$\Rightarrow \quad a^2 4^x + b^2 9^y + 1 + 2ab2^x 3^y + 2b \cdot 3^y + 2a \cdot 2^x$$
$$\leq 4^x a^2 + 4^x b^2 + 4^x + a^2 9^y + b^2 9^y + 9^y + a^2 + b^2 + 1$$
$$\Rightarrow \quad (a^2 9^y - 2ab2^x 3^y + 4^x b^2) + (4^x - 2a2^x + a^2) + (9^y - 2b3^y + b^2) \geq 0$$
$$\Rightarrow \quad (a3^y - b2^x)^2 + (2^x - a)^2 + (3^y - b)^2 \geq 0$$

So, original inequality is true.

Example 24. *There are real numbers a, b, c such that $a \geq b \geq c$. Prove that*

$$\frac{a^2 - b^2}{c} + \frac{c^2 - b^2}{a} + \frac{a^2 - c^2}{b} \geq 3a - 4b + c.$$

Solution From $a \geq b \geq c > 0$, we have

$$\frac{a+b}{c} \geq 2, \; 0 < \frac{b+c}{a} \leq 2$$

and
$$\frac{a+c}{b} \geq 1$$

Now, we get
$$\frac{a^2 - b^2}{c} \geq 2(a-b) \qquad [\because a \geq b]$$
$$\frac{c^2 - b^2}{a} \geq 2(c-b) \qquad [\because c \leq b]$$
$$\frac{a^2 - c^2}{a} \geq (a-c) \qquad [\because a \geq c]$$

After addition of these inequalities, we get

$$\frac{a^2 - b^2}{c} + \frac{c^2 - b^2}{a} + \frac{a^2 - c^2}{b} \geq 2(a-b) + 2(c-b) + (a-c)$$

i.e.,
$$\frac{a^2 - b^2}{c} + \frac{c^2 - b^2}{a} + \frac{a^2 - c^2}{b} \geq 3a - 4b + c$$

Equality holds if $a = b = c > 0$.

Example 25. *Prove that if the number x_1 and x_2 does not exceed 1 in absolute value, then*

$$\sqrt{1 - x_1^2} + \sqrt{1 - x_2^2} \leq 2\sqrt{1 - \left(\frac{x_1 + x_2}{2}\right)^2}.$$

For what number x_1 and x_2 does the equality holds?

Solution Both numbers of the inequality are +ve. On squaring both sides of given inequation, we get

$$1 - x_1^2 + 1 - x_2^2 + 2\sqrt{(1 - x_1^2)(1 - x_2^2)} \leq 4 - (x_1^2 + 2x_1x_2 + x_2^2)$$

i.e.,
$$2\sqrt{(1 - x_1^2)(1 - x_2^2)} \leq 2 - 2x_1x_2$$
$$\Rightarrow \quad \sqrt{(1 - x_1^2)(1 - x_2^2)} \leq 1 - x_1x_2$$

Again squaring both sides

$$1 - x_1^2 - x_2^2 + x_1^2x_2^2 \leq 1 - 2x_1x_2 + x_1^2x_2^2$$

If all transposed to RHS, we have

$$0 \leq (x_1 - x_2)^2$$

Equality holds if $x_1 = x_2$.

Example 26. *If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 1$.*

Prove that $-1/2 \le ab + bc + ca \le 1$.

Solution $\because$ $$a^2 + b^2 + c^2 = 1$$

then $$-\frac{1}{2}(a^2 + b^2 + c^2) \le ab + bc + ca \le a^2 + b^2 + c^2$$

or $$-(a^2 + b^2 + c^2) \le 2(ab + bc + ca) \le 2(a^2 + b^2 + c^2)$$

These inequalities are indeed true.

$\because$ $$2(ab + bc + ca) + (a^2 + b^2 + c^2)$$
$$= (a + b + c)^2 \ge 0$$

and $$2(a^2 + b^2 + c^2) - 2(ab + bc + ca)$$
$$= (a - b)^2 + (a - c)^2 + (b - c)^2 \ge 0$$

Example 27. *Show that $(n!)^2 > n^n$, if n is a +ve integer.*

Solution If r is a +ve integer lying between 1 and n, then we know that
$$r(n - r) > (n - r)$$
or $$r(n - r) + r > n \text{ or } r(n - r + 1) > n$$

Putting $r = 1, 2, 3, \dots n$, we have
$$1(n) = n$$
$$2(n - 1) > n$$
$$3(n - 2) > n$$
$$4(n - 3) > n$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$(n - 1)(2) > n$$
$$n(1) = n$$

Multiplying the above n rows, we get
$$[1 \cdot 2 \cdot 3 \dots (n - 1)n][n(n - 1)(n - 2) \dots 2] > n^n$$

i.e., $$n! \times n! > n^n \text{ or } (n!)^2 > n^n$$

Hence proved.

Example 28. *Show that*
$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{1}{2}$$
where n is a +ve integer.

Solution We have
$$\frac{1}{2n} = \frac{1}{2n}, \frac{1}{2n-1} > \frac{1}{2n}, \dots, \frac{1}{n+2} > \frac{1}{2n}, \frac{1}{n+1} > \frac{1}{2n}$$

Adding these inequalities termwise, we find
$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \dots + \frac{1}{2n}$$
$$= \frac{n}{2n} = \frac{1}{2}$$

Hence proved.

Example 29. *Prove that the sum of any number of fractions taken from among the sequence* $\frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots$ *is always less than unity.*

Solution Let us have n fractions ($n \geq 1$)

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \ldots, \frac{1}{k}, \frac{1}{l}$$

Let us assume $2 \leq a < b < c < d < \ldots\ldots < k < l$

Then $$b \geq a+1, c \geq b+1, d \geq c+1, \ldots, l \geq k+1$$

Consequently

$$b \geq a+1, c \geq a+2, d \geq a+3, \ldots, l \geq a+n-1$$

$\therefore$ $$\frac{1}{a^2}+\frac{1}{b^2}+\ldots+\frac{1}{l^2} \leq \frac{1}{a^2}+\frac{1}{(a+1)^2}+\ldots+\frac{1}{(a+n-1)^2} < \frac{1}{a-1}-\frac{1}{a+n-1}$$

Hence, $$\frac{1}{a^2}+\frac{1}{b^2}+\ldots+\frac{1}{l^2} < \frac{n}{(a-1)(a+n-1)}$$

But $$a-1 \geq 1, a+n-1 \geq n+1$$

$$(a-1)(a+n-1) \geq n+1 \quad \text{and} \quad \frac{1}{a^2}+\frac{1}{b^2}+\ldots+\frac{1}{l^2} \leq \frac{n}{n+1} < 1$$

Hence proved.

Example 30. *Prove that*

$$\frac{n}{2} < \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1} < n$$

Solution We have,

$$\frac{1}{2}=\frac{1}{2}$$
$$\frac{1}{2}<\frac{2}{3}$$
$$\frac{1}{2}<\frac{3}{4}$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$\frac{1}{2}<\frac{n}{n+1}$$

Adding these inequalities we get

$$\frac{n}{2} < \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1} \quad \ldots\text{(i)}$$

Again, $$\frac{1}{2}<1, \frac{2}{3}<1, \frac{3}{4}<1, \ldots \frac{n}{n+1}<1$$

Adding all these inequalities, we get

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1} < n \quad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\frac{n}{2} < \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1} < n$$

Example 31. *If $n \in N$, prove that*

$$\frac{n}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^n - 1} < n$$

Solution We have,

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^n - 1}$$

$$= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right)$$

$$+ \dots + \left(\frac{1}{2^{n-1} + 1} + \frac{1}{2^{n-1} + 2} + \dots + \frac{1}{2^n - 1}\right)$$

$$= \left(\frac{1}{2^0} + \frac{1}{2^1}\right) + \left(\frac{1}{2^1 + 1} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2 + 1} + \dots + \frac{1}{2^3}\right) + \left(\frac{1}{2^3 + 1} + \dots + \frac{1}{2^4}\right)$$

$$+ \dots + \left(\frac{1}{2^{n-1} + 1} + \frac{1}{2^{n-1} + 2} + \dots + \frac{1}{2^n - 1}\right)$$

Now,

$$\frac{1}{2^0} + \frac{1}{2^1} > \frac{1}{2}$$

$$\frac{1}{2^1 + 1} + \frac{1}{2^2} > \frac{2}{2^2}$$

$$\frac{1}{2^2 + 1} + \frac{1}{2^2 + 2} + \dots + \frac{1}{2^3} > \frac{2^2}{2^3}$$

....

....

....

$$\frac{1}{2^{n-1} + 1} + \dots + \frac{1}{2^n - 1} > \frac{2^{n-1}}{2^n}$$

Adding these inequalities, we get $S > \frac{n}{2}$

Again, $$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \dots + \frac{1}{15}\right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^{n-1}}\right)$$

$$= 1 + \left(\frac{1}{2^1} + \frac{1}{2^2 - 1}\right) + \left(\frac{1}{2^2} + \dots + \frac{1}{2^3 - 1}\right)$$

$$+ \left(\frac{1}{2^3} + \dots + \frac{1}{2^4 - 1}\right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^n - 1}\right)$$

Now,

$$1 = 1$$

$$\frac{1}{2^1} + \dots + \frac{1}{2^2 - 1} < 1$$

$$\frac{1}{2^2} + \dots + \frac{1}{2^3 - 1} < 1$$

$$\frac{1}{2^3} + \dots + \frac{1}{2^4 - 1} < 1$$

Hence proved.

Example 32. *Let m and n be +ve integers. Prove that*

$$\frac{1}{n+1}-\frac{1}{n+m+1}<\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+m)^2}<\frac{1}{n}-\frac{1}{n+m}.$$

Solution For any +ve integer, r satisfying $0<r<m$, we have $n+r-1<n+r<n+r+1$

$$\Rightarrow \quad (n+r)(n+r-1)<(n+r)^2<(n+r)(n+r+1)$$

$$\Rightarrow \quad \frac{1}{(n+r)(n+r-1)}>\frac{1}{(n+r)^2}>\frac{1}{(n+r)(n+r+1)}$$

$$\Rightarrow \quad \frac{1}{(n+r-1)}-\frac{1}{(n+r)}>\frac{1}{(n+r)^2}>\frac{1}{(n+r)}-\frac{1}{(n+r+1)}$$

$$\Rightarrow \quad \frac{1}{n+r}-\frac{1}{n+r+1}<\frac{1}{(n+r)^2}<\frac{1}{n+r-1}-\frac{1}{n+r}$$

Putting $r=1,2,3,\ldots,m$, we get

$$\frac{1}{n+1}-\frac{1}{n+2}<\frac{1}{(n+1)^2}<\frac{1}{n}-\frac{1}{n+1}$$

$$\frac{1}{n+2}-\frac{1}{n+3}<\frac{1}{(n+2)^2}<\frac{1}{(n+1)}-\frac{1}{(n+2)}$$

....

....

....

Similarly,

$$\frac{1}{n+m}-\frac{1}{n+m+1}<\frac{1}{(n+m)^2}<\frac{1}{n+m-1}-\frac{1}{n+m}$$

Adding the above inequalities, we get

$$\frac{1}{n+1}-\frac{1}{n+m+1}<\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+m)^2}<\frac{1}{n}-\frac{1}{n+m}$$

Hence proved.

Example 33. *Determine the largest number in the infinite sequence* $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots, \sqrt[n]{n}$.

Solution We find that $3^{1/3}$ to be the largest.

So, we will prove that $(n^{1/n})$, $n \geq 3$ is a decreasing sequence.

$$n^{1/n}>(n+1)^{\frac{1}{n+1}}$$

$$n^{n+1}>(n+1)^n$$

$$n>\left(1+\frac{1}{n}\right)^n$$

Now,

$$\left(1+\frac{1}{n}\right)^n=1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{6}\cdot\frac{1}{n^3}+\ldots$$

$$=1+1+\frac{1}{2}\left(1-\frac{1}{n}\right)+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots<1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots<3$$

or

$$3>\left(1+\frac{1}{n}\right)^n$$

$\therefore$ If

$$n\geq 3,\ n^{1/n}>(n+1)^{\frac{1}{n+1}}$$

i.e., $(n^{1/n})$ is decreasing for $n \geq 3$

But $3^{1/3}$ is also greater than 1 and $2^{1/2}$.

Therefore, $3^{1/3}$ is the largest.

Example 34. *For all $n \in N$. Prove that*

$$2\sqrt{n} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1$$

Solution Now, first we will prove that

$$2\sqrt{p+1} - 2\sqrt{p} < \frac{1}{\sqrt{p}} < 2\sqrt{p} - 2\sqrt{p-1}, \forall p \in N$$

We have

$$2\sqrt{p+1} - 2\sqrt{p} = 2(\sqrt{p+1} - \sqrt{p})$$

$$= \frac{2(\sqrt{p+1} - \sqrt{p})(\sqrt{p+1} + \sqrt{p})}{(\sqrt{p+1} + \sqrt{p})}$$

$$= \frac{2}{\sqrt{p+1} + \sqrt{p}} < \frac{2}{\sqrt{p} + \sqrt{p}} \qquad \left[\begin{array}{l} \because \sqrt{p+1} > \sqrt{p} \\ \Rightarrow \sqrt{p+1} + \sqrt{p} > \sqrt{p} + \sqrt{p} \\ \Rightarrow \dfrac{1}{\sqrt{p+1} + \sqrt{p}} < \dfrac{1}{\sqrt{p} + \sqrt{p}} \end{array}\right]$$

$\therefore$
$$2\sqrt{p+1} - 2\sqrt{p} < \frac{1}{\sqrt{p}} \qquad \ldots(i)$$

and
$$2\sqrt{p} - 2\sqrt{p-1} = \frac{2(\sqrt{p} - \sqrt{p-1})(\sqrt{p} + \sqrt{p-1})}{(\sqrt{p} + \sqrt{p-1})}$$

$$= \frac{2}{\sqrt{p} + \sqrt{p-1}} > \frac{2}{\sqrt{p} + \sqrt{p}} \qquad \left[\begin{array}{l} \because \sqrt{p-1} > \sqrt{p} \therefore \sqrt{p} + \sqrt{p-1} < \sqrt{p} + \sqrt{p} \\ \Rightarrow \dfrac{1}{\sqrt{p} + \sqrt{p-1}} > \dfrac{1}{\sqrt{p} + \sqrt{p}} \end{array}\right]$$

$\therefore$
$$2\sqrt{p} - 2\sqrt{p-1} > \frac{1}{\sqrt{p}} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$2\sqrt{p+1} - 2\sqrt{p} < \frac{1}{\sqrt{p}} < 2\sqrt{p} - 2\sqrt{p-1} \qquad \ldots(iii)$$

$\forall p \in N$

Putting $p = 2, 3, \ldots, n$ in (iii), we have

$$2\sqrt{3} - 2\sqrt{2} < \frac{1}{\sqrt{2}} < 2\sqrt{2} - 2\sqrt{1}$$

$$2\sqrt{4} - 2\sqrt{3} < \frac{1}{\sqrt{3}} < 2\sqrt{3} - 2\sqrt{2}$$

$$2\sqrt{5} - 2\sqrt{4} < \frac{1}{\sqrt{4}} < 2\sqrt{4} - 2\sqrt{3}$$

....

....

....

$$2\sqrt{n} - 2\sqrt{n-1} < \frac{1}{\sqrt{n-1}} < 2\sqrt{n-1} - 2\sqrt{n-2}$$

$$2\sqrt{n+1}-2\sqrt{n}<\frac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{n-1}$$

Adding all inequalities, we get

$$2\sqrt{n+1}-2\sqrt{2}<\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots+\frac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{1}$$

Adding 1 throughout, we have

$$2\sqrt{n+1}-2\sqrt{2}+1<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1 \qquad \ldots\text{(iv)}$$

But we have

$$[2(\sqrt{n})-2]-(2\sqrt{n+1}-2\sqrt{2}+1)$$
$$=2(\sqrt{n}-\sqrt{n+1})-3+2\sqrt{2}$$
$$=2(\sqrt{n}-\sqrt{n+1})+(2\sqrt{2}-3)<0 \qquad [\because \sqrt{n}<\sqrt{n+1} \text{ and } 2\sqrt{2}-3<0]$$

$\therefore$
$$2\sqrt{n}-2<2\sqrt{n+1}-2\sqrt{2}+1 \qquad \ldots\text{(v)}$$

From (iv) and (v), we get

$$2\sqrt{n}-2<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1$$

Example 35. *Let $a>1$ and n be a +ve integer.*

Prove that
$$a^n-1\geq n\left(a^{\frac{n+1}{2}}-a^{\frac{n-1}{2}}\right).$$

Solution Let $a=\alpha^2$

It is required to prove that

$$\alpha^{2n}-1\geq n(\alpha^{n+1}-\alpha^{n-1})$$

or which is the same.

$$\alpha^{2n}-1\geq n\alpha^{n-1}(\alpha^2-1)$$

$$\frac{\alpha^{2n}-1}{\alpha^2-1}\geq n\alpha^{n-1}$$

But
$$\frac{\alpha^{2n}-1}{\alpha^2-1}=\alpha^{2(n-1)}+\alpha^{2(n-2)}+\ldots+\alpha^2+1$$

$$\geq n\sqrt[n]{\alpha^2\cdot\alpha^4\ldots\alpha^{2n-2}}$$

(Using theorem on AM and GM of several numbers)

$\because$
$$2+4+\ldots+(2n-2)=n(n-1)$$

we have
$$\frac{\alpha^{2n}-1}{\alpha^2-1}\geq n\alpha^{n-1}$$

$\therefore$
$$a^n-1\geq n\left(a^{\frac{n+1}{2}}-a^{\frac{n-1}{2}}\right)$$

Hence proved.

Example 36. *Let $s > 1$ and n be a +ve integer. Prove that*

$$\frac{1}{1^s} + \frac{1}{2^s} + \dots + \frac{1}{n^s} < \frac{1}{1 - 2^{1-s}}.$$

Solution $1^s + \frac{1}{2^s} + \dots + \frac{1}{n^s} < \frac{1}{s} + \frac{1}{2^s} + \dots + \frac{1}{n^s} + 1$

$$= \frac{1}{1^s} + \left(\frac{1}{2^s} + \frac{1}{3^s}\right) + \left(\frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s}\right) + \dots < \frac{1}{1^s} + \frac{2}{2^s} + \frac{4}{4^s} + \frac{8}{8^s} + \dots$$

$$= \frac{1}{1^{s-1}} + \frac{1}{2^{s-1}} + \frac{1}{4^{s-1}} + \frac{1}{8^{s-1}} + \dots = \frac{1}{1 - 2^{1-s}}$$

Example 37. *If a, b, c are +ve real numbers, prove that*

$$6abc \le a^2(b + c) + b^2(c + a) + c^2(a + b) \le 2(a^3 + b^3 + c^3)$$

Solution We have $a^2 + b^2 > 2ab$

$$\Rightarrow \quad a^2 + b^2 - ab > ab$$

$$\Rightarrow \quad (a + b)(a^2 + b^2 - ab) > ab(a + b)$$

$$\Rightarrow \quad a^3 + b^3 > ab(a + b)$$

Similarly,

$$b^3 + c^3 > bc(b + c)$$

$$c^3 + a^3 > ca(c + a)$$

Adding all these, we get

$$2(a^3 + b^3 + c^3) > ab(a + b) + bc(b + c) + ca(c + a)$$

Again, AM> GM

$$\Rightarrow \quad \frac{ab(a + b) + bc(b + c) + ca(c + a)}{6} > (a^2b \cdot ab^2 \cdot b^2c \cdot bc^2 \cdot c^2a \cdot ca^2)^{1/6}$$

$$\Rightarrow \quad ab(a + b) + bc(b + c) + ca(c + a) > 6abc$$

Hence,

$$6abc \le a^2(b + c) + b^2(c + a) + c^2(a + b) \le 2(a^3 + b^3 + c^3)$$

Example 38. *If $x + y + z = 1$ and $x, y, z > 0$, then show that*

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right) \ge 64.$$

Solution Consider LHS

Open the brackets

Let

$$S = 1 + \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + \left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right) + \frac{1}{xyz}$$

We know

$$\left(\frac{x + y + z}{3}\right) \ge (xyz)^{1/3}$$

$$x + y + z \ge 3(xyz)^{1/3}$$

On cubing both sides, we get

$$1 \ge 3(xyz)^{1/3}$$

$$\Rightarrow \quad (xyz)^{1/3} \le \frac{1}{3} \qquad \dots(i)$$

$\Rightarrow$ $$xyz \le \frac{1}{27} \quad \text{...(ii)}$$

or $$\frac{1}{xyz} \ge 27 \quad \text{...(iii)}$$

or $$\frac{2}{xyz} \ge 54 \quad \text{...(iv)}$$

Also, $$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3} \ge \left(\frac{1}{xyz}\right)^{1/3}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \ge 3\left(\frac{1}{xyz}\right)^{1/3}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \ge 9 \quad \text{[from (iii)] ...(v)}$$

Now, (v) becomes

$$1+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\left(\frac{x+y+z}{xyz}\right)+\frac{1}{xyz}$$

$$=1+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{2}{xyz} \quad [\because x+y+z=1]$$

Adding (iv) and (v), we get

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{2}{xyz} \ge 63$$

Adding 1 on both sides, we get

$$1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{2}{xyz} \ge 64$$

or $$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \ge 64$$

Example 39. *Show that*

$$xyz > (y+z-x)(z+x-y)(x+y-z).$$

Solution Consider two quantities $(y+z-x)$ and $(z+x-y)$.

$\because$ $$\text{AM} > \text{GM}$$

$\therefore$ $$\frac{(y+z-x)+(z+x-y)}{2} > \sqrt{(y+z-x)(z+x-y)}$$

or $$z > \sqrt{(y+z-x)(z+x-y)}$$

Similarly, $$y > \sqrt{(y+z-x)(x+y-z)}$$

and $$x > \sqrt{(x+y-z)(x+z-y)}$$

Multiplying these three results, we get

$$xyz > \sqrt{(y+z-x)^2(z+x-y)^2(x+y-z)^2}$$

$$xyz > (y+z-x)(z+x-y)(x+y+z)$$

Hence proved.

Example 40. *If $a+b+c+d=s$, show that $(s-a)(s-b)(s-c)(s-d)>81\,abcd$.*

Solution $\because a+b+c+d=s$

$\therefore$ $$s-a=b+c+d$$

As $$\text{AM} > \text{GM} \;\therefore\; \frac{b+c+d}{3} > (bcd)^{1/3}$$

or $$(b+c+d) > 3(bcd)^{1/3} \text{ or } (s-a) > 3(bcd)^{1/3} \quad \ldots(\text{i})$$

$$(s-b) > 3(cad)^{1/3} \quad \ldots(\text{ii})$$

$$(s-c) > 3(abd)^{1/3} \quad \ldots(\text{iii})$$

$$(s-d) > 3(abc)^{1/3} \quad \ldots(\text{iv})$$

Multiplying (i), (ii), (iii) and (iv), we get

$$(s-a)(s-b)(s-c)(s-d) > 81\,(bcd)^{1/3}(cad)^{1/3}(abd)^{1/3}(abc)^{1/3}$$

or $$(s-a)(s-b)(s-c) > 81(a^3b^3c^3d^3)^{1/3}$$

or $$(s-a)(s-b)(s-c)(s-d) > 81abcd$$

Hence proved.

Example 41. *Prove that*

$$\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} > \frac{9}{a+b+c}.$$

Solution $\because$ $$\text{AM} > \text{GM}$$

$\therefore$ $$\frac{1}{3}\left(\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\right) > \left(\frac{2}{b+c}\times\frac{2}{c+a}\times\frac{2}{a+b}\right) \quad \ldots(\text{i})$$

Also, $$\frac{1}{3}\left(\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}\right) > \left(\frac{b+c}{2}\times\frac{c+a}{2}\times\frac{a+b}{2}\right) \quad \ldots(\text{ii})$$

Multiplying (i) and (ii), we have

$$\frac{1}{9}\left(\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\right)$$

$$\left[\frac{(b+c)+(c+a)+(a+b)}{2}\right] > 1$$

or $$\left(\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\right)\left(\frac{2a+2b+2c}{2}\right) > 9$$

or $$\left(\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\right)(a+b+c) > 9$$

or $$\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} > \frac{9.}{a+b+c}$$

Hence proved.

Example 42. *If $a_i > 0$, $\forall\, i = 1, 2, \ldots, n$ and $s = a_1 + a_2 + \ldots + a_n$, then prove that*

$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\frac{s}{s-a_3}+\ldots+\frac{s}{s-a_n} > \frac{n^2}{n-1}.$$

Solution $\because$ AM > GM we have

$$\frac{(s-a_1)+(s-a_2)+\ldots+(s-a_n)}{n} > [(s-a_1)(s-a_2)\ldots(s-a_n)]^{1/n}$$

and
$$\frac{\frac{1}{s-a_1}+\frac{1}{s-a_2}+\ldots+\frac{1}{s-a_n}}{n} > \left[\frac{1}{(s-a_1)(s-a_2)..(s-a_n)}\right]^{1/n}$$

$\Rightarrow$
$$\frac{ns-(a_1+a_2+\ldots+a_n)}{n} > [(s-a_1)(s-a_2)\ldots(s-a_n)]^{1/n}$$

and
$$\frac{1}{n}\left(\frac{1}{s-a_1}+\frac{1}{s-a_2}+\ldots+\frac{1}{s-a_n}\right) > \frac{1}{[(s-a_1)(s-a_2)\ldots(s-a_n)]^{1/n}}$$

$\Rightarrow$
$$\frac{(n-1)s}{n} > [(s-a_1)(s-a_2)\ldots(s-a_n)]^{1/n}$$

and
$$\frac{1}{n}\left(\frac{1}{s-a_1}+\frac{1}{s-a_2}+\ldots+\frac{1}{s-a_n}\right) > \frac{1}{[(s-a_1)(s-a_2)\ldots(s-a_n)]^{1/n}}$$

$\Rightarrow$
$$\left(\frac{n-1}{n}\right)s\times\frac{1}{n}\left(\frac{1}{s-a_1}+\frac{1}{s-a_2}+\ldots+\frac{1}{s-a_n}\right) > 1$$

$\Rightarrow$
$$\left(\frac{n-1}{n^2}\right)\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\ldots+\frac{s}{s-a_n}\right) > 1$$

$\Rightarrow$
$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\ldots+\frac{s}{s-a_n} > \frac{n^2}{n-1}$$

Example 43. *If a, b, c are +ve real numbers representing the sides of a triangle, prove that*
$$ab+bc+ca < a^2+b^2+c^2 < 2(ab+bc+ca)$$

or
$$1 < \frac{a^2+b^2+c^2}{ab+bc+ca} < 2$$

Hence prove that
$$3(ab+bc+ca) < (a+b+c)^2 < 4(ab+bc+ca)$$

or
$$3 < \frac{(a+b+c)^2}{ab+bc+ca} < 4.$$

Solution
$$\frac{a^2+b^2}{2} > ab,\ \frac{b^2+c^2}{2} > bc$$

and
$$\frac{c^2+a^2}{2} > ca \qquad [\because \text{AM} > \text{GM}]$$

$\Rightarrow$
$$a^2+b^2 > 2ab,\ b^2+c^2 > 2bc \text{ and } c^2+a^2 > 2ca$$

$\Rightarrow$
$$a^2+b^2+b^2+c^2+c^2+a^2 > 2(ab+bc+ca)$$

$\Rightarrow$
$$a^2+b^2+c^2 > ab+bc+ca$$

$\Rightarrow$
$$ab+bc+ca < a^2+b^2+c^2 \qquad \ldots\text{(i)}$$

In a $\Delta\, ABC$ with sides $BC = a, CA = b, AB = c$, we have $b^2+c^2-a^2 = 2bc\cos A$

$\Rightarrow$
$$b^2+c^2-a^2 < 2bc \qquad [\because \cos A < 1]$$

Similarly, we have $\quad c^2+a^2-b^2 < 2ca$ and $a^2+b^2-c^2 < 2ab$

Adding these three, we get
$$a^2+b^2+c^2 < 2(ab+bc+ca) \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$ab + bc + ca < a^2 + b^2 + c^2 < 2(ab + bc + ca) \quad \text{...(iii)}$$

$$\Rightarrow \quad 1 < \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2$$

Adding $2(ab + bc + ca)$ throughout in (iii)

$$3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$$

$$3 < \frac{(a + b + c)^2}{ab + bc + ca} < 4$$

Hence proved.

Example 44. *If a, b, c are +ve real numbers, prove*

$$\frac{a^3 + b^3 + c^3}{9} > \left(\frac{a + b + c}{3}\right)\left(\frac{a^2 + b^2 + c^2}{3}\right)$$

Solution

$$\frac{a^2 + b^2}{2} > ab \quad [\because \text{AM} > \text{GM}]$$

$$\Rightarrow \quad a^2 + b^2 > 2ab$$

$$\Rightarrow \quad a^2 + b^2 - ab > ab$$

$$\Rightarrow \quad (a^2 + b^2 - ab)(a + b) > ab(a + b)$$

$$\Rightarrow \quad a^3 + b^3 > ab(a + b)$$

Similarly, we have

$$b^3 + c^3 > bc(b + c)$$

$$c^3 + a^3 > ca(c + a)$$

Adding all these, we get

$$2(a^3 + b^3 + c^3) > ab(a + b) + bc(b + c) + ca(c + a)$$

Adding $a^3 + b^3 + c^3$ on both sides, we get

$$3(a^3 + b^3 + c^3) > (a^3 + b^3 + c^3) + ab(a + b) + bc(b + c) + ca(c + a)$$

$$\Rightarrow \quad 3(a^3 + b^3 + c^3) > (a + b + c)(a^2 + b^2 + c^2)$$

$$\Rightarrow \quad \frac{a^3 + b^3 + c^3}{9} > \left(\frac{a + b + c}{3}\right)\left(\frac{a^2 + b^2 + c^2}{3}\right)$$

Hence proved.

Example 45. *If a, b, c, d are four +ve real numbers such that $abcd = 1$, prove that*

$$(1 + a)(1 + b)(1 + c)(1 + d) \geq 16.$$

Solution $\because$

$$\text{AM} > \text{GM}$$

$$\therefore \quad \frac{1 + a}{2} \geq \sqrt{1 \cdot a}, \frac{1 + b}{2} \geq \sqrt{1 \cdot b}$$

$$\frac{1 + c}{2} \geq \sqrt{1 \cdot c}, \frac{1 + d}{2} \geq \sqrt{1 \cdot d}$$

Multiplying corresponding sides of the above inequalities, we have

$$(1 + a)(1 + b)(1 + c)(1 + d) \geq 16\sqrt{abcd} \geq 16 \quad [\because abcd = 1]$$

Example 46. *If a, b, c are +ve real numbers such that $a + b + c = 1$. Prove that*

$$(1 + a)(1 + b)(1 + c) \geq 8(1 - a)(1 - b)(1 - c)$$

Solution $\because$ $a + b + c = 1$

The given inequality is equivalent to

$$(2a + b + c)(a + 2b + c)(a + b + 2c) \geq 8(b + c)(c + a)(a + b)$$

If we write $\frac{b+c}{2} = x, \frac{c+a}{2} = y, \frac{a+b}{2} = z$

Then $x + y + z = \frac{b+c}{2} + \frac{c+a}{2} + \frac{a+b}{2}$

$$= a + b + c = 1$$

$$x, y, z > 0$$

and given inequality becomes

$$(y + z)(z + x)(x + y) \geq 8xyz \quad \ldots(i)$$

In order to prove the given inequality, it is enough to prove (i)

$\because$ x, y, z are all +ve.

$\therefore$ $\frac{y+z}{2} \geq (yz)^{1/2}, \frac{z+x}{2} \geq (zx)^{1/2},$

$$\frac{x+y}{2} \geq (xy)^{1/2}$$

By the inequality of the means (AM $\geq$ GM)

Now, $(y + z)(z + x)(x + y) \geq [2(yz)^{1/2}] \cdot [2(zx)^{1/2}] \cdot [2(xy)^{1/2}]$

$$= 8xyz$$

Example 47. *If a, b, c are real numbers such that $0 < a < 1, 0 < b < 1, 0 < c < 1, a + b + c = 2$.*

Prove that $\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$

Solution Put $a = y + z, b = z + x, c = x + y$

so that $x + y + z = 1$

$\because$ $x = 1 - a, y = 1 - b, z = 1 - c$

and $0 < a < 1, 0 < b < 1, 0 < c < 1$

It follows that $x, y, z > 0$

Now, $\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c}$

$$= \frac{[(y + z)(z + x)(x + y)]}{xyz} \quad \ldots(i)$$

By the inequality of the means

$$\frac{y+z}{2} \geq \sqrt{yz}\,; \frac{z+x}{2} \geq \sqrt{zx}; \frac{x+y}{2} \geq \sqrt{xy}$$

It follows that

$$\frac{[(y + z)(z + x)(x + y)]}{(xyz)} \geq 8 \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$$

Example 48. *Let a, b, c be the sides of a triangle, show that $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$ must lie between the limits $3/2$ and 2. Can equality hold at either limit?*

Solution $\because$ a, b, c denotes the sides of a triangle

$$\therefore \quad 0<a<b+c, 0<b<c+a, 0<c<a+b$$

$$\therefore \quad 0<\frac{a}{b+c}<1, 0<\frac{b}{c+a}<1, 0<\frac{c}{a+b}<1$$

Now, we know that if x, y are +ve real numbers such that $\frac{x}{y}<1$ and z be any +ve real number, then

$$\frac{x}{y}<\frac{x+z}{y+z} \qquad \left[\text{for } \frac{x+z}{y+z}-\frac{x}{y}=\frac{z(y-x)}{y(y+z)}>0\right]$$

Using the fact, we find that

$$\frac{a}{b+c}<\frac{a+a}{b+c+a} \text{ so that } \frac{a}{b+c}<\frac{2a}{a+b+c}$$

Similarly,
$$\frac{b}{c+a}<\frac{2b}{a+b+c}$$

and
$$\frac{c}{a+b}<\frac{2c}{a+b+c}$$

Adding the above inequalities, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<\frac{2a+2b+2c}{a+b+c}$$

so that
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2 \qquad \text{...(i)}$$

$\because$ AM $\geq$ GM to the +ve numbers $b+c, c+a, a+b$, we have

$$\frac{1}{3}[(b+c)+(c+a)+(a+b)] \geq [(b+c)(c+a)(a+b)]^{1/3} \qquad \text{...(ii)}$$

$\because$ AM $\geq$ GM to the +ve numbers $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ we have

$$\frac{1}{3}\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq \left(\frac{1}{b+c} \cdot \frac{1}{c+a} \cdot \frac{1}{a+b}\right)^{1/3} \qquad \text{...(iii)}$$

Multiplying (ii) and (iii), we get

$$\frac{2}{9}(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 1$$

or
$$(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq \frac{9}{2}$$

or
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2} \qquad \text{...(iv)}$$

$\because$ The inequalities in (ii) and (iii) are both equalities when $a=b=c$. $\therefore$ Inequality in (iv) is an quality when a, b, c are all equal.

Example 49. *If a, b, c are +ve distinct integers, show that* $(a + b + c)^2 > \sqrt{abc}$
$[(ab)^{1/4} + (bc)^{1/4} + (ca)^{1/4}]$

Solution $\because$

$$a^2 + b^2 > 2ab \qquad \text{[AM > GM]}$$

$$b^2 + c^2 > 2bc, c^2 + a^2 > 2ca \qquad \text{(unequal numbers)}$$

Adding these three inequation, we get

$$a^2 + b^2 + c^2 > ab + bc + ca \qquad \text{...(i)}$$

We have, also $ab + bc > 2b\sqrt{ac}$

Similarly, $bc + ca > 2c\sqrt{ab}$

and $ac + ab > 2a\sqrt{bc}$

Adding these three inequation

$$ab + bc + ca > a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$$

$$= \sqrt{abc}\,[\sqrt{a} + \sqrt{b} + \sqrt{c}] \qquad \text{...(ii)}$$

Also, $\sqrt{a} + \sqrt{b} > 2(ab)^{1/4}$

$$\sqrt{b} + \sqrt{c} > 2(bc)^{1/4}, \sqrt{c} + \sqrt{a} > 2(ca)^{1/4}$$

Adding these three relations

$$\sqrt{a} + \sqrt{b} + \sqrt{c} > (ab)^{1/4} + (bc)^{1/4} + (ca)^{1/4} \qquad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$a^2 + b^2 + c^2 > \sqrt{abc}\,[(ab)^{1/4} + (bc)^{1/4} + (ca)^{1/4}]$$

$\because$ $(a + b + c)^2 > a^2 + b^2 + c^2$

$\therefore$ $(a + b + c)^2 > \sqrt{abc}$

$$[(ab)^{1/4} + (bc)^{1/4} + (ca)^{1/4}]$$

Hence proved.

Example 50. *If a, b, c are the sides of a triangle. Prove that*

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq 3.$$

Solution a, b, c are sides of triangle.

We have a, b, c are all > 0

Also, $c + a - b, a + b - c, b + c - a$ are all > 0.

By AM-GM inequality

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq 3\left[\frac{abc}{(c+a-b)(a+b-c)(b+c-a)}\right]^{1/3} \qquad \text{...(i)}$$

We have $a^2 \geq a^2 - (b - c)^2$

$\Rightarrow$ $a^2 \geq (a + b - c)(a - b + c)$ ...(ii)

Similarly, $b^2 \geq (b + c - a)(b - c + a)$ ...(iii)

and $c^2 \geq (c + a - b)(c - a + b)$ ...(iv)

Multiplying (ii), (iii) and (iv), we get

$$a^2b^2c^2 \geq (a + b - c)^2(b + c - a)^2\,(c + a - b)^2 \qquad \text{...(v)}$$

Taking +ve square root of (v), we get

$$abc \ge (a+b-c)(b+c-a)(c+a-b)$$

$$\Rightarrow \quad \frac{abc}{(a+b-c)(b+c-a)(c+a-b)} \ge 1$$

$$\Rightarrow \quad \left[\frac{abc}{(a+b-c)(b+c-a)(c+a-b)}\right]^{1/3} \ge 1$$

So, (i) implies that

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \ge 3$$

Example 51. *Find the largest constant k such that* $\dfrac{kabc}{a+b+c} \le (a+b)^2 + (a+b+4c)^2, \forall a, b, c > 0.$

Solution By AM-GM inequality

$$(a+b)^2 + (a+b+4c)^2$$

$$= (a+b)^2 + (a+2c+b+2c)^2 \ge (2\sqrt{ab^2}) + (2\sqrt{2ac} + 2\sqrt{2bc^2})$$

$$= 4ab + 8ac + 8bc + 16c\sqrt{ab}$$

$$\therefore \quad \frac{(a+b)^2 + (a+b+4c)^2}{abc}(a+b+c) \ge \frac{4ab + 8ac + 8bc + 16c\sqrt{ab}}{abc}(a+b+c)$$

$$= \left(\frac{4}{c} + \frac{8}{b} + \frac{8}{a} + \frac{16}{\sqrt{ab}}\right)(a+b+c)$$

$$= 8\left(\frac{1}{2c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{ab}}\right)$$

$$\left(\frac{a}{2} + \frac{a}{2} + \frac{b}{2} + \frac{b}{2} + c\right) \ge 8\left(5\sqrt[5]{\frac{1}{2a^2b^2c}}\right)\left(5\sqrt[5]{\frac{a^2b^2c}{2^4}}\right) = 100$$

Again, by AM - GM inequality.

Hence, largest constant k is 100.

For $k = 100$ equality holds, if $a = b = 2c > 0$.

Example 52. *If $a, b, c, \ldots, k$ are +ve quantities.*

Prove that

$$\left(\frac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k} < a^a b^b c^c \ldots k^k.$$

Solution Consider a quantities each equal to $(1/a)$, b quantities each equal to $(1/b)$ .... and k quantities each equal to $(1/k)$.

As AM > GM

$$\frac{\left(\frac{1}{a} + \frac{1}{a} + \ldots \text{ to } a \text{ terms}\right) + \left(\frac{1}{b} + \frac{1}{b} + \ldots \text{ to } b \text{ terms}\right) + \left(\frac{1}{k} + \frac{1}{k} + \ldots \text{ to } k \text{ terms}\right)}{a+b+\ldots+k}$$

$$> \left[\left(\frac{1}{a} \cdot \frac{1}{a} \ldots \text{ to } a \text{ factors}\right)\left(\frac{1}{b} \cdot \frac{1}{b} \ldots \text{ to } b \text{ factors}\right) \ldots \left(\frac{1}{k} \cdot \frac{1}{k} \text{ to } k \text{ factors}\right)\right]^{1/(a+b+\ldots+k)}$$

or $$\frac{\left(\frac{1}{a}\cdot a\right)+\left(\frac{1}{b}\cdot b\right)+\ldots+\left(\frac{1}{k}\cdot k\right)}{a+b+c+\ldots+k} > \left[\left(\frac{1}{a}\right)^a\cdot\left(\frac{1}{b}\right)^b\cdots\left(\frac{1}{k}\right)^k\right]^{\frac{1}{(a+b+\ldots+k)}}$$

or $$\frac{1+1+\ldots+1}{a+b+\ldots+k} > \left[\frac{1}{a^a b^b\ldots k^k}\right]^{\frac{1}{(a+b+\ldots+k)}}$$

or $$\frac{n}{a+b+c+\ldots+k} > \left(\frac{1}{a^a b^b\ldots k^k}\right)^{\frac{1}{(a+b+\ldots+k)}}$$

or $$\left(\frac{n}{a+b+c+\ldots+k}\right)^{a+b+c+\ldots+k} > \left(\frac{1}{a^a b^b\ldots k^k}\right)$$

[Raising both sides to power $(a+b+c+\ldots+k)$]

or $$\left(\frac{a+b+c+\ldots+k}{n}\right)^{a+b+\ldots+k} < (a^a b^b\ldots k^k)$$

Example 53. *Prove that* $a^p b^q > \left(\frac{ap+bq}{p+q}\right)^{p+q}$.

Solution Consider p quantities each equal to a and q quantities each equal to b.

$\because$ AM > GM

$\therefore$ $$\frac{(a+a+\ldots\text{ to } p \text{ terms})+(b+b+\ldots\text{ to } q\text{ terms})}{p+q} > [(a\times a\ldots\text{ to } p\text{ factors})$$
$$(b\times b\ldots\text{ to } q\text{ factors})]^{\frac{1}{p+q}}$$

or $$\frac{ap+bq}{p+q} > (a^p b^q)^{\frac{1}{p+q}}$$

Raising both sides to power $p+q$, we get
$$\left(\frac{ap+bq}{p+q}\right)^{p+q} > (a^p b^q)$$

Hence proved.

Example 54. *Prove that unless* $p=q=r$ *or* $x=1$,
$$px^{q-r}+qx^{r-p}+rx^{p-q} > p+q+r.$$

Solution Consider p quantities each equal to x^{q-r}

q quantities each equal to x^{r-p}

r quantities each equal to x^{p-q}

$\because$ AM> GM

$\therefore$ $$\frac{(x^{q-r}+x^{q-r}\ldots\text{ to } p\text{ terms})+(x^{r-p}+x^{r-p}\ldots\text{ to } q\text{ terms})+(x^{p-q}+x^{p-q}\ldots\text{ to } r\text{ terms})}{p+q+r} > [(x^{q-r}\cdot x^{q-r}\ldots\text{ to } p\text{ factors})(x^{r-p}\cdot x^{r-p}\ldots$$
$$\text{to } q\text{ factors})(x^{p-q}\,x^{p-q}\ldots\text{ to } r\text{ factors})]^{\frac{1}{p+q+r}}$$

or $$px^{q-r}+qx^{r-p}+rx^{p-q} > [x^{p(q-r)}\times x^{q(r-p)}\times x^{r(p-q)}]^{1/(p+q+r)}$$

or $$\frac{rx^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > [x^{pq-pr-qp+rp-rq}]^{\frac{1}{p+q+r}}$$

or $$\frac{px^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > (x^0)^{\frac{1}{p+q+r}}$$

or $$\frac{px^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > 1$$

or $$px^{q-r}+qx^{r-p}+rx^{p-q} > p+q+r$$

Example 55. *Prove that* $(n!)^3 < n^n\left(\frac{n+1}{2}\right)^{2n}$.

Solution $\because$ AM > GM

$\therefore$ $$\frac{1^3+2^3+3^3+\ldots+n^3}{n} > (1^3\cdot 2^3\cdot 3^3\ldots n^3)^{1/n}$$

or $$\left(\frac{1^3+2^3+3^3+\ldots+n^3}{n}\right)^n > (1\cdot 2\cdot 3\ldots n)^3 \quad \ldots(i)$$

Now, $$1^3+2^3+3^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$$

Substituting this value of $1^3+2^3+\ldots+n^3$ in (i), we get

$$\left[\frac{n^2(n+1)^2}{4n}\right]^n > (n!)^3 \text{ or } n^n\left[\left(\frac{n+1}{2}\right)^2\right]^n > (n!)^3$$

or $$n^n\left(\frac{n+1}{2}\right)^{2n} > (n!)^3 \text{ or } (n!)^3 < n^n\left(\frac{n+1}{2}\right)^n$$

Example 56. *Prove that* $2^n > 1 + n\sqrt{s^{n-1}}$

Solution Consider n quantities $1, 2, 2^2, \ldots, 2^{n-1}$. Then as AM > GM

$$\frac{1+2+2^2+\ldots\ldots+2^{n-1}}{n} > (1\cdot 2\cdot 2^2\ldots\ldots 2^{n-1})^{1/n}$$

$\therefore$ $$\frac{1}{n}\left[\frac{1(2^n-1)}{(2-1)}\right] > [2^{1+2+3+(n-1)}]^{1/n} \quad (\because \text{LHS is sum of GP})$$

or $$\frac{2^n-1}{n} > [2^{(n-1)n/2}]^{1/n} \quad \left[\because 1+2+3+\ldots+n = \frac{1}{2}n(n+1)\right]$$

or $$\frac{2^n-1}{n} > 2^{(n-1)/2}$$

or $$\frac{2^n-1}{n} > \sqrt{2^{n-1}}$$

or $$2^n - 1 > n\sqrt{2^{n-1}}$$

or $$2^n > 1 + n\sqrt{2^{n-1}}$$

Example 57. *If $n > 1$. Prove that*

$$1 + 2 + 2^2 + 2^3 + \dots + 2^{n-1} > n\sqrt{2^{n-1}}.$$

Solution Consider n quantities,

$$1, 2, 2^2, 2^3, \dots, 2^{n-1}$$

Then as AM > GM

$$\therefore \quad \left(\frac{1 + 2 + 2^2 + 2^3 + \dots + 2^{n-1}}{n}\right) > (1 \cdot 2 \cdot 2^2 \cdot 2^3 \dots 2^{n-1})^{1/n}$$

or $$(1 + 2 \quad 2^2 + 2^3 + \dots + 2^{n-1} > n[2^{1+2+3+\dots+(n-1)}]^{1/n}) \qquad \dots(i)$$

Now, $$1 + 2 + 3 + \dots + (n-1) = \frac{1}{2}(n-1)$$

$$[2(1) + (n-2)] = \frac{1}{2}(n-1)n$$

$\therefore$ From (i), we get

$$1 + 2 + 2^2 + \dots + 2^{n-1} > n[2^{n(n-1)/2}]^{1/n}$$

$$\Rightarrow \quad 1 + 2 + 2^2 + \dots + 2^{n-1} > n[2^{(n-1)/2}]$$

$$\Rightarrow \quad 1 + 2 + 2^2 + \dots + 2^{n-1} > n\sqrt{2^{n-1}}$$

Hence proved.

Example 58. *Prove that*

$$1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \dots n^n < \left(\frac{2n+1}{3}\right)^{\frac{(n+1)}{2}}.$$

Solution $\because$ $$1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \dots n^n$$

$$= 1 \cdot (2 \cdot 2) \cdot (3 \cdot 3 \cdot 3) \cdot (4 \cdot 4 \cdot 4 \cdot 4) \dots (n \cdot n \cdot n \cdot \dots n)$$

Total number $= 1 + 2 + 3 + 4 + \dots + n$

$$= \Sigma n = \frac{n(n+1)}{2}$$

$\because$ AM > GM

$$\therefore \quad \frac{\begin{array}{c}1 + (2+2) + (3+3+3) + (4+4+4+4) \\ \dots + (n + n + n \dots n \text{ times})\end{array}}{\frac{n(n+1)}{2}} > (1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \dots n^n)^{\frac{1}{n(n+1)/2}}$$

$$\Rightarrow \quad \frac{1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2}{\frac{n(n+1)}{2}} > (1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \dots n^n)^{\frac{2}{n(n+1)}}$$

$$\Rightarrow \quad \frac{\frac{n(n+1)(2n+1)}{6n(n+1)}}{2} > (1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \dots n^n)^{\frac{2}{n(n+1)}}$$

$$\Rightarrow \quad \left(\frac{2n+1}{3}\right) > (1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \dots n^n)^{\frac{2}{n(n+1)}}$$

$$\Rightarrow \quad \left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}} > 1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \dots n^n$$

Hence, $$1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \dots n^n < \left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}$$

Hence proved.

Example 59. *For any $n \in N$. Prove that*

$$[(n+1)!]^{1/n+1} < 1 + \frac{n}{n+1}(n!)^{1/n}.$$

Solution $\because$ AM > GM we have

$$[(n+1) + (n!)^{1/n} + (n!)^{1/n} + \ldots + (n!)^{1/n}]$$
$$> (n+1)[(n+1)(n!)^{1/n}(n!)^{1/n}(n!)^{1/n}\ldots(n!)^{1/n}]^{1/n+1}$$

$\Rightarrow$ $$[(n+1) + n(n!)^{1/n}] > (n+1)$$
$$[(n+1)\cdot n!]^{1/n+1}$$

$\Rightarrow$ $$[(n+1) + n(n!)^{1/n}] > (n+1)[(n+1)!]^{1/n+1}$$

$\Rightarrow$ $$1 + \frac{n}{n+1}(n!)^{1/n} > [(n+1)!]^{1/n+1}$$

$\Rightarrow$ $$[(n+1)!]^{1/n+1} < 1 + \frac{n}{n+1}(n!)^{1/n}$$

Hence proved.

Example 60. *Prove that for each natural number n*

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} > 1.$$

Solution

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{3n+1}$$
$$= \left(\frac{1}{n+1} + \frac{1}{3n+1}\right) + \left(\frac{1}{n+2} + \frac{1}{3n}\right) + \ldots + \left(\frac{1}{2n} + \frac{1}{2n+2}\right) + \frac{1}{2n+1}$$

Since, the AM of two unequal +ve numbers exceeds their HM.

$\therefore$ $$\frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{3n+1}\right) > \left[\frac{(n+1)+(3n+1)}{2}\right]^{-1}$$
$$= \frac{1}{2n+1}$$

$$\frac{1}{2}\left(\frac{1}{n+2} + \frac{1}{3n}\right) > \left[\frac{(n+2)+3n}{2}\right]^{-1} = \frac{1}{2n+1}$$

$$\frac{1}{2}\left(\frac{1}{2n} + \frac{1}{2n+2}\right) > \left[\frac{2n+(2n+2)}{2}\right]^{-1} = \frac{1}{2n+1}$$

Adding corresponding sides of inequalities, we get

$$\frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} + \frac{1}{2n+2} + \ldots + \frac{1}{3n+1}\right) > \frac{n}{2n+1}$$

Multiplying throughout by 2 and adding $\frac{1}{2n+1}$ to both sides, we get

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} > 1$$

Hence proved.

Example 61. *Prove that*

$$1 < \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \ldots + \frac{1}{3001} < 1\frac{1}{3}.$$

Solution AM of any n distinct +ve numbers always exceeds their HM. We shall apply this result to the 2001 numbers.

$$1001, 1002, \ldots, 3000, 3001$$

For these numbers

$$\text{AM} = \frac{1001 + 1002 + \ldots + 3001}{2001}$$

$$= \frac{2001}{2}(1001 + 3001) \cdot \frac{1}{2001}$$

$$= 2001$$

$$\text{HM} = \left[\frac{\frac{1}{1001} + \frac{1}{1002} + \ldots + \frac{1}{3000} + \frac{1}{3001}}{2001}\right]^{-1}$$

$$= 2001 \Big/ \left(\frac{1}{1001} + \frac{1}{1002} + \ldots + \frac{1}{3001}\right)$$

$\because$ $\quad$ AM > HM

$\therefore$ $\quad 2001 > 2001 \Big/ \left(\frac{1}{1001} + \frac{1}{1002} + \ldots + \frac{1}{3001}\right)$

i.e., $\quad \frac{1}{1001} + \frac{1}{1002} + \ldots + \frac{1}{3001} > 1$

Now, let us observe that

$$\frac{1}{1001} + \frac{1}{1002} + \ldots + \frac{1}{1250} < \frac{250}{1000} = \frac{1}{4}$$

$$\frac{1}{1251} + \frac{1}{1252} + \ldots + \frac{1}{1500} < \frac{250}{1000} = \frac{1}{5}$$

$$\frac{1}{1501} + \ldots + \frac{1}{2000} < \frac{500}{1500} = \frac{1}{3}$$

$$\frac{1}{2001} + \frac{1}{2002} + \ldots + \frac{1}{3001} < \frac{1001}{2000}$$

Adding throughout, we have

$$\frac{1}{1001} + \frac{1}{1002} + \ldots + \frac{1}{3001} < \frac{1}{4} + \frac{1}{5} + \frac{1}{3} + \frac{1001}{2000} = \frac{7703}{6000} < 1\frac{1}{3}$$

Example 62. *If n be a +ve integer. Prove*

(*i*) $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$ $\quad$ (*ii*) $\left(1 - \frac{1}{n}\right)^n < \left(1 - \frac{1}{n+1}\right)^{n+1}$

(*iii*) $\left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{n+1}\right)^{n+2}$

Solution (i) If $x_1, \ldots, x_{n+1}$ be +ve numbers (not all equal), then by the inequality of the means, we have

$$(x_1 \ldots x_{n+1})^{\frac{1}{n+1}} < \frac{x_1 + \ldots + x_{n+1}}{n+1} \qquad \ldots(i)$$

Putting $$x_1 = x_2 = \dots = x_n = 1 + \frac{1}{n};\ x_{n+1} = 1$$

we get

$$\left(1+\frac{1}{n}\right)^{n/(n+1)} < \frac{n+2}{n+1}$$

$$\left(1+\frac{1}{n}\right)^{n/(n+1)} < 1 + \frac{1}{n+1}$$

Raising both sides to power $n+1$, we get

$$\left(1+\frac{1}{n}\right)^{n} < \left(1+\frac{1}{n+1}\right)^{n+1}$$

(ii) Putting $x_1 = x_2 = \dots = x_n = 1 - \frac{1}{n};\ x_{n+1} = 1$ in (i), we get

$$\left[\left(1-\frac{1}{n}\right)^{n}\right]^{1/(n+1)} < \frac{n}{n+1}$$

or $$\left(1-\frac{1}{n}\right)^{n} < \left(1-\frac{1}{n+1}\right)^{n+1} \quad \dots\text{(ii)}$$

(iii) $$\left(1+\frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} = \frac{1}{\left(\frac{n}{n+1}\right)^{n+1}} = \frac{1}{\left(1-\frac{1}{n+1}\right)^{n+1}} \quad \dots\text{(iii)}$$

Replacing n by $n+1$ in (iii), we get

$$\left(1+\frac{1}{n+1}\right)^{n+2} = \frac{1}{\left(1-\frac{1}{n+2}\right)^{n+2}} \quad \dots\text{(iv)}$$

Replacing n by $n+1$ in (ii), we get

$$\left(1-\frac{1}{n+1}\right)^{n+1} < \left(1-\frac{1}{n+2}\right)^{n+2}$$

or $$\frac{1}{\left(1-\frac{1}{n+1}\right)^{n+1}} > \frac{1}{\left(1-\frac{1}{n+2}\right)^{n+2}}$$

or $$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2} \quad \text{[using (iii) and (iv)]}$$

Example 63. *Show that*

$$1!3!5!\dots(2n-1)! > (n!)^n.$$

Solution First let us prove the following

$$\frac{r!(2n-r)!}{n} > n!,\ n > r$$

$$\because \quad \frac{r!(2n-r)!}{n!}$$

$$= \frac{(1\cdot 2\cdot 3\dots r)(2n-r)(2n-r-1)\dots(n+1)n!}{n!}$$

$$= (1 \cdot 2 \cdot 3 \ldots r)(2n - r)(2n - r - 1) \ldots (n + 1)$$

$$> (1 \cdot 2 \cdot 3 \ldots r)[(r + 1)(r + 2) \ldots n] \qquad [\because n > r]$$

Also,
$$(n + 1) > r + 1$$
$$n - r > 0$$
$$2n - r > n$$

$\therefore$
$$1 \cdot 2 \cdot 3 \ldots r(r + 1)(r + 2) \ldots > n!$$

$\therefore$
$$\frac{r!(2n - r)!}{n!} > n! \text{ or } r!(2n - r)! > (n!)^2$$

Now, giving r successively the value 1, 3, 5, ..., $(2n - 1)$

We get
$$1!(2n - 1)! > (n!)^2$$
$$3!(2n - 3)! > (n!)^2$$
$$5!(2n - 5)! > (n!)^2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(2n - 1)!\, 1! > (n!)^2$$

Multiplying both sides, we have
$$[1!3!5!7! \ldots (2n - 1)!]^2 > (n!)^{2n}$$

Taking square root of both sides
$$1!3!5! \ldots (2n - 1) > (n!)^n$$

Hence proved.

Example 64. *If $a, b, c, d, \ldots$ are p +ve integers, whose sum is equal to n. Show that the least value of $a!b!c!d! \ldots$ is $(q!)^{p-r}(q! + 1)!)^r$, where q is the quotient and r the remainder when n is divided by p.*

Solution Take the expression $a!b!c!d! \ldots$

Then, if any two of the quantities a and b say, are not equal, we can diminish $a!b!$ by taking a, b equal and keeping sum of a and b to be the same.

Hence, the value of $a!b!c!d! \ldots$ is least when all the quantities $a, b, c, d, \ldots$ are equal. If however n is not exactly divisible by p, this will not be the case.

Suppose r is the remainder, q is quotient when n is divided by p.

Thus,
$$n = pq + r$$
$$= (p - r)q + r(q + 1)$$

$\therefore$ $p - r$ of the quantities $a, b, c, d, \ldots$ will be equal and the remainder r will be equal to $(q + 1)$.

So, the least value of the expression is $(q!)^{p-r}((q! + 1)!)^r$.

Example 65. *For any natural number n, prove the following inequality.*

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right) \geq \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\right) \qquad \text{(RMO 1998)}$$

Solution We will prove it by induction.

Given inequality is true for $n = 1$. Assume it to be true as such and show that when n is replaced by $(n + 1)$ too it is true.

Rewriting, the inequality is

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)\geq\frac{1}{n}\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)$$

$$=\frac{n+1}{n}\cdot\frac{1}{n+1}\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)$$

$$=\left(1+\frac{1}{n}\right)\frac{1}{n+1}\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)$$

which is equivalent to

$$\frac{1}{n+1}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{5}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}\right)\geq\frac{1}{n(n+1)}\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)$$

i.e.,

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{5}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}\right)\geq\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)$$

(As $n\to\infty$. LHS converges to log 2, while RHS converges to $\lim\limits_{n\to\infty}\frac{1}{2}\cdot\frac{\log n}{n}=0$). Assuming the last inequality we have to prove.

$$1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{1}{2n-1}-\frac{1}{2n}+\frac{1}{2n+1}-\frac{1}{2n+2}\geq\frac{1}{n+1}\left(\frac{1}{2}+\ldots+\frac{1}{2n+2}\right)$$

LHS (By induction hypothesis)

$$\geq\frac{1}{n}\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)+\frac{1}{2n+1}-\frac{1}{2n+2}$$

$$=\frac{n+1}{n(n+1)}\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)+\frac{1}{2n+1}-\frac{1}{2n+2}$$

$$=\left(1+\frac{1}{n}\right)\frac{1}{n+1}\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)+\frac{1}{2n+1}-\frac{1}{2n+2}\geq\frac{1}{n+1}\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)+\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)$$

$$\frac{1}{n(n+1)}+\frac{1}{2n+1}-\frac{1}{2n+2}$$

It suffices to show that

$$\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)\frac{1}{n(n+1)}+\frac{1}{2n+1}-\frac{1}{2n+2}\geq\frac{1}{n+1}\cdot\frac{1}{2n+2}$$

$$\text{LHS}\geq\frac{1}{2n(2n+1)}+\frac{1}{2n+1}-\frac{1}{2n+2}$$

$$=\frac{1}{2n(n+1)}+\frac{1}{(2n+1)(2n+2)}$$

$$=\frac{1}{2n(n+1)}+\frac{1}{2(n+1)(2n+2)}$$

$$=\frac{1}{2(n+1)}\left(\frac{1}{n}+\frac{1}{2n+1}\right)$$

$$=\frac{3n+1}{2(n+1)n(2n+1)}$$

$$=\frac{3n+1}{2n(n+1)(2n+1)}\geq\frac{1}{n+1}\cdot\frac{1}{2n+2}$$

$\because$ $3n+1>2n$ and $2n+2>2n+1$

$\therefore$ Given inequality is true by induction.

Example 66. *Prove that for any +ve integer n we have* $2 \le \left(1+\frac{1}{n}\right)^n < 3.$ **(IMO)**

Solution Let us prove that

$$1+\frac{k}{n} \le \left(1+\frac{1}{n}\right)^k < 1+\frac{k}{n}+\frac{k^2}{n^2}$$

for any +ve integer k such that $k \le n$

We shall use the method of mathematical induction for $k = 1$, the required relation obviously holds. Now, let us assume that it holds for $k + 1$ as well. We have

$$\left(1+\frac{1}{n}\right)^{k+1} = \left(1+\frac{1}{n}\right)^k\left(1+\frac{1}{n}\right) \ge \left(1+\frac{k}{n}\right)\left(1+\frac{1}{n}\right)$$

$$= 1+\frac{k+1}{n}+\frac{k}{n^2} > 1+\frac{k+1}{n}$$

It should be noted that here we have not used the relation $k \le n$.

Consequently, this inequality holds for any +ve integer k.

Let us put $k \le n$. Then, we get

$$\left(1+\frac{1}{n}\right)^{k+1} = \left(1+\frac{1}{n}\right)^k\left(1+\frac{1}{n}\right) < \left(1+\frac{k}{n}+\frac{k^2}{n^2}\right)\left(1+\frac{1}{n}\right)$$

$$= 1+\frac{k+1}{n}+\frac{k^2+2k+1}{n^2}-\frac{k+1}{n^2}+\frac{k^2}{n^3}$$

$$= 1+\frac{k+1}{n}+\frac{(k+1)^2}{n^2}-\frac{n(k+1)-k^2}{n^3} < 1+\frac{k+1}{n}+\frac{(k+1)^2}{n^2}$$

as $n(k+1) > k^2$ for $n \ge k$

On substituting the value $k = n$ into the inequalities we have derived we get

$$2 = 1+\frac{n}{n} \le \left(1+\frac{1}{n}\right)^n < 1+\frac{n}{n}+\frac{n^2}{n^2} = 3$$

Example 67. *If* $abc = 1$ $a, b, c > 0$. *Show that*

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$ **(IMO 1995)**

Solution Let

$$x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$$

$$xyz = \frac{1}{abc} = 1, \therefore xyz = 1$$

Now,

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}$$

$$= \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}$$

Let

$$S = \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}$$

Consider $(x+y+z)^2$

$$= \left(\frac{x\sqrt{y+z}}{\sqrt{y+z}}+\frac{y\sqrt{z+x}}{\sqrt{z+x}}+\frac{z\sqrt{x+y}}{\sqrt{x+y}}\right)^2$$

Using Cauchy's inequality

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

$$\left(\frac{x}{\sqrt{y+z}}\sqrt{y+z} + \frac{y}{\sqrt{z+x}}\sqrt{z+x} + \frac{z}{\sqrt{x+y}}\sqrt{x+y}\right)^2$$

$$\le \left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right)\cdot(y+z+z+x+x+y)$$

$\Rightarrow$ $$(x+y+z)^2 \le S(y+z+z+x+x+y)$$

or $$S \ge \frac{x+y+z}{2}$$

We know, $$\frac{x+y+z}{3} \ge (xyz)^{1/3}$$

or $$x+y+z \ge 3(xyz)^{1/3}$$

or $$\frac{x+y+z}{2} \ge \frac{3(xyz)^{1/3}}{2}$$

$\therefore$ We get $$S \ge \frac{3}{2}(xyz)^{1/3}$$

$$S \ge \frac{3}{2} \qquad [\because xyz = 1]$$

$\therefore$ $$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}$$

Example 68. *If* $C_r = \dfrac{n!}{r!(n-r)!}$. *Prove that*

$$\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n} < \sqrt{[n(2^n - 1)]}$$

Solution AM of (1/2)th powers < (1/2)th power of AM

$\therefore$ $$\frac{(C_1)^{\frac{1}{2}} + (C_2)^{\frac{1}{2}} + \ldots + (C_n)^{\frac{1}{2}}}{n} < \left(\frac{C_1 + C_2 + \ldots + C_n}{n}\right)^{1/2}$$

$\Rightarrow$ $$\frac{\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n}}{n} < \left(\frac{2^n - 1}{n}\right)^{1/2}$$

$\Rightarrow$ $$\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n} < \frac{n\sqrt{(2^n - 1)}}{\sqrt{n}}$$

Hence, $$\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n} < \sqrt{[n(2^n - 1)]}$$

Example 69. *If* $x + y + z = 1, x, y, z$ *are +ve, show that*

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 + \left(z + \frac{1}{z}\right)^2 > \frac{100}{3}$$

Solution AM of (2)th power >(2)th power of AM

$$\frac{\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 + \left(z + \frac{1}{z}\right)^2}{3} > \left[\frac{\left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) + \left(z + \frac{1}{z}\right)}{3}\right]^2$$

$$\Rightarrow \quad \frac{\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2}{3} > \frac{1}{3}\left(x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2$$

$$\Rightarrow \quad \frac{\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2}{3} > \frac{1}{9}\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2 \qquad [\because x+y+z=1]$$

Again, $$\frac{x^{-1}+y^{-1}+z^{-1}}{3} > \left(\frac{x+y+z}{3}\right)^{-1}$$

or $$x^{-1}+y^{-1}+z^{-1} > 9$$

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2 (1+9)^2 > \frac{100}{3}$$

Example 70. *If x, y, z are unequal +ve quantities such that sum of any two is greater than the third. Show that*

$$\frac{1}{y+z-x}+\frac{1}{z+x-y}+\frac{1}{x+y-z} > \frac{9}{x+y+z}.$$

Solution Consider three quantities $(y+z-x)$, $(z+x-y)$ and $(x+y-z)$ and taking $m=-1$

$\because$ AM of mth power > mth power of AM

$$\therefore \quad \frac{(z+x-y)^{-1}+(y+z-x)^{-1}+(x+y-z)^{-1}}{3} > \left[\frac{(y+z-x)+(z+x-y)+(x+y-z)}{3}\right]^{-1}$$

or $$\frac{1}{3}\left(\frac{1}{y+z-x}+\frac{1}{z+x-y}+\frac{1}{x+y-z}\right) > \left(\frac{x+y+z}{3}\right)^{-1} = \frac{3}{x+y+z}$$

or $$\frac{1}{y+z-x}+\frac{1}{z+x-y}+\frac{1}{x+y-z} > \frac{9}{x+y+z}$$

Hence proved.

Example 71. *Prove that*

$$(b+c-a)^2+(c+a-b)^2+(a+b-c)^2 \geq (bc+ca+ab).$$

Solution Taking $m=2$ and considering three quantities $b+c-a$, $c+a-b$ and $a+b-c$.

$\because$ AM of mth power > mth power of AM

$$\therefore \quad \frac{(b+c-a)^2+(c+a-b)^2+(a+b-c)^2}{3} > \left[\frac{(b+c-a)+(c+a-b)+(a+b-c)}{3}\right]^2$$

or $$\frac{(b+c-a)^2+(c+a-b)^2+(a+b-c)^2}{3} > \left(\frac{a+b+c}{3}\right)^2 \qquad \text{...(i)}$$

Now, considering a^2 and b^2

$\because$ $$\text{AM} > \text{GM}$$

$\therefore$ $$\frac{1}{2}(a^2+b^2) > \sqrt{a^2b^2}$$

or $$a^2+b^2 > 2ab$$

Similarly, $$b^2+c^2 > 2bc$$

and $$c^2+a^2 > 2ac$$

Adding these results, we get

$$(a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2) > (2ab + 2bc + 2ca)$$

or $$(a^2 + b^2 + c^2) > ab + bc + ca$$

Now, add $2ab + 2bc + 2ca$ to both sides, we have

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca > (3ab + 3bc + 3ca)$$

or $$(a + b + c)^2 > 3(ab + bc + ca)$$

or $$\left(\frac{a + b + c}{3}\right)^2 > \frac{3(ab + bc + ca)}{9}$$ [on dividing by 9]

or $$\left(\frac{a + b + c}{3}\right)^2 > \frac{ab + bc + ca}{3} \quad ...(i)$$

From (i) and (ii), we get

$$(b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2 > ab + bc + ca$$

If $a = b = c$, then

$$(b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2$$
$$= ab + bc + ca$$

Example 72. *Prove that* $\frac{a^8 + b^8 + c^8}{a^3b^3c^3} > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Solution Take $m = 8$ and consider a, b, c

AM of mth power > mth power of AM

$\therefore$ $$\left(\frac{a^8 + b^8 + c^8}{3}\right) > \left(\frac{a + b + c}{3}\right)^8$$

$$\left(\frac{a^8 + b^8 + c^8}{3}\right) > \left(\frac{a + b + c}{3}\right)^6 \left(\frac{a + b + c}{3}\right)^2 \quad ...(i)$$

We know that $$\left(\frac{a + b + c}{3}\right)^3 > abc \quad ...(ii)$$

$$\left(\frac{a + b + c}{3}\right)^2 > \left(\frac{ab + bc + ca}{3}\right) \quad ...(iii)$$

Squaring both sides of (ii), we get

$$\left(\frac{a + b + c}{3}\right)^6 > a^2b^2c^2 \quad ...(iv)$$

Putting the value of (iii) and (iv) in (i), we have

$$\frac{a^8 + b^8 + c^8}{3} > a^2b^2c^2\left(\frac{ab + bc + ca}{3}\right)$$

Dividing both sides by $a^3b^3c^3$

$$\frac{a^8 + b^8 + c^8}{a^3b^3c^3} > \frac{ab + bc + ca}{abc}$$

or $$\frac{a^8 + b^8 + c^8}{a^3b^3c^3} > \frac{1}{c} + \frac{1}{a} + \frac{1}{b}$$

Hence proved.

Example 73. *Prove that*

$$a^5 + b^5 + c^5 > abc\,(ab + bc + ca)$$

Solution $\because$ AM of mth power > mth power of AM

$\because$
$$\frac{a^5 + b^5 + c^5}{3} > \left(\frac{a+b+c}{3}\right)^5$$

$$\frac{a^5 + b^5 + c^5}{3} > \left(\frac{a+b+c}{3}\right)^3\left(\frac{a+b+c}{3}\right)^2 \quad \text{...(i)}$$

$\because$
$$\text{AM} > \text{GM}$$

$\therefore$
$$\frac{a+b+c}{3} > (abc)^{1/3} \text{ or } \left(\frac{a+b+c}{3}\right)^3 > abc$$

From (i), we have

$$\frac{a^5 + b^5 + c^5}{3} > abc\left(\frac{a+b+c}{3}\right)^2 \quad \text{...(ii)}$$

$\because$
$$\text{AM} > \text{GM}$$

$\therefore$
$$\frac{a^2+b^2}{2} > \sqrt{a^2b^2} \text{ or } a^2 + b^2 > 2ab$$

Similarly, $b^2 + c^2 > 2bc$ and $c^2 + a^2 > 2ac$

Adding these results, we have

$$(a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2) > 2ab + 2bc + 2ac$$

$$2(a^2 + b^2 + c^2) > 2(ab + bc + ca)$$

or
$$a^2 + b^2 + c^2 > ab + bc + ca$$

Adding $2ab + 2bc + 2ca$ to both sides, we get

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca > 3(ab + bc + ca)$$

$$(a + b + c)^2 > 3(ab + bc + ca)$$

$$\frac{(a+b+c)^2}{9} > \frac{3(ab+bc+ca)}{9} \quad \text{[on dividing by 9]}$$

$$\left(\frac{a+b+c}{3}\right)^2 > \left(\frac{ab+bc+ca}{3}\right)$$

From (ii), we get $\dfrac{a^5 + b^5 + c^5}{3} > abc$

$$\left(\frac{ab+bc+ca}{3}\right)$$

$\therefore$
$$a^5 + b^5 + c^5 > abc\,(ab + bc + ca)$$

Example 74. *By Cauchy's inequality. Prove that*

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

where a, b, c are all +ve.

Solution Let two sets of numbers are $\sqrt{a}, \sqrt{b}, \sqrt{c}$ and $\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}$. Now, from Cauchy-Schwartz inequality, we have

$$[(\sqrt{a})^2 + (\sqrt{b})^2 + (\sqrt{c})^2]$$
$$\left[\left(\frac{1}{\sqrt{a}}\right)^2 + \left(\frac{1}{\sqrt{b}}\right)^2 + \left(\frac{1}{\sqrt{c}}\right)^2\right] \geq \left(\sqrt{a}\cdot\frac{1}{\sqrt{a}} + \sqrt{b}\cdot\frac{1}{\sqrt{b}} + \sqrt{c}\cdot\frac{1}{\sqrt{c}}\right)^2$$

or
$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq (1+1+1)^2$$

or
$$\frac{a+b+c}{a} + \frac{a+b+c}{b} + \frac{a+b+c}{c} \geq 9$$

or
$$1 + \frac{b+c}{a} + 1 + \frac{c+a}{b} + 1 + \frac{a+b}{c} \geq 9$$

or
$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 9-3$$

Hence,
$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

Example 75. *If $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ be any two sets of +ve numbers, then by using Tchebychef 's inequality show that*

$$\left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + ... + \frac{x_n}{y_n}\right)\left(\frac{y_1}{x_1} + \frac{y_2}{x_2} + ... + \frac{y_n}{x_n}\right) \geq n^2.$$

Solution Assume that

$$\frac{x_1}{y_1} \leq \frac{x_2}{y_2} \leq ... \leq \frac{x_n}{y_n}, \text{ then}$$
$$\frac{y_1}{x_1} \geq \frac{y_2}{x_2} \geq ... \geq \frac{y_n}{x_n} \text{ for two sets}$$

i.e.,
$$\frac{x_1}{y_1}, \frac{x_2}{y_2}, ..., \frac{x_n}{y_n} \text{ and } \frac{y_1}{x_1}, \frac{y_2}{x_2}, ..., \frac{y_n}{x_n}$$

By applying Tchebychef 's inequality, we get

$$\left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + ... + \frac{x_n}{y_n}\right)\left(\frac{y_1}{x_1} + \frac{y_2}{x_2} + ... + \frac{y_n}{x_n}\right) \geq n\left(\frac{x_1}{y_1}\cdot\frac{y_1}{x_1} + \frac{x_2}{y_2}\cdot\frac{y_2}{x_2} + ... + \frac{x_n}{y_n}\cdot\frac{y_n}{x_n}\right)$$

or
$$\left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + ... + \frac{x_n}{y_n}\right)\left(\frac{y_1}{x_1} + \frac{y_2}{x_2} + ... + \frac{y_n}{x_n}\right) \geq n(1 + 1 + ... + 1)$$

$\therefore$
$$\left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + ... + \frac{x_n}{y_n}\right)\left(\frac{y_1}{x_1} + \frac{y_2}{x_2} + ... + \frac{y_n}{x_n}\right) \geq n^2$$

Example 76. *Prove that*

$$\frac{\sqrt{1} + \sqrt{\frac{1}{2}} + ... + \sqrt{\frac{1}{n}}}{\sqrt{n}} < (2n-1)^{1/4}$$

where n is any +ve integer.

Solution $\because$ n is +ve integer such that $\sqrt{1} > \sqrt{\frac{1}{2}} > ... > \sqrt{\frac{1}{n}}$ for two sets of number

$$\sqrt{1}, \sqrt{\frac{1}{2}}, ..., \sqrt{\frac{1}{n}}$$

and
$$\sqrt{1}, \sqrt{\frac{1}{2}}, ..., \sqrt{\frac{1}{n}}$$

Applying Tchebychef 's inequality

$$\left(\sqrt{1}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)\left(\sqrt{1}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)<n\left(\sqrt{1}\cdot\sqrt{1}+\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\cdot\sqrt{\frac{1}{n}}\right)$$

$$\Rightarrow \quad \left(\sqrt{1}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)^2<n\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) \qquad \ldots\text{(i)}$$

Again applying Tchebychef 's inequality for two sets.

$$1,\frac{1}{2},\ldots,\frac{1}{n} \text{ and } 1,\frac{1}{2},\ldots,\frac{1}{n}$$

$$\because \quad 1>\frac{1}{2}>\ldots>\frac{1}{n}$$

$$\therefore \quad \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)<n\left(1\cdot 1+\frac{1}{2}\cdot\frac{1}{2}+\ldots\frac{1}{n}\cdot\frac{1}{n}\right)$$

$$\therefore \quad \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)^2<n\left(1^2+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right) \qquad \ldots\text{(ii)}$$

$$\because \quad n>0 \text{ and } -n<0$$

or $$n^2-n<n^2$$

$$n(n-1)<n^2$$

or $$\frac{1}{n(n-1)}<\frac{1}{n^2}$$

Putting $n = 2, 3, 4, \ldots, n$

$$\frac{1}{1\cdot 2}>\frac{1}{2^2}$$

$$\frac{1}{2\cdot 3}>\frac{1}{3^2}$$

$$\frac{1}{3\cdot 4}>\frac{1}{4^2}$$

$$\frac{1}{(n-1)\cdot n}>\frac{1}{n^2}$$

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Adding all corresponding sides, we get

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(n-1)n}>\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}$$

or $$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n-1}-\frac{1}{n}>\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}$$

$$\Rightarrow \quad 1-\frac{1}{n}>\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}$$

or $$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}<2-\frac{1}{n}$$

$$n\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right)<(2n-1) \qquad \ldots\text{(iii)}$$

From (ii) and (iii), we get

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)^2 < (2n-1)$$

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) < (2n-1)^{1/2} \qquad \ldots\text{(iv)}$$

From (i) and (iv), we have

$$\left(\sqrt{1}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)^2 < n(2n-1)^{1/2}$$

$$\left(\sqrt{1}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right) < \sqrt{n}(2n-1)^{1/4}$$

Hence,

$$\frac{\sqrt{1}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}}{\sqrt{n}} < (2n-1)^{1/4}$$

Example 77. *Prove that*

$$\frac{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n} > \frac{1}{(n+1)}$$

where n is +ve integer.

Solution Let

$$a_1=\frac{1}{2}, a_2=\frac{1}{4}, a_3=\frac{1}{6}, \ldots, a_n=\frac{1}{2n}$$

then

$$a_1+a_2+a_3+\ldots+a_n=S_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}<1$$

Now, apply second inequality of Weierstress's inequality, then

$$(1+S_n)<(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n)$$

$$(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n)>1+S_n$$

$$\Rightarrow \quad \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\ldots\left(1+\frac{1}{2n}\right)>1+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}$$

$$\Rightarrow \quad \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\ldots\frac{(2n+1)}{2n}>1+2^{-1}+4^{-1}+6^{-1}+\ldots+(2n)^{-1} \qquad \ldots\text{(i)}$$

$\because$ AM of (-1)th power $>$ (-1)th power of AM

i.e.,

$$\frac{2^{-1}+4^{-1}+6^{-1}+\ldots+(2n)^{-1}}{n} > \left(\frac{2+4+6+\ldots+2n}{n}\right)^{-1}$$

$$=\left(\frac{n}{2+4+6+\ldots+2n}\right)$$

$$2^{-1}+4^{-1}+6^{-1}+\ldots+(2n)^{-1} > \frac{n^2}{\frac{n}{2}(2+2n)}$$

$$2^{-1}+4^{-1}+6^{-1}+\ldots+(2n)^{-1} > \frac{n}{n+1}$$

$$1+2^{-1}+4^{-1}+6^{-1}+\ldots+(2n)^{-1} > 1+\frac{n}{n+1}$$

$$1+2^{-1}+4^{-1}+6^{-1}+\ldots+(2n)^{-1} > \frac{2n+1}{n+1} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{(2n+1)}{2n}>\frac{(2n+1)}{n+1}$$

$$\Rightarrow \quad \frac{3\cdot 5\cdot 7\ldots(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots 2n}>\frac{(2n+1)}{n+1}$$

$$\Rightarrow \quad \frac{3\cdot 5\cdot 7\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}>\frac{1}{n+1}$$

Example 78. *If x, y, z are each +ve and $x+y+z=6$. Show that*

$$\left(x+\frac{1}{y}\right)^2+\left(y+\frac{1}{z}\right)^2+\left(z+\frac{1}{x}\right)^2\geq\frac{75}{4}. \qquad \textbf{(IMO)}$$

Solution By using HM inequality

$$\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}{3}\geq\frac{3}{x+y+z}$$

$$=\frac{3}{6}=\frac{1}{2}$$

So,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{3}{2}$$

Now,

$$\sqrt{\frac{\left(x+\frac{1}{y}\right)^2+\left(y+\frac{1}{z}\right)^2+\left(z+\frac{1}{x}\right)^2}{3}}\geq\frac{x+\frac{1}{y}+y+\frac{1}{z}+z+\frac{1}{x}}{3}$$

$$\geq\frac{6+\frac{3}{2}}{3}=\frac{5}{2}$$

Now, squaring both sides and multiplying by 3, we get

$$\left(x+\frac{1}{y}\right)^2+\left(y+\frac{1}{z}\right)^2+\left(z+\frac{1}{x}\right)^2\geq\frac{75}{4}$$

Example 79. *If $x, y, z>0$, prove that*

$$(xy+yz+zx)\left[\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right]\geq\frac{9}{4}. \qquad \textbf{(IMO)}$$

Solution Let $y+z=a$,

$$z+x=b \text{ and } x+y=c$$

We have

$$x=b+c-\frac{a}{2}$$

Thus,

$$yz=\frac{a^2-(b-c)^2}{4}$$

$$=\frac{a^2-b^2-c^2+2bc}{4}$$

$$\Sigma yz=\frac{1}{4}\Sigma(2bc-a^2)$$

Now, assume $a \geq b \geq c$ without loss of generality, then

$$2bc - a^2 \leq 2ca - b^2 \leq 2ab - c^2$$

$\therefore$ By Tchebychef 's inequality and AM-GM inequality

$$\frac{4}{9}(\Sigma yz)\left[\Sigma\frac{1}{(y+z)^2}\right] = \frac{1}{3}\Sigma(2bc - a^2)\frac{1}{3}\Sigma\frac{1}{a^2}$$

$$\geq \frac{1}{3}\Sigma\left[(2bc - a^2)\cdot\frac{1}{a^2}\right] = \frac{1}{3}\left(\Sigma\frac{2bc}{a^2}\right) - 1 \geq \left(\pi\frac{2bc}{a^2}\right)^{1/3} - 1 = 2 - 1 = 1$$

Example 80. *Determine all real numbers satisfying the inequality*

$$\frac{1}{2}\log(2x-1) + \log\sqrt{x-9} > 1.$$

[log *denotes logarithm to base* 10]

Solution $\frac{1}{2}\log(2x-1) + \log\sqrt{x-9} > 1.$

$\Rightarrow$ $$\frac{1}{2}\log(2x-1) + \frac{1}{2}\log(x-9) > 1$$

$\Rightarrow$ $$\log[(2x-1)(x-9)] > 2$$

$\Rightarrow$ $$(2x-1)(x-9) > 10^2$$

$\Rightarrow$ $$2x^2 - 19x - 91 > 0$$

$\Rightarrow$ $$(x-13)(2x+7) > 0$$

$\Rightarrow$ $$\text{Either } x < \frac{-7}{2} \text{ or } x > 13$$

$\because$ x cannot be less than $-7/2$ (as if $x < 9$, then $\log\sqrt{x-9}$ has no meaning)

$\therefore$ x must be > 13

The desired set is $\{x : x > 13 \text{ and } x \in R\}$.

Example 81. *Without evaluating square roots, find the larger of two surds :* $\sqrt{12} - \sqrt{11}, \sqrt{7} - \sqrt{6}$

Solution $\because$ $(\sqrt{12} - \sqrt{11})(\sqrt{12} + \sqrt{11}) = 1$

$$(\sqrt{7} - \sqrt{6})(\sqrt{7} + \sqrt{6}) = 1$$

$\therefore$ $$(\sqrt{12} - \sqrt{11})(\sqrt{12} + \sqrt{11}) = (\sqrt{7} - \sqrt{6})(\sqrt{7} + \sqrt{6})$$

i.e., $$\frac{(\sqrt{12} - \sqrt{11})}{(\sqrt{7} - \sqrt{6})} < \frac{(\sqrt{7} + \sqrt{6})}{(\sqrt{12} + \sqrt{11})} \quad \text{...(i)}$$

$\therefore$ $$\sqrt{7} + \sqrt{6} < 6, \sqrt{12} + \sqrt{11} > 6$$

$\therefore$ $$\frac{\sqrt{7} + \sqrt{6}}{\sqrt{12} + \sqrt{11}} < 6 \cdot \frac{1}{6} = 1$$

Consequently, $$\frac{\sqrt{12} - \sqrt{11}}{\sqrt{7} - \sqrt{6}} < 1$$

so that $$\sqrt{7} - \sqrt{6} > \sqrt{12} - \sqrt{11}$$

Example 82. *Let* $a_1, a_2, \ldots, a_n$ *be real numbers all greater than* 1 *and such that* $|a_k - a_{k+1}| < 1$ *for* $1 \leq k \leq n-1$. *Show that*

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2n - 1.$$

Solution By using mathematical induction, we will prove the result.

(i) We will prove that $\dfrac{a_1}{a_2} + \dfrac{a_2}{a_1} < 3$

Assume $a_1 < a_2$

$\because$ $|a_1 - a_2| < 1$, we have $a_2 - a_1 < 1$

i.e., $a_1 < a_2 < a_1 + 1$

So $$\frac{a_1}{a_2} + \frac{a_2}{a_1} < 1 + \frac{a_1 + 1}{a_1} = 2 + \frac{1}{a_1} < 3 \qquad [\because a_1 > 1]$$

Assume $a_2 < a_1$

$\because$ $|a_1 - a_2| < 1$, we have $a_2 < a_1 < a_2 + 1$

$$\frac{a_1}{a_2} + \frac{a_2}{a_1} < \frac{a_2 + 1}{a_2} + 1 = 2 + \frac{1}{a_2} < 3 \qquad [\because a_2 > 1]$$

Assume $a_1 = a_2$, then $\dfrac{a_1}{a_2} + \dfrac{a_2}{a_1} = 2 < 3$

(ii) Now assume

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_k}{a_1} < 2k - 1$$

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_{k+1}}{a_1}$$

$$= \left(\frac{a_1}{a_2} + \ldots + \frac{a_{k-1}}{a_k} + \frac{a_k}{a_1}\right) + \frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_1} - \frac{a_k}{a_1} < (2k-1) + \left(\frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_1} - \frac{a_k}{a_1}\right)$$

Now it is enough to show that

$$\frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_1} - \frac{a_k}{a_1} \text{ is less than 2.}$$

If $a_k < a_{k+1}$, then $a_{k+1} - a_k < 1$ and we have

$$\frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_1} - \frac{a_k}{a_1} < 1 + \frac{1}{a_1} < 2 \qquad [\because a_1 > 1]$$

If $a_k > a_{k+1}$, then $a_k - a_{k+1} < 1$, we have

$$\frac{a_k}{a_{k+1}} < 2, \frac{a_{k+1}}{a_1} - \frac{a_k}{a_1} < 0$$

If $a_k = a_{k+1}$, then $\left(\dfrac{a_k}{a_{k+1}} + \dfrac{a_{k+1}}{a_1} - \dfrac{a_k}{a_1}\right) = 1$

So in all the cases

$$\frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_1} - \frac{a_k}{a_1} < 2$$

Example 83. *If a, b are +ve real numbers. Prove that*

$$[(a+b)(a^2+b^2)\ldots(a^n+b^n)]^2 > (a^{n+1} + b^{n+1})^n,$$

for every +ve integer n.

Solution $[(a+b)(a^2+b^2)\ldots(a^n+b^n)]^2$

$$= [(a+b)(a^2+b^2)\ldots(a^n+b^n)]$$

$$[(a^n+b^n)(a^{n-1}+b^{n-1})\ldots(a+b)]$$

$$= (a+b)(a^n+b^n)(a^2+b^2)(a^{n-1}+b^{n-1})\ldots(a^n+b^n)(a+b)$$

If k be any +ve integer such that $1 \le k \le n$, then

$$(a^k + b^k)(a^{n-k+1} + b^{n-k+1})$$
$$= a^{n+1} + b^{n+1} + a^k b^{n-k+1} + b^k a^{n-k+1} > a^{n+1} + b^{n+1}$$

Applying this result to each of the n products we find

$$[(a+b)(a^2+b^2)\dots(a^n+b^n)^2]^2$$
$$> (a^{n+1}+b^{n+1})(a^{n+1}+b^{n+1})\dots \text{ to } n \text{ factors}$$
$$= (a^{n+1}+b^{n+1})^n$$

Hence,

$$[(a+b)(a^2+b^2)\dots(a^n+b^n)]^2 > (a^{n+1}+b^{n+1})^n$$

Hence proved.

Example 84. *If $a_1 \ge a_2 \ge \dots \ge a_n$ be real numbers such that $a_1^k + a_2^k + \dots + a_n^k \ge 0$ for all integers $k > 0$ and $p = max\{|a_1|, |a_2|, \dots, |a_n|\}$. Prove that $p = |a_1| = a_1$ and that $(x - a_1)(x - a_2)\dots$*

$$(x - a_n) \le x_n - a_1^n$$

Solution Take $k = 1$

$\because$ $$a_1^k + a_2^k + \dots + a_n^k \ge 0$$

and for $k = 1$, we have

$$a_1 + a_2 + \dots + a_n \ge 0 \qquad \dots\text{(i)}$$

$\because$ $$a_1 \ge a_2 \ge a_3 \ge \dots \ge a_n, a_1 \ge 0 \qquad \dots\text{(ii)}$$

and if all $a_i, i = 1, 2, \dots n$ are +ve

a_1 is the maximum of all a_i's.

$\therefore$ $$p = |a_1| = a_1$$

Suppose some of the a_i's are -ve and $p \ne a$, then $a_n < 0$

Hence, $p = |a_n|$

Let r be an index such that

$$a_n = a_{n-1} = \dots = a_{r+1} < a_r \le a_{r-1} \le \dots \le a_1$$

Then $$a_1^k + a_2^k + \dots + a_{r-1} k + a_r k + \dots + a_n^k$$

$$= a_n^k \left[\left(\frac{a_1}{a_n}\right)^k + \left(\frac{a_2}{a_n}\right)^k + \dots + \left(\frac{a_{r-1}}{a_n}\right)^k + \left(\frac{a_r}{a_n}\right)^k + (n-r) \right]$$

$$= a_n^k X$$

where $$X = \left[\left(\frac{a_1}{a_n}\right)^k + \left(\frac{a_2}{a_n}\right)^k + \dots + \left(\frac{a_{r-1}}{a_n}\right)^k + \left(\frac{a_r}{a_n}\right)^k + (n-r) \right]$$

$\because$ $$\left|\frac{a_1}{a_n}\right|, \left|\frac{a_2}{a_n}\right|, \dots, \left|\frac{a_r}{a_n}\right| < 1$$

So their kth powers < these fractions and by taking k sufficiently large which would make $X > 0$ and $Xa_n^k < 0$ for k odd.

A contradiction and hence $p = a_1$

Let $$x > a_1$$

By AM-GM inequality

$$(x - a_2)(x - a_3)(x - a_4) \dots (x - a_n)$$

$$\le \left[\frac{\sum_{i=2}^{n} (x - a_i)}{n-1}\right]^{n-1} \le \left[\frac{(n-1)x + a_1}{n-1}\right]^{n-1} \qquad \left[\because \sum_{i=1}^{b} a_i \ge 0\right]$$

$$= \left[x + \frac{a_1}{n-1}\right]^{n-1}$$

$$\le x^{n-1} + x^{n-2}a_1 + x^{n-2}a_1^2 + \dots + a_1^{n-1} \qquad \left[\because \binom{n-1}{r} \le (n-1)^r, r \ge 1\right]$$

Multiplying both sides by $(x - a_1)$, we get

$$(x - a_1)(x - a_2)(x - a_3) \dots (x - a_n)$$

$$\le (x - a_1)(x^{n-1} + x^{n-2}a_1 + \dots + a_1^{n-1}) = x^n - a_1^n$$

Example 85. *If x, y, z be +ve real numbers and $xyz = 1$, prove that*

$$\frac{xy}{x^5 + y^5 + xy} + \frac{yz}{y^5 + z^5 + yz} + \frac{xz}{z^5 + x^5 + zx} \le 1,$$

when does equality hold. **(IMO 1996)**

Solution $x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

$$= (x + y)[x^4 + x^2y^2 + y^4 - xy(x^2 + y^2)]$$

$$= (x + y)[(x^2 + xy + y^2)(x^2 - xy + y^2) - xy(x^2 + xy + y^2) + x^2y^2]$$

$$= (x + y)[(x^2 + xy + y^2)(x^2 - 2xy + y^2) + x^2y^2]$$

$$= (x + y)[(x^2 + xy + y^2)(x - y)^2 + x^2y^2]$$

$$\ge (x + y) \times x^2y^2 \qquad [\because (x - y)^2(x^2 + xy + y^2) \ge 0]$$

i.e., $$x^5 + y^5 \ge x^2y^2(x + y)$$

Equality holds if $a = b$

$$\frac{1}{x^5 + y^5} \le \frac{1}{x^2y^2(x + y)}$$

$$\Rightarrow \quad \frac{x^5 + y^5}{xy} \ge xy\,(x + y)$$

$$\Rightarrow \quad \frac{x^5 + y^5 + 1}{xy} \ge xy\,(x + y) + 1$$

$$\Rightarrow \quad \frac{x^5 + y^5 + xy}{xy} \ge xy\,(x + y) + 1$$

$$\Rightarrow \quad \frac{xy}{x^5 + y^5 + xy} \le \frac{1}{1 + xy\,(x + y)}$$

Example 86. *If x, y, z are sides of a triangle and $x + y + z = 2$, show that*

$$x^2 + y^2 + z^2 + 2xyz < 2.$$

(INMO 1993)

Solution It is given $x + y + z = 2$

On squaring both sides of above equation, we get

$$\Rightarrow \quad 4 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = 2(2 - xy - yz - zx)$$

Adding $2xyz$ on both sides, we get

$$x^2 + y^2 + z^2 + 2xyz = 2(2 - xy - yz - zx + xyz)$$

Now, to show that $x^2 + y^2 + z^2 + 2xyz < 2$ it is enough to show that

$$2(2 - xy - yz - zx + xyz) < 2$$

or $$2 + xyz - xy - yz - zx < 1$$

or $$xy + yz + zx - xyz - 1 > 0$$

$\because$ $$x + y + z = 2s = 2$$

So $$s = 1$$

Now, $1(1 - x)(1 - y)(1 - z) > 0$ as the expression in the LHS is square of the area of triangle with sides x, y, z.

$$\Rightarrow \quad 1^3 - (x + y + z)1^2 + (xy + yz + zx)1 - xyz > 0$$

or $$1 - 2 + xy + yz + zx - xyz > 0$$

or $xy + yz + zx - xyz - 1 > 0$ as desired.

Example 87. *Let a, b, c be +ve real numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1$$

Solution Now, $$\left(b - 1 + \frac{1}{c}\right) = b\left(1 - \frac{1}{b} + \frac{1}{bc}\right)$$

$$= b\left(1 + a - \frac{1}{b}\right) \qquad \left[\because a = \frac{1}{bc}\right]$$

$$\Rightarrow \quad \left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right) = b\left[a^2 - \left(1 - \frac{1}{b^2}\right)\right]$$

$$\le ba^2$$

So, the square of product of all three

$$\le ba^2cb^2ac^2 = 1$$

Actually that is not quite true. The last sentence would not follow if we had some -ve LHS, as then we couldn't multiply inequalities.

But it is easy to deal separately with the case where $\left(a - 1 + \frac{1}{b}\right), \left(b - 1 + \frac{1}{c}\right), \left(c - 1 + \frac{1}{a}\right)$ are not all +ve. If one of the three terms is -ve then other two must be +ve. For example

if $a - 1 + \frac{1}{b} < 0$, then $a < 1$ so $c - 1 + \frac{1}{a} > 0$

$$b > 1 \text{ so } b - 1 + \frac{1}{c} > 0$$

But if one term is -ve and two are +ve, then their product is -ve and hence less than 1.

Example 88. *Prove that*

$$\frac{1}{k+1}n^{k+1} < 1^k + 2^k + 3^k + .. + n^k < \left(1+\frac{1}{n}\right)^{k+1}\frac{1}{k+1}n^{k+1}$$

(n and k are arbitrary integers).

Solution Now, in

$$S = x^k + x^{k-1} + x^{k-2} + \ldots + x + 1$$

if $x > 1$, then the first term is numerically the greatest but if $x < 1$, then the last term is greatest. It follows that

$$(k+1)x^k > s > k+1, \text{ if } x > 1$$

$$(k+1)x^k < s < k+1, \text{ if } x < 1$$

If both sides of these inequalities are multiplied by $x - 1$, it is found that for $x \neq 1$

$$(k+1)x^k(x-1) > x^{k+1} - 1 > (k+1)(x-1)$$

Assume that $x = \frac{p}{(p-1)}$, then we find

$$\frac{(k+1)p^k}{(p-1)^{k+1}} > \frac{p^{k+1} - (p-1)^{k+1}}{(p-1)^{k+1}} > \frac{(k+1)(p-1)^k}{(p-1)^{k+1}}$$

If we assume $x = \frac{p+1}{p}$, we get

$$\frac{(k+1)(p+1)^k}{p^{k+1}} > \frac{(p+1)^{k+1} - p^{k+1}}{p^{k+1}} > \frac{(k+1)p^k}{p^{k+1}}$$

If follows that

$$(p+1)^{k+1} - p^{k+1} > (k+1)p^k > p^{k+1} - (p-1)^{k+1}$$

Letting p successively have the values 1, 2, 3, ... , n.

$$2^{k+1} - 1^{k+1} > (k+1)1^k > 1^{k+1} - 0$$

$$3^{k+1} - 2^{k+1} > (k+1)2^k > 2^{k+1} - 1^{k+1}$$

$$4^{k+1} - 3^{k+1} > (k+1)3^k > 3^{k+1} - 2^{k+1}$$

....

....

....

$$(n+1)^{k+1} - n^{k+1} > (k+1)n^k > n^{k+1} - (n-1)^{k+1}$$

If these inequalities are added together, we get

$$(n+1)^{k+1} - 1 > (k+1)(1^k + 2^k + 3^k + \ldots + n^k) > n^{k+1}$$

Dividing through these inequalities by $k + 1$

$$\left[\left(1+\frac{1}{n}\right)^{k+1} - \frac{1}{n^{k+1}}\right]\frac{1}{k+1}n^{k+1} > 1^k + 2^k + 3^k + \ldots + n^k > \frac{1}{k+1}n^{k+1}$$

Example 89. *Show that*

$$\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}\le \sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}$$

for all +ve real numbers a and b.

Determine when equality occurs. **(IMO)**

Solution Multiplying both sides of desired inequality by $\sqrt[3]{ab}$ gives

$$\sqrt[3]{a^2}+\sqrt[3]{b^2}\le\sqrt[3]{2(a+b)^2}$$

Put $\sqrt[3]{a}=x$ and $\sqrt[3]{b}=y$ we find that it is sufficient to prove

$$x^2+y^2\le\sqrt[3]{2(x^3+y^3)^2} \text{ for } x,y>0$$

By AM-GM inequality, we have

$$\frac{x^6+x^3y^3+x^3y^3}{3}\ge(x^{12}y^6)^{1/3}$$

or $$3x^4y^2\le x^6+x^3y^3+x^3y^3$$

and $$3x^2y^4\le y^6+x^3y^3+x^3y^3$$

with equality if and only if $x^6=x^3y^3=y^6$ or equivalently if and only if $x=y$. Adding these inequalities and adding x^6+y^6 to both sides yields

$$x^6+y^6+3x^2y^2(x^2+y^2)\le 2(x^6+y^6+2x^3y^3)$$

Taking cube roots of both sides yields.

Equality occurs when $x=y$ or $a=b$.

Example 90. *Prove that*

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\ge\frac{(a+b+c)^3}{3(x+y+z)}$$

a, b, c, x, y, z are +ve real numbers.

Solution By Holder's inequality

$$\prod_{i=1}^{3}(p_i^3+q_i^3+r_i^3)^{1/3}\ge p_1p_2p_3+q_1q_2q_3+r_1r_2r_3$$

for all +ve reals p_i, q_i, r_i

Hence, $$\left(\frac{a^{1/3}}{x}+\frac{b^3}{y}+\frac{c^3}{z}\right)^{1/3}(1+1+1)^{1/3}$$

$$(x+y+z)^{1/3}>a+b+c$$

Cubing both sides and then dividing both sides by $3(x+y+z)$, we get

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\ge\frac{(a+b+c)^3}{3(x+y+z)}$$

Example 91. *Let m and n be +ve integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i+a_j\le n$ for some i, j (possibly the same) we have $a_i+a_j=a_k$ for some k. Prove that* $\frac{a_1+a_2+\ldots+a_m}{m}\ge\frac{n+1}{2}$.

Solution $$A=\{a_1, a_2, \ldots, a_n\}$$

Assume, $$a_1>a_2>\ldots>a_m$$

Assign to any a_i element of A, the element a_{m-i+1} as its pair. This matching is symmetric, the pair of a_{m-i+1} is a_i and if m is odd, then median $a_{\frac{m+1}{2}}$ is equal to its own pair.

First we show that sum of any pair is at least $n+1$. If to the contrary $a_i + a_{m-i+1} < n+1$ for some $(1 \le i \le m)$ then ordering of A gives

$$a_i < a_i + a_m < a_i + a_{m-1} < \ldots < a_i + a_{m-i+1} \le n \quad \ldots(i)$$

Hence, by condition the i sums in (i) produce elements of A. All of them are greater than a_i a contradiction since there are only $i-1$ elements in A beyond a_i.

$\therefore$

$$a_1 + a_m \ge n+1$$
$$a_2 + a_{m-1} \ge n+1$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$a_m + a_1 \ge n+1$$

Adding these, we get

$$2(a_1 + a_2 + \ldots + a_m) \ge m(n+1)$$

Example 92. *Let $\{a_k\}$ be a sequence of distinct +ve integers $(k = 1, 2, \ldots, n, \ldots)$. Prove that for every +ve integer n, $\sum_{k=1}^{n} \frac{a_k}{k^2} \ge \sum_{k=1}^{n} \frac{1}{k}$.* (INMO 1978)

Solution Let $x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n$ be real numbers

where $\quad x_1 \ge x_2 \ge \ldots \ge x_n, y_1 \le y_2 \le \ldots \le y_n$

Moreover $z_1, z_2, \ldots, z_n$ *a* permutations of the y_i's, then

$$\sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{k} x_i z_i \quad \ldots(i)$$

In (i) equality holds if the order of z_i's agrees with the order of the y_i's.

If to the contrary, we assume that they first differ at the kth place *i.e.*, $y_1 = z_1, y_2 = z_2, \ldots, y_{k-1} = z_{k-1}$

But $\quad y_k \ne z_k$

Let $\quad z_k = y_r$ and $y_k = z_s$

where r and s are greater than k, $r > k$ and $s > k$ using this notation $x_k z_k + x_s z_s = x_k y_r + x_s y_k$.

Now, change the LHS by substituting

$x_k z_k + x_s z_s = x_k y_r + x_s y_k$ by $x_k y_k + x_s y_r$. This way we increased the sum on the left.

$$\because \quad (x_k y_k + x_s y_r) - (x_k y_r + x_s y_k) = (x_k - x_s)(y_k - y_r) \ge 0$$

This implies that the sum $\Sigma x_i y_i$, where $x_i - s$ are in decreasing order is the smallest if $y_i - s$ are in increasing order. There is a minimal among the above sums as there are finitely many of them and that can only occur when the $y_i - s$ are in increasing order otherwise we could decrease the sum using our arguments.

So, for a given n let $a_{i1} \le a_{i2} \le \ldots \le a_{in}$ be the first k elements in increasing order.

But $$\frac{1}{1^2} > \frac{1}{2^2} > \frac{1}{3^2} > \ldots > \frac{1}{n^2}$$

and hence by (i)

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \ge \sum_{k=1}^{n} \frac{a_{ik}}{k^2} \quad \ldots(ii)$$

$a_{i1} \ge 1, a_{i2} \ge 2, \ldots, a_{ik} \ge k, \ldots, a_{in} \ge n$ and so (3) implies that

$$\sum_{k=1}^{n} \frac{a_k}{2} \ge \sum_{k=1}^{n} \frac{k}{k^2} = \sum_{k=1}^{n} \frac{1}{k}$$

Example 93. *Consider the infinite non-increasing sequence $\{x_i\}$ of +ve reals such that $x_0 = 1$. Prove that for every such sequence there is an $n \geq 1$ such that*

$$S_n = \frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3999.$$

Also find such a sequence for which

$$S_n = \frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4. \qquad \textbf{(INMO 1982)}$$

Solution It is given

$$S_n = \frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \leq 3999 \qquad \ldots(i)$$

$\because$ $(x_i - 2x_{i+1})^2 \geq 0\,; x_i^2 \geq 4x_i x_{i+1} - 4x_{i+1}^2$

follows and so $\dfrac{x_i^2}{x_{i+1}} \geq 4(x_i - x_{i+1})$

Applying this to sum in (i)

$$S_n = \frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 4(x_0 - x_1 + x_1 - x_2 + \ldots + x_{n-1} - x_n)$$

$$= 4(1 - x_n)$$

If the limit of the sequence x_n is zero, then there exists an n such that $x_n \leq \dfrac{1}{4000}$ and for this n,

$$S_n \geq 4\left(1 - \frac{1}{4000}\right) = 3999$$

Now, let the limit c be different from $0, c > b > 0$ so every element of the sequence is greater than b. Let $n \geq 4/b$, then

$$S_n = \sum_{i=0}^{n} \frac{x_i^2}{x_{i+1}} > \sum_{i=0}^{n} \frac{x_i x_{i+1}}{x_{i+1}} = \sum_{i=0}^{n} x_i > nb \geq 4$$

Now, let $x_i = 2^{-i} (i = 0, 1, \ldots)$. Now

$$\frac{x_i^2}{x_{i+1}} = \frac{2^{-2i}}{2^{-(i+1)}} = 2^{1-i}$$

and so $$S_n = 2 + 1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}$$

Extending it to an infinite geometric sequence the sum is 4.
Hence, the finite sum S_n never reaches 4.

Example 94. *If a, b, c be the length of the sides of a triangle. Prove that*

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0. \qquad \textbf{(INMO 1983)}$$

Solution Let

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) = 0 \qquad \ldots(i)$$

Let $x = -a + b + c\,; y = a - b + c$

and $z = a + b - c$

Thus, x, y, z are twice the length of the segments between the vertices and the touching point of the incircles. So

$$a = \frac{y+z}{2}, b = \frac{z+x}{2}, c = \frac{x+y}{2}$$

Substitute them to (i) and multiply the inequality by 16.

$$(y+z)^2(z+x)(y-x)+(z+x)^2(x+y)(z-y)+(x+y)^2(y+z)(x-z) \ge 0$$

$\Rightarrow$ $$x^3z + y^3x + z^2y \ge x^2yz + y^2zx + z^2xy$$

and so $$x^3z + y^3x + z^3y - x^2yz - y^2zx - z^2xy$$

$$= zx(x-y)^2 + xy(y-z)^2 + yz(z-x)^2 \ge 0$$

As $x > 0, y > 0, z > 0$, equality holds if and only if $x = y = z$.

i.e., if $a = b = c$, then triangle is equilateral.

Example 95. *Prove that* $0 \le xy + yz + zx - 2xyz \le \frac{7}{27}$, *where* x, y, z *are non-negative real numbers for which* $x + y + z = 1$. **(INMO 1984)**

Solution We have to prove

$$0 \le xy + yz + zx - 2xyz \le \frac{7}{27} \quad \text{...(i)}$$

$$x + y + z = 1 \quad \text{...(ii)}$$

To prove LHS of inequality observe that (ii) implies

$0 \le x, y, z \le 1$ and so

$$xy + yz + zx - 2xyz = xy(1-z) + yz(1-x) + zx \ge 0$$

($\because$ All terms of LHS are non -ve.)

In order to show the RHS of inequality.

Let $$x = a + \frac{1}{3}, y = b + \frac{1}{3}, z = c + \frac{1}{3}$$

Now, (ii) gives

$$a + b + c = 0 \quad \text{...(iii)}$$

and $$-\frac{1}{3} \le a, b, c \le \frac{2}{3}$$

After substituting $b + c = -a$, we get

$$xy + yz + zx - 2xyz = \frac{7}{27} + \frac{1}{3}$$

$$(ab + bc + ca - 6abc) = \frac{7}{27} + \frac{1}{3}(bc - a^2 - 6abc) \quad \text{...(iv)}$$

Our original expression is symmetric in a, b, c.

Hence we may assume that $a \le b \le c$.

According to (iii), a, b, c cannot be all +ve or -ve at the same time so there are 2 possibilities.

(a) $a \le b \le 0 \le c$ or (b) $-\frac{1}{3} \le a \le 0 \le b \le c$

Now, $bc - a^2 - 6abc$ is not +ve. In case (a) every expression is +ve, so 2 holds.

Now to prove (b), let us arrange (iv)

$$bc - a^2 - 6abc = bc - (b+c)^2 - 6abc$$

$$= -(b-c)^2 - 3bc(1+2a)$$

Now, as $1 + 2a > 0$ and $bc \ge 0$, the expression cannot be +ve, so (ii) holds.

Aliter

(ii) implies that at least one of x, y, z, let us say x is not greater than 1/3. Thus, $1 - 2x > 0$ and so

$$xy + yz + zx - 2xyz = xy + yz(1 - 2x) + zx > 0$$

Hence we proved LHS of (i)

To prove RHS. We consider 2 cases.

(a) One of x, y, z, let us say z is at least 1/2; $z \geq 1/2$

(b) $0 \leq x, y, z \leq 1/2$

In (a) case $x + y = 1 - z$ we get

$$\begin{aligned} xy + yz + zx - 2xyz &= z(x + y) + xy(1 - 2z) \\ &= z(1 - z) + xy(1 - 2z) \\ &\leq z(1 - z) \leq \frac{1}{4} \leq \frac{7}{27} \end{aligned}$$

In (b) case, apply the following substitution

$$a = 1 - 2x, b = 1 - 2y, x = 1 - 2z$$

These values gives $a, b, c \geq 0, a + b + c = 1$ ...(v)

Substituting and applying (v), we get

$$xy + yz + zx - 2xyz = \frac{1 + abc}{4} \qquad \text{...(vi)}$$

Now, AM-GM inequality gives

$$abc \leq \left(\frac{a + b + c}{3}\right)^3 = \frac{1}{27}$$

Hence, (vi) implies

$$xy + yz + zx - 2xyz \leq \frac{\frac{28}{27}}{4} = \frac{7}{27}$$

So, we proved (i) in every case.

Aliter

The problem involves the so called elementary symmetric polynomials of x, y, z that lead us the following idea. Consider the polynomial $f(t)$ of degree (3)

$$f(t) = (t - x)(t - y)(t - z) = t^3 - t^2(x + y + z) + t(xy + yz + zx) - xyz \qquad \text{...(vii)}$$

From (ii), we have

$$2f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + (xy + yz + zx) - 2xyz$$

$$xy + yz + zx - 2xyz = 2f\left(\frac{1}{2}\right) + \frac{1}{4}$$

Now,

$$0 \leq 2f\left(\frac{1}{2}\right) + \frac{1}{4} \leq \frac{7}{27}$$

i.e.,

$$-\frac{1}{8} \leq f\left(\frac{1}{2}\right) \leq \frac{1}{216} \qquad \text{...(viii)}$$

Assume $\qquad x \geq y \geq z \geq 0$

Choose x, y, z such that $x \geq 1/2, y \leq 1/2, z \leq 1/2$ holds.

(vii) implies that $f\left(\frac{1}{2}\right) \leq 0$.

Hence,

$$-f\left(\frac{1}{2}\right) = \left(x - \frac{1}{2}\right)\left(\frac{1}{2} - y\right)\left(\frac{1}{2} - z\right)$$

Applying AM-GM inequality and using $x \le 1$

$$-f\left(\frac{1}{2}\right) \le \left(\frac{x-y-z+\frac{1}{2}}{3}\right)^3 = \left(\frac{2x-\frac{1}{2}}{3}\right)^3 \le \frac{1}{8}$$

$$\Rightarrow \qquad f\left(\frac{1}{2}\right) \ge -\frac{1}{8}$$

If we choose the values of the variables such that $x \le 1/2$ and so $y \le 1/2$, $z \le 1/2$ holds.

The previous method gives

$$0 \le f\left(\frac{1}{2}\right) \le \left(\frac{1}{2}-x\right)\left(\frac{1}{2}-y\right)\left(\frac{1}{2}-z\right) \le \left[\frac{\frac{3}{2}-(x+y+z)}{3}\right]^3$$

$$= \frac{1}{216}$$

Hence proved.

Example 96. *Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Prove that for every integer $k \ge 2$ there are integers a_i ($i = 1, 2, \ldots, n$) not all zero such that $|a_i| \le k-1$ for all i and*

$$|a_1x_1 + a_2x_2 + \ldots + a_nx_n| \le \frac{(k-1)\sqrt{n}}{k^n - 1}$$ **(INMO 1987)**

Solution

$$x_1^2 + x_2^2 + \ldots + x_n^2 = 1 \qquad \ldots\text{(i)}$$

$$|a_1x_1 + a_2x_2 + \ldots + a_nx_n| \le \frac{(k-1)\sqrt{n}}{k^n - 1} \qquad \ldots\text{(ii)}$$

Applying AM-GM inequality for (i), yields

$$\frac{1}{\sqrt{n}} = \frac{\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}}{n} \ge \frac{|x_1| + |x_2| + \ldots + |x_n|}{n}$$

i.e.,

$$|x_1| + |x_2| + \ldots + |x_n| \le \sqrt{n}$$

Consider, now the $k^n - 1$ sequence of length n, none of them is constantly zero that are composed from the numbers $(0, 1, 2, \ldots, k-1)$.

Let one of them be $(a_1, a_2, \ldots, a_n)$. Set now the sign of a_i in such a way that the product a_ix_i is not -ve ($i = 1, 2, \ldots, n$), clearly

$$a_1x_1 + \ldots + a_nx_n = |a_1||x_1| + |a_2||x_2| + \ldots + |a_n||x_k|$$
$$\le (k-1)(|x_1| + |x_2| + \ldots + |x_n|)$$
$$\le (k-1)\sqrt{n} \qquad \ldots\text{(iii)}$$

Split the interval $[0, (k-1)\sqrt{n}]$ into $k^n - 1$ equal parts the length of each part is $\frac{(k-1)\sqrt{n}}{k^n - 1}$. If any one of the sums above is inside the first interval, then (ii) clearly holds for this sum and we are done. If there is no such a sum, then there are at least two of them in some of the remaining $k^n - 2$ intervals. If these sums are

$$b_1x_1 + b_2x_2 + \ldots + b_nx_n$$

and

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad [|b_i|, |c_i| \le k-1]$$

Then obviously their difference cannot exceed the length of the interval

$$|b_1x_1 + b_2x_2 + \dots + b_nx_n - c_1x_1 - c_2x_2 - c_nx_n|$$
$$= |(b_1 - c_1)x_1 + (b_2 - c_2)x_2 + \dots + (b_n - c_n)x_n|$$
$$\le \frac{(k-1)\sqrt{n}}{k^n - 1}$$

$\because$ b_i and c_i are of same sign, $|b_i - c_i| \le k - 1$

Clearly, $a_i = b_i - c_i$ are integer of (ii).

Example 97. *Let ABC be a triangle with sides a, b, c. Consider a $\Delta A_1B_1C_1$ with sides equal to $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$. Show that*

$$[A_1B_1C_1] \ge 9/4[ABC]$$

where $[XYZ]$ denotes area of ΔXYZ. **(INMO 2003)**

Solution It is readily seen that $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$ can serve as the sides of a Δ. Heron's formula gives

$$16[ABC]^2 = (a + b + c)(a + b - c)(b + c - a)(c + a - b)$$

$$16[A_1B_1C_1]^2 = \frac{3}{16}(a + b + c)(-a + b + 3c)(-b + c + 3a)(-c + a + 3b)$$

Now, a, b, c are the sides of a triangle, we may assume that

$$a = q + r; b = r + p; c = p + q$$

p, q, r are +ve real numbers.

So, we have

$$\frac{[ABC]^2}{[A_1B_1C_1]^2} = \frac{16pqr}{3(2p + q)(2q + r)(2r + p)}$$

$\therefore$ It is enough to prove that

$$(2p + q)(2q + r)(2r + p) \ge 27pqr$$

for +ve real numbers p, q, r.

Now, AM-GM inequality gives

$$2p + q \ge 3(p^2q)^{1/3},$$
$$2q + r \ge 3(q^2r)^{1/3}$$
and
$$2r + p \ge 3(r^2p)^{1/3}$$

If we multiply all these inequalities, we get

$$(2q + r)(2r + p)(2p + q) \ge 27pqr$$

Equality holds if $p = q = r$

i.e., when Δ *ABC* is equilateral.

Example 98. *The diagonal connecting two opposite vertices of a rectangular parallelopiped is $\sqrt{73}$. Prove that if the square of the edges of the parallelopiped are integers, its volume does not exceed 120.*

Solution Let a, b, c be the lengths of the edges of the parallelopiped.

We have $$a^2 + b^2 + c^2 = 73$$

By AM-GM inequality,

$$a^2b^2c^2 \le \left(\frac{a^2 + b^2 + c^2}{3}\right)^3 = \left(\frac{73}{3}\right)^3$$

$\because$ a^2, b^2, c^2 are integers.

$$\therefore \qquad a^2b^2c^2 \le \left[\left(\frac{73}{3}\right)^3\right] = 14408$$

$$\Rightarrow \qquad a^2b^2[73 - (a^2 + b^2)] \le 14408$$

We need the largest integer N such that $N \le 14408$.
N has 3 factors whose sum is 73.
14401 and 14407 are primes.

$$14402 = 7201 \times 2 \Rightarrow 7201 \text{ is a prime.}$$
$$14403 = 4801 \times 3 \Rightarrow 4801 \text{ is a prime.}$$
$$14404 = 4 \times 13 \times 277$$
$$14405 = 43 \times 67 \times 5$$
$$14406 = 2 \times 3 \times 7^4$$
$$14408 = 1801 \times 8$$

None of these numbers have 3 factors with sum = 73

Now, $\qquad 14400 = 24 \times 24 \times 25$

and $\qquad 24 + 24 + 25 = 73$

$\therefore$ Maximum volume is $\sqrt{14400} = 120$

Example 99. *Let x_1, x_2 be the roots of the equation $x^2 + px - \dfrac{1}{2p^2} = 0$, where x is unknown and p is a real parameter. Prove that $x_1^4 + x_2^4 \ge 2 + \sqrt{2}$.*

Solution We have $x_1 + x_2 = -p$

$$x_1x_2 = -\frac{1}{2p^2}$$

Thus,
$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = p^2 + \frac{1}{p^2}$$
$$x_1^4 + x_2^4 = (x_1^2 + x_2^2)^2 - 2x_1^2x_2^2$$
$$= \left(p^2 + \frac{1}{p^2}\right)^2 - \frac{1}{2p^4} = p^4 + \frac{1}{2p^4} + 2$$

For $\alpha \ge 0, \beta \ge 0$, we have $\alpha + \beta \ge 2\sqrt{\alpha\beta}$

So
$$x_1^4 + x_2^4 = p^4 + \frac{1}{2p^4} + 2 \ge 2\sqrt{p^4\left(\frac{1}{2p^4}\right)} + 2 = \sqrt{2} + 2$$

Hence proved.

Example 100. *Prove that for every three non -ve numbers a, b, c the inequality $a^3 + b^3 + c^3 + 6abc \ge \dfrac{1}{4}(a + b + c)^3$ holds and that equality occurs only when two of these numbers are equal and the third is zero.*

Solution Without loss of generality, assume $a \ge b \ge c$

$$a^3 + b^3 + c^3 + 6abc \ge \frac{1}{4}(a + b + c)^3$$

$$\Leftrightarrow \qquad 4(a^3 + b^3 + c^3 + 6abc) \ge (a^3 + b^3 + c^3) + 3ab(a + b) + 3bc(b + c) + 3ca(c + a) + 6abc$$

$\Leftrightarrow \quad (a^3 + b^3 + c^3 + 6abc) \geq ab(a+b) + bc(b+c) + ca(c+a)$

We have $\quad a - c \geq b - c$

$\Rightarrow \quad a(a-c) \geq b(b-c)$

$\Rightarrow \quad a(a-c)(a-b) \geq b(b-c)(a-b)$

$\Rightarrow \quad a(a-b)(a-c) + b(b-c)(b-a) \geq 0$

Also, $\quad c - a \leq 0$ and $c - b \leq 0$

and $\quad c(c-a)(c-b) \geq 0$

Adding these, we get

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0$$

Expanding LHS, we get

$$a^3 + b^3 + c^3 - ab(a+b) - bc(b+c) - ca(c+a) + 3abc \geq 0$$

Thus, $\quad ab(a+b) + bc(b+c) + ca(c+a)$

$$\leq a^3 + b^3 + c^3 + 3abc \leq a^3 + b^3 + c^3 + 6abc$$

Equality holds if and only if

$$a(a-b)(a-c) + (b-c)b(b-a) = 0$$

and $\quad c(c-a)(c-b) = 0$

Second equality holds if either $c = 0$ or $c - a = 0$

or $\quad c - b = 0$

If $c = 0$, from first equality, we get

$$a^2(a-b) + b^2(b-a) = 0$$

$\Rightarrow \quad (a-b)(a^2+b^2) = 0$

$\Rightarrow \quad a = b$

If $c - a = 0 \because a \geq b \geq c$ it follows that $a = b = c$

Substituting in the equality we see that $a = b = c = 0$

If $c = b$ then $a(a-b)(a-b) = 0 \Rightarrow a = 0$ or $a = b$

If $a = b$ then $a = b = c = 0$.

This completes the proof.

Example 101. *Find the largest integer n for which $n^{200} < 5^{300}$.*

Solution $\quad n^{200} < 5^{300}$

$\Rightarrow \quad n^{200} < \left(5^{\frac{3}{2}}\right)^{200}$

$\Rightarrow \quad n < 5^{3/2} = (\sqrt{5})^3 = 5\sqrt{5}$

To find n such that $n < 5\sqrt{5} \leq n + 1$

$\Leftrightarrow \quad n^2 \leq 125 \leq (n+1)^2$

By inspection we get 11.

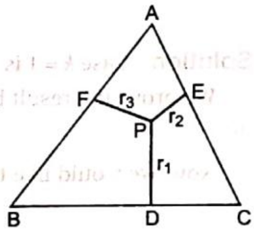
$\therefore$ Answer is 11.

Example 102. *Let P be a point inside a ΔABC. Let r_1, r_2, r_3 denote the distances from P to the sides BC, CA, AB respectively. If R is the circumradius of ΔABC, show that*

$$\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} < \frac{1}{\sqrt{2}R}(a^2 + b^2 + c^2)^{1/2}$$

where BC = a, CA = b and AB = C.

Solution $$\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} = \sqrt{ar_1}\cdot\frac{1}{\sqrt{a}} + \sqrt{br_2}\cdot\frac{1}{\sqrt{b}} + \sqrt{cr_3}\cdot\frac{1}{\sqrt{c}}$$



Applying Cauchy-Schwartz inequality for the sets

$$\{\sqrt{ar_1}, \sqrt{br_2}, \sqrt{cr_3}\} \text{ and } \left\{\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right\},$$

we have $\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}$

$$\leq (ar_1 + br_2 + cr_3)^{1/2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{1/2} \quad \text{...(i)}$$

Now, $$ar_1 + br_2 + cr_3 = 2(\Delta PBC + \Delta PCA + \Delta PAB)$$

$$= 2\Delta ABC = \frac{abc}{2R} \qquad \left(\text{for } \Delta ABC \text{ Area} = \frac{abc}{4R}\right) \quad \text{...(ii)}$$

Now, (ii) reduces (i) to

$$\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} \leq \left(\frac{abc}{2R}\right)^{1/2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{1/2} \quad \text{...(iii)}$$

Again by Cauchy-Schwartz inequality, we get

$$(ab + bc + ca) \leq (a^2 + b^2 + c^2)^{1/2}$$

$$(b^2 + c^2 + a^2)^{1/2} = a^2 + b^2 + c^2 \quad \text{...(iv)}$$

Applying (iv) to (iii)

$$\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} \leq \left(\frac{abc}{2R}\right)^{1/2}\left(\frac{bc + ca + ab}{abc}\right)^{1/2}$$

$$\leq \left(\frac{abc}{2R}\right)^{1/2}\left(\frac{a^2 + b^2 + c^2}{abc}\right)^{1/2}$$

$$\therefore \quad \sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} \leq \frac{1}{\sqrt{2R}}(a^2 + b^2 + c^2)^{1/2}$$

Example 103. *Let a, b, c be +ve real numbers such that $a + b + c \geq abc$. Prove that*

$$a^2 + b^2 + c^2 \geq \sqrt{3}\, abc.$$

Solution Assume that $a^2 + b^2 + c^2 < \sqrt{3}\, abc$

By AM-GM inequality

$$3\sqrt[3]{a^2b^2c^2} \leq a^2 + b^2 + c^2 < \sqrt{3}\, abc$$

It follows that $$abc > 3\sqrt{3} \quad \text{...(i)}$$

Also, by given condition on a, b, c

$$\frac{a^2b^2c^2}{3} \le \frac{(a+b+c)^2}{3} \le a^2+b^2+c^2 < \sqrt{3}\,abc$$

$\Rightarrow$ $$abc < 3\sqrt{3} \quad \text{...(ii)}$$

A contradiction to (i)

$\therefore$ $$a^2+b^2+c^2 \ge \sqrt{3}\,abc.$$

Example 104. *Let k and $n_1 < n_2 < \ldots < n_k$ be odd +ve integers, show that*

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + \ldots + n_k^2 \ge 2k^2 - 1.$$

Solution Case $k = 1$ is clearly true.

We prove the result by induction. Assume that

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + \ldots + n_k^2 \ge 2k^2 - 1 \quad \text{...(i)}$$

Now, we would like to prove

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + \ldots + n_k^2$$
$$n_{k+1}^2 + n_{k+2}^2 \ge 2(k+2)^2 - 1 = 2k^2 - 1 + 8k + 8$$

It is enough to prove that

$$n_{k+2}^2 - n_{k+1}^2 \ge 8k + 8 \quad \text{...(ii)}$$

Now, $\quad (n_{k+2} - n_{k+1}) \ge (n_{k+2} + n_{k+1})$

and $\quad n_k \ge 2k - 1$

$\therefore$ We get $\quad (n_{k+2} - n_{k+1})(n_{k+2} + n_{k+1})$

$$= n_{k+2}^2 - n_{k+1}^2 \ge 2[(2k+1) + (2k+3)] = 8k + 8$$

This proves (ii) and hence result is true for $k + 1$.

Example 105. *If a, b be +ve numbers and p, q be rational numbers ($p > 1$) such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Solution $\because$ $$p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1$$

$\therefore$ $$q > 1$$

Let $$p = \frac{\alpha}{\beta},\ q = \frac{\alpha'}{\beta'}$$

where $\alpha, \beta, \alpha', \beta'$ are +ve integers.

Then $$\frac{a^p}{p} + \frac{b^q}{q} = \frac{a^{\alpha/\beta}}{(\alpha/\beta)} + \frac{b^{\alpha'/\beta'}}{(\alpha'/\beta')}$$

$$= \frac{\beta a^{\alpha/\beta}}{\alpha} + \frac{\beta' b^{\alpha'/\beta'}}{\alpha'}$$

$$= \frac{\alpha'\beta a^{\alpha/\beta} + \alpha\beta' b^{\alpha'/\beta'}}{\alpha\alpha'} \ge (a^{\alpha\alpha'} b^{\alpha\alpha'})^{1/\alpha\alpha'} > ab$$

$\therefore$ $$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

Example 106. *If $a+b+c=n$, where a, b, c are +ve integers. Prove that*

$$(a^a \cdot b^b \cdot c^c)^{\frac{1}{n}} + (a^b \cdot b^c \cdot c^a)^{\frac{1}{n}} + (a^c \cdot b^a \cdot c^b)^{\frac{1}{n}} \leq n.$$

Solution Let $a^a = (a \cdot a \ldots a \text{ times})$,

$$b^b = (b \cdot b \ldots b \text{ times})$$

and $$c^c = (c \cdot c \ldots c \text{ times})$$

Now, using AM-GM inequality

$$\frac{(a + a \ldots a \text{ times}) + (b + b \ldots b \text{ times}) + (c \cdot c \ldots c \text{ times})}{a+b+c} \geq (a^a \cdot b^b \cdot c^c)^{\frac{1}{a+b+c}}$$

But $$a+b+c=n$$

$\therefore$ $$\frac{a^2+b^2+c^2}{n} \geq (a^a b^b c^c)^{\frac{1}{n}} \quad \ldots\text{(i)}$$

Similarly, $a^b = (a \cdot a \cdot a \ldots b \text{ times})$, $b^c = (b \cdot b \cdot b \ldots c \text{ times})$

and $$c^a = (c \cdot c \cdot c \ldots a \text{ times})$$

Again, using AM-GM inequality

$$\frac{(a + a \ldots b \text{ times}) + (b + b \ldots c \text{ times}) + (c \cdot c \cdot c \ldots a \text{ times})}{b+c+a} \geq (a^b \cdot b^c \cdot c^a)^{\frac{1}{a+b+c}}$$

$$\frac{(ab+bc+ca)}{n} \geq (a^b \cdot b^c \cdot c^a)^{\frac{1}{n}} \quad \ldots\text{(ii)}$$

Similarly, we prove

$$\frac{ab+bc+ca}{n} \geq (a^c \cdot b^a \cdot c^b)^{\frac{1}{n}} \quad \ldots\text{(iii)}$$

Adding (i), (ii) and (iii), we get

$$\frac{(a^2+b^2+c^2+2ab+2bc+2ca)}{n} \geq (a^a \cdot b^b \cdot c^c)^{\frac{1}{n}} + (a^b \cdot b^c \cdot c^a)^{\frac{1}{n}} + (a^c \cdot b^a \cdot c^b)^{\frac{1}{n}}$$

$\Rightarrow$ $$\frac{(a+b+c)^2}{n} \geq (a^a \cdot b^b \cdot c^c)^{\frac{1}{n}} + (a^b \cdot b^c \cdot c^a)^{\frac{1}{n}} + (a^c \cdot b^a \cdot c^b)^{\frac{1}{n}}$$

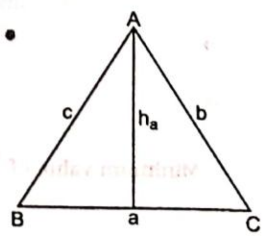
$\Rightarrow$ $$\frac{n^2}{n} \geq (a^a \cdot b^b \cdot c^c)^{\frac{1}{n}} + (a^b \cdot b^c \cdot c^a)^{\frac{1}{n}} + (a^c \cdot b^a \cdot c^b)^{\frac{1}{n}}$$

or $$(a^a \cdot b^b \cdot c^c)^{\frac{1}{n}} + (a^b \cdot b^c \cdot c^a)^{\frac{1}{n}} + (a^c \cdot b^a \cdot c^b)^{\frac{1}{n}} \leq n$$

Hence proved.

Example 107. *A triangle has sides of length a, b, c and altitudes of length h_a, h_b, h_c respectively. If $a \geq b \geq c$, show that $a + h_a \geq b + h_b \geq c + h_c$.*

Solution Area of $\Delta = \frac{1}{2} a \cdot h_a = \frac{1}{2} b \cdot h_b = \frac{1}{2} c \cdot h_c$



Now, $$h_a = \frac{2\Delta}{a}$$

$$h_b = \frac{2\Delta}{b}$$

$$h_c = \frac{2\Delta}{c}$$

Let us assume

$$a + h_a \geq b + h_b$$
$$a - b \geq h_b - h_a$$
$$a - b \geq \frac{2\Delta}{b} - \frac{2\Delta}{a}$$
$$a - b \geq 2\Delta\left(\frac{1}{b} - \frac{1}{a}\right)$$
$$a - b \geq 2\Delta\left(\frac{a-b}{ab}\right)$$
$$(a-b)\left(1 - \frac{2\Delta}{ab}\right) \geq 0$$
$$(a-b)(ab - 2\Delta) \geq 0$$

Area of $\Delta = \frac{1}{2} ab \sin C$

$$2\Delta = ab \sin C$$
$$2\Delta = ab \sin C \leq ab \quad \therefore \quad 2\Delta \leq ab$$
$$ab - 2\Delta \geq 0 \quad \therefore \quad (a-b) \geq 0$$

which is true as it is given $a \geq b$

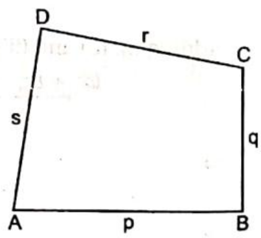
Similarly, we can prove other.

Example 108. *If p, q, r, s are the sides of a quadrilateral. Find the minimum value of*

$$\frac{p^2 + q^2 + r^2}{s^2}.$$

Solution We have

$$AB = p$$
$$BC = q$$
$$CD = r$$
$$AD = s$$



We know that

$$(p-q)^2 + (q-r)^2 + (r-p)^2 \geq 0$$

$\Rightarrow$ $$2(p^2 + q^2 + r^2) \geq 2(pq + qr + rp)$$

$\Rightarrow$ $$3(p^2 + q^2 + r^2) \geq (p^2 + q^2 + r^2) + 2(pq + qr + rp)$$

[on adding $p^2 + q^2 + r^2$ to both sides]

$\Rightarrow$ $$3(p^2 + q^2 + r^2) \geq (p + q + r)^2$$

[$\because$ sum of any three sides of a quadrilateral is greater than fourth one]

$\Rightarrow$ $$3(p^2 + q^2 + r^2) \geq (p + q + r)^2 > s^2$$

$\Rightarrow$ $$\frac{p^2 + q^2 + r^2}{s^2} > \frac{1}{3}$$

$\therefore$ Minimum value of $\frac{p^2 + q^2 + r^2}{s^2}$ is $\frac{1}{3}$.

Example 109. *If n is a +ve integer > 1. Prove that*

$$2^n + n \cdot 6^{\frac{n-1}{2}} < 3^n.$$

Solution $\because$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1})$$

$\therefore$

$$3^n - 2^n = 3^{n-1} + 3^{n-2} \cdot 2 + 3^{n-3} \cdot 2^2 + \ldots + 2^{n-1} \quad \ldots\text{(i)}$$

$\because$ AM > GM for distinct +ve numbers

$\therefore$

$$\frac{3^{n-1} + 3^{n-2} \cdot 2 + \ldots + 2^{n-1}}{n} > [(3 \cdot 3^2 \ldots 3^{n-1})(2 \cdot 2^2 \ldots 2^{n-1})]^{\frac{1}{n}}$$

$$= 3^{\frac{n-1}{2}} \cdot 2^{\frac{n-1}{2}} = 6^{\frac{n-1}{2}} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$3^n > 2^n + n \cdot 6^{\frac{n-1}{2}}$$

Example 110. *If $A > 0$, $B > 0$ and $A + B = \pi/3$, find maximum value of* $\tan A \cdot \tan B$.

Solution Let

$$y = \tan A \tan B$$

$\because$ AM $\geq$ GM

$\therefore$

$$\frac{\tan A + \tan B}{2} \geq \sqrt{\tan A \tan B}$$

$\because$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan\frac{\pi}{3} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$\Rightarrow$

$$\sqrt{3}(1 - \tan A \tan B) = \tan A + \tan B$$

$\therefore$

$$\frac{\sqrt{3}(1 - \tan A \tan B)}{2} \geq \sqrt{\tan A \tan B}$$

$\therefore$

$$\sqrt{3}(1-y) \geq 2\sqrt{y} \quad (\because \tan A \tan B = y)$$

$\therefore$

$$3(1-y)^2 \geq 4y$$

$\Rightarrow$

$$3y^2 - 10y + 3 \geq 0$$

$\therefore$

$$y \leq 1/3 \text{ or } y \geq 3$$

But

$$A + B = \pi/3 \text{ and } A > 0, B > 0$$

$\therefore$

$$y \leq 1/3$$

$\therefore$ Maximum value of $\tan A \cdot \tan B = \frac{1}{3}$

Example 111. *Let a, b, c be the length of the sides of a triangle and r be its radius. Show that*

$$r < \frac{a^2 + b^2 + c^2}{3(a+b+c)}.$$

Solution We know for a ΔABC

$$r = \frac{\text{Area}}{\text{Semi-perimeter}} = \frac{\Delta}{\frac{a+b+c}{2}}$$

$\Rightarrow \quad r(a+b+c) = 2\Delta = bc \sin A = ca \sin B = ab \sin C$

$\Rightarrow \quad 3r(a+b+c) = 6\Delta = bc \sin A + ca \sin B + ab \sin C \quad \text{...(A)}$

Now,

$$bc \sin A < bc \quad \text{...(i)}$$

$$ca \sin B < ca \quad \text{...(ii)}$$

$$ab \sin C < ab \quad \text{...(iii)}$$

From (i), (ii) and (iii), we have

$$bc \sin A + ca \sin B + ab \sin C < bc + ca + ab$$

$\Rightarrow \quad 3r(a+b+c) < bc + ca + ab \quad \text{...(B)}$

$$bc \le \frac{b^2+c^2}{2}$$

$$ca \le \frac{c^2+a^2}{2}$$

$$ab \le \frac{a^2+b^2}{2}$$

$\Rightarrow \quad bc + ca + ab \le \frac{2}{2}(a^2+b^2+c^2) \quad \text{...(C)}$

From (B) and (C), we get

$$3r(a+b+c) < (a^2+b^2+c^2) \Rightarrow r < \frac{a^2+b^2+c^2}{3(a+b+c)}$$

Example 112. *In any acute angle ΔABC, show that*

(*i*) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{3\sqrt{3}}{8}$ *or* $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$ (*ii*) $\tan A + \tan B + \tan C \ge 3\sqrt{3}$.

Solution (i) We have to show

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{3\sqrt{3}}{8} \quad \text{...(i)}$$

If $A+B+C=\pi$, we know that

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \quad \text{...(ii)}$$

Using (ii) in (i), it is sufficient to show that

$$\frac{1}{4}(\sin A + \sin B + \sin C) \le \frac{3\sqrt{3}}{8}$$

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$$

or

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$$

(ii) If $A+B+C=\pi$, we know that

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C \quad \text{...(i)}$$

Consider three non -ve quantities

$$\tan A, \tan B, \tan C$$

$\because \quad \text{AM} \ge \text{GM}$

$\therefore \quad \frac{1}{3}(\tan A + \tan B + \tan C) \ge (\tan A + \tan B + \tan C)^{1/3}$

$\therefore \quad \tan A + \tan B + \tan C \ge 3(\tan A + \tan B + \tan C)^{1/3}$

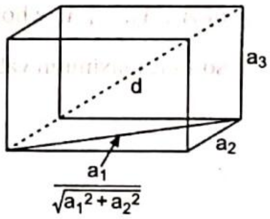
Cubing both sides, we get

$$(\tan A + \tan B + \tan C)^3 \geq 27$$

$$\therefore \qquad \tan A + \tan B + \tan C \geq 3\sqrt{3}$$

Example 113. *In a rectangular parallelopiped, the length of the edges are a_1, a_2, a_3 and length of the diagonal is d. Prove that*

$$d < a_1 + a_2 + a_3 < 2d$$



Solution From

$$d < \sqrt{a_1^2 + a_2^2} + a_3 < a_1 + a_2 + a_3$$

Also, $$(a_1 + a_2 + a_3)^2 - (a_1 + a_2 - a_3)^2 = 4(a_1 + a_2)\,a_3$$

so that $$(a_1 + a_2 + a_3)^2 \geq 4(a_1 + a_2)\,a_3$$

Similarly, $$(a_1 + a_2 + a_3)^2 \geq 4(a_2 + a_3)\,a_1$$

and $$(a_1 + a_2 + a_3)^2 \geq 4(a_3 + a_1)\,a_2$$

Adding these inequalities, we get

$$3(a_1 + a_2 + a_3)^2 \geq 8\,(a_1a_2 + a_2a_3 + a_3a_1)$$

$$(a_1 + a_2 + a_3)^2 \geq \frac{8}{3}\,(a_1a_2 + a_2a_3 + a_3a_1)$$

Adding $\frac{4d^2}{3}$ to both sides, where

$d^2 = a_1^2 + a_2^2 + a_3^2$, we get

$$\frac{4d^2}{3} + (a_1 + a_2 + a_3)^2 \geq \frac{4}{3}\,[(a_1^2 + a_2^2 + a_3^2) + 2(a_1a_2 + a_2a_3 + a_3a_1)$$

$$= \frac{4}{3}\,(a_1 + a_2 + a_3)^2$$

or $$\frac{4d^2}{3} \geq \frac{1}{3}\,(a_1 + a_2 + a_3)^2$$

$$2d \geq a_1 + a_2 + a_3 \text{ or } a_1 + a_2 + a_3 < 2d$$

Equality sign hold if and only if

$$a_1 + a_2 - a_3 = 0\,;\, a_2 + a_3 - a_1 = 0$$

or $$a_3 + a_1 - a_2 = 0$$

i.e., if $$a_1 + a_2 + a_3 = 0$$

so we get $d < a_1 + a_2 + a_3 < 2d$

Example 114. *In any ΔABC. Prove that*

$$\sum_{A,B,C} \frac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}} \geq 3.$$

Solution Let $$\sin A = \frac{a}{2R},$$

$$\sin B = \frac{b}{2R} \quad \text{and} \quad \sin C = \frac{c}{2R}.$$

Using the above concept, we get the desired result.

Example 115. *Show that if $A_1, A_2, \ldots, A_n$ are the angles of a convex polygon, then*

$$\sin A_1 + \sin A_2 + \ldots + \sin A_n \le n \sin \frac{2\pi}{n}$$

and
$$\tan \frac{A_1}{2} + \tan \frac{A_2}{2} + \ldots + \tan \frac{A_n}{2} \ge n \cot \frac{\pi}{n}.$$

Solution $\because$ The sum of exterior angles of a convex polygon is 2π, the sum of the interior angles is $(n-2)\pi$.

If $0 < x < \pi$, then $\sin x$ a concave function of x so that $\sin A_1 + \sin A_2 + \ldots + \sin A_n$ is a concave function of $A_1, A_2, \ldots, A_n$ whose maximum value occurs when $A_1 = A_2 = \ldots = A_n = \frac{(n-2)\pi}{n}$

so that maximum value of $\sin A_1 + \ldots + \sin A_n$ is

$$n \sin \frac{(n-2)\pi}{n}$$

$$\therefore \quad \sin A_1 + \sin A_2 + \ldots + \sin A_n \le n \sin\left(\frac{(n-2)\pi}{n}\right)$$

$$= n \sin \frac{2\pi}{n}$$

Similarly $\because$ $A_1, A_2, \ldots, A_n$ are all $< \pi$, $\tan \frac{A_1}{2}$, $\tan \frac{A_2}{2}, \ldots, \tan\left(\frac{A_n}{2}\right)$ are all convex functions so that

$\tan \frac{A_1}{2} + \tan \frac{A_2}{2} + \ldots + \tan \frac{A_n}{2}$ is a convex function.

$\therefore$ Its maximum value occurs when

$$A_1 = A_2 = \ldots = A_n = \frac{(n-2)\pi}{2n} \text{ so that}$$

$$\tan \frac{A_1}{2} + \tan \frac{A_2}{2} + \ldots + \tan \frac{A_n}{2} \ge n \tan \frac{(n-2)\pi}{2n}$$

$$= n \tan\left(\frac{\pi}{2} - \frac{\pi}{2n}\right)$$

$$= n \cot \frac{\pi}{2n}$$

Example 116. *Prove that if p, q, k are all $> 1, a_1, a_2, a_3, \ldots, a_n$ are all +ve numbers whose sum is unity, then*

$$\sum_{i=1}^{n} (a_i^p + a_i^q)^k \ge \frac{(n^p + n^q)^k}{(n)^{kp+kq-1}}.$$

Solution x^p is convex, if $p > 1$

$$x^q \text{ is convex, if } q > 1$$

$$x^p + x^q \text{ is convex, if } p, q > 1$$

Let
$$f(x) = (x^p + x^q)k$$

$$f'(x) = k(x^p + x^q)^{k-1}(px^{p-1} + qx^{q-1})$$

$$f''(x) = k(k-1)(x^p + x^q)^{k-2}$$

$$[p(p-1)x^{p-2} + q(q-1)x^{q-2}]$$

If p, q, k are all >1, then

$f''(x) > 0$ and $f(x)$ is a convex function.

$$\frac{f(a_1) + f(a_2) + \ldots + f(a_n)}{n} \ge f\left[\frac{(a_1 + a_2 + \ldots + a_n)}{n}\right]$$

but $a_1 + a_2 + \ldots + a_n = 1$ so that

$$\frac{f(a_1) + f(a_2) + \ldots + f(a_n)}{n} \geq f\left(\frac{1}{x}\right)$$

Put $f(x) = (x^p + x^q)^k$, we get

$$\frac{1}{n}\sum_{i=1}^{n}(a_i^p + a_i^q)^k \geq \left(\frac{1}{n^p} + \frac{1}{n^q}\right)^k$$

or

$$\sum_{i=1}^{n}(a_i^p + a_i^q)^k \geq \frac{(n^p + n^q)^k}{n^{(p+q)k-1}}$$

Equality sign holds only when

$$a_1 = a_2 = \ldots = a_n = \frac{1}{n}$$

Example 117. *For a acute angle* $\Delta ABC \left]0, \frac{\pi}{2}\right]$. *Prove that*

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}.$$

Solution Let

$$f(x) = \log \sin x$$

$$f'(x) = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f''(x) < 0$$

$$f\left(\frac{A + B + C}{3}\right) \geq \frac{f(A) + f(B) + f(C)}{3}$$

$$\log \sin\left(\frac{\pi}{3}\right) \geq \frac{\log \sin A + \log \sin B + \log \sin C}{3}$$

$$3 \log\left(\frac{\sqrt{3}}{2}\right) \geq \log(\sin A \sin B \sin C)$$

$$\log\left(\frac{3\sqrt{3}}{8}\right) \geq \log(\sin A \sin B \sin C)$$

$$\therefore \quad \sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$$

Let us Practice

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Level 1

1. If $x > a$, then prove that
$$x^3 + 13a^2x > 5ax^2 + 9a^3.$$

2. If $x, y > 0$, then show that
$$x^n + y^n > x^{n-1}y + xy^{n-1}.$$

3. Prove that $a^2(a-b)(a-c) + b^2(b-c)(b-a)$
$$+ c^2(c-a)(c-b) \geq 0$$

4. If $a, b, c > 0$, then prove that
$$a + b + c > \sqrt{ab} + \sqrt{bc} + \sqrt{ac}.$$

5. Without using AM-GM show that
$$a^8 + b^8 + c^8 \geq a^2b^2c^2(a^2 + b^2 + c^2).$$

6. Prove $\dfrac{a^4b^4}{c^4} + \dfrac{b^4c^4}{a^4} + \dfrac{c^4a^4}{b^4} > (abc)(a + b + c)$
without using AM-GM

7. If x is any real number show $\dfrac{x^2}{1 + x^4} \leq \dfrac{1}{2}$.

8. If $A + B + C = \pi$, show that
$$\frac{\tan^2 A + \tan^2 B}{\tan A + \tan B} + \frac{\tan^2 B + \tan^2 C}{\tan B + \tan C} + \frac{\tan^2 C + \tan^2 A}{\tan C + \tan A} \geq \tan A \tan B \tan C.$$

9. If x, y, z are positive, show that
$$\frac{x^3 + y^3}{x + y} + \frac{y^3 + z^3}{y + z} + \frac{z^3 + x^3}{z + x} \geq (xy + yz + zx).$$

10. If $m > 1$ and $n \in N$. Prove that
$$1^m + 2^m + 3^m + \ldots + n^m > n\left(\frac{n+1}{2}\right)^m.$$

11. If $m > 1$ and $n \in N$. Prove that
$$1^m + 3^m + 5^m + \ldots + (2n-1)^m > n^{m+1}.$$

12. If $a, b, c \geq 0$, then show that
$$ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$

13. Prove that if $a, b, c, d > 0$, then
$$\sqrt{(a + c)(b + d)} \geq \sqrt{ab} + \sqrt{cd}.$$

14. Find, which is greater $x^3 + 2y^3$ or $3xy^2$

15. Let a, b, c and $a + b - c, a + c - b, b + c - a$ be +ve. Prove that
$$abc \geq (a + b - c)(a + c - b)(b + c - a).$$

16. If $A = \dfrac{1 + 9 + 9^2 + \ldots + 9^{99}}{1 + 9 + 9^2 + \ldots + 9^{100}}$
and $B = \dfrac{1 + 7 + 7^2 + \ldots + 7^{99}}{1 + 7 + 7^2 + \ldots + 7^{100}}$.
Prove $A < B$.

17. Without using AM-GM, prove
$$\frac{a + b + c}{3} \geq \sqrt[3]{abc}.$$

18. Let $a, b, c, \ldots, l$ be n real +ve numbers, p and q be also two real numbers. Prove that if p and q are of same sign, then
$$n(a^{p+q} + b^{p+q} + \ldots + l^{p+q})$$
$$\geq (a^p + b^p + \ldots + l^p)(a^q + b^q + \ldots + l^q)$$
If p and q are of different signs, then
$$n(a^{p+q} + b^{p+q} + \ldots + l^{p+q})$$
$$\leq (a^p + b^p + \ldots + l^p)(a^q + b^q + \ldots + l^q)$$

19. Prove that
(i) $\dfrac{a + b}{2} \geq \sqrt{ab}$ $(a, b > 0)$
(ii) $\dfrac{1}{8} \cdot \dfrac{(a-b)^2}{a} \leq \dfrac{a + b}{2} - \sqrt{ab} \leq \dfrac{1}{8} \cdot \dfrac{(a-b)^2}{b}$
if $a \geq b$.

20. If $x + y = a$, then without AM-GM, show that
$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq \frac{1}{2}\left(a + \frac{4}{a}\right)^2.$$

21. Prove that
$$\frac{a^3 + b^3}{2} \geq \left(\frac{a + b}{2}\right)^3, a > 0, b > 0.$$

22. If $a_1, a_2, \ldots, a_n$ are real numbers show that
$$(\cos a_1 + \cos a_2 + \ldots + \cos a_n)^2$$
$$+ (\sin a_1 + \sin a_2 + \ldots + \sin a_n)^2 \leq n^2.$$

23. For $n \in N, n > 1$ show that
$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n^2} > 1.$$

24. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n+1}>1$

25. If a, b, c are unequal and +ve. Prove that
$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}<\frac{1}{2}(a+b+c).$$

26. Show that $(a+b+c+d)$
$$(a^3+b^3+c^3+d^3)>(a^2+b^2+c^2+d^2)^2.$$

27. Prove that $(a_1b_1+a_2b_2+a_3b_3)$
$$\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}\right)>(a_1+a_2+a_3)^2.$$

28. Show that
$$\frac{3}{b+c+d}+\frac{3}{c+d+a}+\frac{3}{d+a+b}+\frac{3}{a+b+c}>\frac{16}{a+b+c+d}.$$

29. If a, b, c, d are +ve real numbers, prove that
$$\frac{bcd}{a^2}+\frac{cda}{b^2}+\frac{dab}{c^2}+\frac{abc}{d^2}>a+b+c+d.$$

30. Prove that
$$(a+b)(b+c)(a+c)\geq 8abc\ (a, b, c>0).$$

31. Let a, b, c be +ve integers, prove that
$$a^{\frac{a}{a+b+c}}\cdot b^{\frac{b}{a+b+c}}\cdot c^{\frac{c}{a+b+c}}\geq\frac{1}{3}(a+b+c).$$

32. Prove that $(1+\alpha)^n>1+\alpha\lambda$ (α is any the number and $\lambda>1$ is rational).

33. Show that $(1+\alpha)^n<\frac{1}{1-\alpha\lambda}$ real, λ rational and +ve, $\alpha\lambda<1$.

34. If x, y, z are unequal +ve quantities. Prove that
$$\left(\frac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z}>x^xy^yz^z>\left(\frac{x+y+z}{3}\right)^{x+y+z}.$$

35. Prove that $\left(\frac{bc+ca+ab}{a+b+c}\right)^{a+b+c}>\sqrt{(bc)^a(ca)^b(ab)^c}.$

36. If x, y, z are +ve real numbers such that $x^3y^2z^4=7$, show that
$$2x+5y+3z\geq 9\left(\frac{525}{2^7}\right)^{\frac{1}{9}}.$$

37. Show that
$$\frac{b^m+c^m}{(b+c)^{m-1}}+\frac{c^m+a^m}{(c+a)^{m-1}}+\frac{a^m+b^m}{(a+b)^{m-1}}>\frac{a+b+c}{2^{m-2}}$$
where m does not lie between 0 and 1.

38. Show that
$$\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b}>a+b+c$$

39. Find the maximum value of $(8-x)^3(x+6)^4$, if x lies between -6 and 8.

40. Find the maximum value of $(7-x)^4(2+x)^5$, when x lies between -2 and 7.

41. Find the greatest value of $x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}$ when x lies between 0 and 1.

42. Find minimum value of $\frac{(5+x)(2+x)}{1+x}$.

43. If $5x+12y=60$, find the minimum value of $\sqrt{x^2+y^2}$.

44. Find the minimum value of $\frac{9x^2\sin^2 x+4}{x\sin x}$.

45. If a, b are +ve real numbers such that $a^2+b^2=1$ show that
$$a+b+\left(\frac{1}{\sqrt{ab}}\right)^2\geq 2+\sqrt{2}.$$

46. For $a>b>0$. n is a +ve integer >1, show that for $k\geq 0\ \sqrt[n]{a^n+k^n}-\sqrt[n]{b^n+k^n}\leq a-b$.

47. If a, b, c are unequal +ve rational number, show that
$$\left(\frac{a+b+c}{3}\right)^{a+b+c}<a^ab^bc^c<\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}$$

48. If a triangle having base a and ratio of other two sides is $r<1$, show that altitude of triangle is $\frac{ar}{1-r^2}$.

49. Suppose $(x_1, x_2, \ldots, x_n, \ldots)$ is a sequence of positive real numbers such that $x_1\geq x_2\geq x_3\geq\ldots\geq x_n\ldots$, and for all n
$$\frac{x_1}{1}+\frac{x_4}{2}+\frac{x_9}{3}+\ldots+\frac{x_{n^2}}{n}\leq 1.$$

Show that for all k the following inequality is satisfied

$$\frac{x_1}{1}+\frac{x_2}{2}+\frac{x_3}{3}+\ldots+\frac{x_k}{k}\le 3.$$

(RMO 2000)

50. If x, y, z are the sides of a triangle, then prove that

$$|x^2(y-z)+y^2(z-x)+z^2(x-y)|<xyz.$$

(RMO 2001)

51. Find all integers a, b, c, d satisfying the following relations.

(i) $1\le a\le b\le c\le d$;

(ii) $ab+cd=a+b+c+d+3$. **(RMO 2002)**

52. For any natural number $n>1$, prove the inequality **(RMO 2002)**

$$\frac{1}{2}<\frac{1}{n^2+1}+\frac{2}{n^2+2}+\frac{3}{n^2+3}+\ldots+\frac{n}{n^2+n}<\frac{1}{2}+\frac{1}{2n}.$$

53. Suppose the integers $1, 2, 3, \ldots, 10$ are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that

$a_1<a_2<a_3<a_4<a_5$, $b_1>b_2>b_3>b_4>b_5$.

(i) Show that the larger number in any pair $\{a_j, b_j\}$, $1\le j\le 5$, is at least 6.

(ii) Show that

$$|a_1-b_1|+|a_1-b_2|+|a_3-b_3|+|a_4-b_4|+|a_5-b_5|=25$$

for every such partition. **(RMO 2002)**

54. Let a, b, c be three positive real numbers such that $a+b+c=1$. Prove that among the three numbers $a-ab, b-bc, c-ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$. **(RMO 2003)**

55. Let x and y be positive real numbers such that $y^3+y\le x-x^3$. Prove that

(i) $y<x<1$;

(ii) $x^2+y^2<1$. **(RMO 2004)**

56. Let a, b, c be three positive real numbers such that $a+b+c=1$. Let

$\lambda=\min\{a^3+a^2bc, b^3+ab^2c, c^3+abc^2\}$.

Prove that the roots of the equation

$x^2+x+4\lambda=0$ are real. **(RMO 2005)**

57. If a, b, c are three real numbers such that

$$|a-b|\ge|c|, |b-c|\ge|a|, |c-a|\ge|b|,$$

then prove that one of a, b, c is the sum of the other two. **(RMO 2005)**

58. If a, b, c are three positive real numbers, prove that

$$\frac{a^2+1}{b+c}+\frac{b^2+1}{c+a}+\frac{c^2+1}{a+b}\ge 3.$$ **(RMO 2006)**

59. Prove that

(i) $5<\sqrt{5}+\sqrt[3]{5}+\sqrt[4]{5}$;

(ii) $8>\sqrt{8}+\sqrt[3]{8}+\sqrt[4]{8}$;

(iii) $n>\sqrt{n}+\sqrt[3]{n}+\sqrt[4]{n}$ for all integers $n\ge 9$.

(RMO 2007)

60. Prove that there exist two infinite sequences $\langle a_n\rangle_{n\ge 1}$ and $\langle b_n\rangle_{n\ge 1}$ of positive integers such that the following conditions hold simultaneously

(i) $1<a_1<a_2<a_3<\ldots$;

(ii) $a_n<b_n<a_n^2$, for all $n\ge 1$;

(iii) a_n-1 divides b_n-1, for all $n\ge 1$;

(iv) a_n^2-1 divides b_n^2-1, for all $n\ge 1$.

(RMO 2008)

Level 2

1. If a, b, c are real numbers such that

$$a^2+b^2+c^2=1$$

prove the inequalities

$$-\frac{1}{2}\le ab+bc+ca\le 1$$

2. If a, b and c are integers with $c>0$, then

$$\left[\frac{2a}{c}\right]+\left[\frac{2b+1}{c}\right]\ge\left[\frac{a}{c}\right]+\left[\frac{b}{c}\right]+\left[\frac{a+b+1}{c}\right].$$

3. Prove that for every three non-negative numbers a, b, c, the inequality $a^3+b^3+c^3+6abc\ge\frac{1}{4}(a+b+c)^3$ holds and that equality occurs only when two of these numbers are equal and the third is zero.

4. Show that for all natural numbers $n>1$ the inequality

$$\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{(n-1)}>\left(\frac{1+n^n}{n+1}\right)^n \text{ is valid.}$$

5. Show that if $x, y, z > 0$,

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}$$

6. Find the largest constant k such that

$$\frac{kabc}{a+b+c} \leq (a+b)^2 + (a+b+4c)^2$$

for all $a, b, c > 0$

7. Given positive real numbers a, b and c such that $a + b + c = 1$, show that

$$a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \leq 1.$$

8. If a, b, c, x, y and z are real and $a^2 + b^2 + c^2 = 25, x^2 + y^2 + z^2 = 36$, and $ax + by + cz = 30$, compute $\frac{a+b+c}{x+y+z}$.

9. Prove that

$$\frac{1}{k+1} n^{k+1} < 1^k + 2^k + 3^k + \dots + n^k < \left(1 + \frac{1}{n}\right)^{k+1} \frac{1}{k+1} n^{k+1}$$

(n and k are arbitrary integers).

10. There are real numbers a, b, c such that $a \geq b \geq c$. Prove that

$$\frac{a^2 - b^2}{c} + \frac{c^2 - b^2}{a} + \frac{a^2 - c^2}{b} \geq 3a - 4b + c.$$

11. Prove that $1 < \frac{1}{1001} + \frac{1}{1002} + \dots + \frac{1}{3001} < 1\frac{1}{3}$

12. Let $a_1, a_2, a_3, \dots, a_n$ be real numbers all greater than 1 and such that $|a_k - a_{k+1}| < 1$ for $1 \leq k \leq n - 1$.

Show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2n - 1.$$

13. For any natural number n, prove the following inequality

$$\frac{1}{n+1}\left\{1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}\right\} \geq \frac{1}{n}\left\{\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2a}\right\}.$$

14. Prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots + \frac{1}{n^2} < 2$$

for any n.

15. Let a, b, c be three real numbers such that $1 \geq a \geq b \geq c \geq 0$. Prove that if λ is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$ (real or complex), then $|\lambda| \leq 1$. (INMO 2000)

16. If a, b, c are positive real numbers such that $abc = 1$, prove that

$$a^{b+c} b^{c+a} c^{a+b} \leq 1. \quad \text{(INMO 2001)}$$

17. Let x, y be positive reals such that $x + y = 2$. Prove that

$$x^3 y^3 (x^3 + y^3) \leq 2. \quad \text{(INMO 2002)}$$

18. Let ABC be a triangle with sides a, b, c. Consider a $\Delta A_1B_1C_1$ with sides equal to $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$. Show that

$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$

where $[XYZ]$ denotes the area of the ΔXYZ.

(INMO 2003)

19. Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}.$$

Find the minimum possible value of β.

(INMO 2005)

20. (i) Prove that if n is a positive integer such that $n \geq 4011^2$, then there exists an integer l such that $n < l^2 < \left(1 + \frac{1}{2005}\right)n$.

(ii) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$, there exists an integer l, such that

$$n < l^2 < \left(1 + \frac{1}{2005}\right)n. \quad \text{(INMO 2006)}$$

21. If x, y, z are positive real numbers, prove that

$$(x + y + z)^2 (yz + zx + xy)^2 \leq 3 (y^2 + yz + z^2)(z^2 + zx + x^2)(x^2 + xy + y^2). \quad \text{(INMO 2007)}$$

22. Let a, b, c be positive real numbers such that $a^3 + b^3 = c^3$. Prove that

$$a^2 + b^2 - c^2 > 6(c - a)(c - b). \quad \text{(INMO 2009)}$$

Indian National Mathematics Olympiad Conducted by HBCSE with Solved Papers of RMO INMO 2016-2019 Part 2 from Page 336 Inequalities Solutions Rajeev Manocha

Indian National MATHEMATICS OLYMPIAD

Conducted by Homi Bhabha Centre for Science Education

With
SOLVED PAPERS
RMO & INMO
2016-2019

Rajeev Manocha

Indian National

MATHEMATICS OLYMPIAD

Conducted by Homi Bhabha Centre for Science Education

In India, National Board of Higher Mathematics (NBHM) started National Mathematics Olympiad in 1986. It worked with Homi Bhabha Centre for Science Education, Mumbai. One aim of this activity is to support the mathematical talent among the senior secondary students in the country. The problems of the Olympiads, chosen from various areas of secondary school mathematics, require exceptional mathematical ability and mathematical knowledge on the part of the candidates.

This book is a collection of lessons, and includes challenging and stimulating problems from various National and Regional Mathematics Olympiad. The problems are provided with complete solutions, occasionally accompanied by remarks, alternative ways of doing the same problem or the generalization of the problem.

This book is an ideal source of study for all those aiming to participate in National and Regional Mathematics Olympiads as well as ones, who are preparing for entrances such as JEE Main & Advanced, where complex and tricky problems are routine...

Indian National MATHEMATICS OLYMPIAD

Conducted by **Homi Bhabha Centre for Science Education**

With

SOLVED PAPERS

RMO & INMO

2016-2019

RAJEEV MANOCHA

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ARIHANT PRAKASHAN (SERIES), MEERUT

PREFACE

Since the year 1989 when India started participating in the Mathematical Olympiads, the interest of the school students in the country has tremendously increased in India National Mathematics Olympiad (INMO). Now, we find a number of really talented students in almost all the prestigious schools in India who are excited about this mega event and sincerely desire to participate in it, win a handful of medals and make the country proud. We just need to educate them about the competition, and provide them the relevant study material.

The Mathematical Olympiad tests the participant's level of mastery of the methods of Mathematics and the strategizing and tactical skills in plenty. The Olympiad is an open challenge to all those who love the problem solving.

This book has been written, keeping in mind the orientation required on the parts of students to face the Olympiads at national or regional level. This book has designed to give the student an insight and proficiency into almost all the areas of Mathematics. Exhaustive theory has been provided of selected and relevant chapters to clarify the basic concepts. Problems from recently held Olympiads have been given to increase awareness of what to expect in the event.

Revised Edition of This Book Has

1. Complete theory with support of good number of solved examples and exactly on the pattern and level of Indian National Mathematics Olympiads.
2. Each chapter has two level exercises divided according to RMO Regional Mathematics Olympiad and INMO (Indian National Mathematical Olympiad)
3. Solutions have been provided for selected questions.

First of all, I would like to thank Mr Deepesh Jain, Director Arihant Group the man with a distinct vision, for the idea to write this book, and then bringing it to reality. I am also thankful to my colleagues and students for the moral support they provided. I take this an opportunity to thank Sunil Chugh, Director, HMA, for the inspiration to write the book of this nature, and Sumit Malviya for the assistance he provided in the preparation of the manuscript.

It is hoped, this book will charge you up for the Olympiad juggernaut. I have tried my best to keep this book error-free. However, if any error or whatsoever is left I request the readers to bring forward to my notice. Suggestions for the further improvement of the book are welcome.

With best wishes

Rajeev Manocha

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Dedicated to My Parents
I.P.F. Manocha, Bimla Manocha
& My sons Robit and Sarthak

INTRODUCTION

MATHEMATICS OLYMPIADS

ABOUT THE EXAM
& HOW TO SUCCEED IN IT ?

International Mathematical Olympiad

International Mathematical Olympiads was created by Dr. George Lenchner, a prominent Maths educator, in 1977 with the aim of stimulating enthusiasm and love for Mathematics. About 1,50,000 students from 50 states of USA and 25 other countries participate in this competitive examination every year. Through this examination, effort has been made to introduce important mathematical concepts and teach major strategies for problem solving. The examination also seeks to foster Mathematical creativity and ingenuity and provide satisfaction, joy and thrill through meeting challenges.

Every participant team comprises 35 students. Some schools send more than one team in the contest. However, the rule indicates that only schools or home-school associations may participate individuals are barred from entering into the contest.

Every team enters into competition in just one division. Those teams which have members from more than one school are called 'district teams' or 'institute teams' and they are not eligible for team awards. The team score is the total of ten highest individual scores taken after the fifth contest in a series of contests which involve selection.

The International Mathematics Olympiad invites students from 2nd to 12th class to participate and excel at international level.

Indian National Mathematical Olympiads

In India, National Board for Higher Mathematics (NBHM) started National Mathematics Olympiads in 1986. It worked with Homi Bhabha Centre for Science Education, Mumbai, for this purpose. One aim of this activity is to support the mathematical talent among the high school students in the country. NBHM take the responsibility of selecting, training and grooming of Indian team for participation in the International Mathematical Olympiad every year. There are certain regional bodies which provide voluntary service and play an important role at various stages. The country has been divided into 23 regions for conducting Mathematical Olympiads.

Stages

Stage I Regional Mathematical Olympiad (RMO)

A regional coordinator conducts tests in each region. Regional Mathematical Olympiads (RMO) is held in each region between September and first Sunday of December every year. The regional coordinator makes sure that at least one centre in each district of the region is provided for the test. All the students of high school upto Class XII can appear at Regional Mathematical Olympiads. The qualifying test is of three hours and consists of six to seven problems.

The regional coordinator has the liberty of making his own paper or obtaining paper from NBHM. The regions which choose to go for paper by NBHM conduct this contest on the first Sunday of December. The performance in RMO is judged and a certain number of students from every region are chosen to appear for the next round. There is a nominal fee to meet the expenses of the organising the contest.

Stage II Indian National Mathematical Olympiad

The second stage of selection involves holding of the contest at the national level. Indian National Mathematical Olympiads is (INMO) organized on the first Sunday of every February at various centres in different regions. Only those students who have been selected in Regional Mathematical Olympiads can appear at this contest. At this level, there is a 4-hour test which is common to all regions. Those who rank among the top 35, receive a certificate of merit.

Stage III International Mathematical Olympiad Training Camp (IMOTC)

The INMO awardees are invited to a month long training camp held in April-May each year at the Homi Bhabha Centre for Science Education (HBCSE), Mumbai. INMO awardees of the previous year who have satisfactorily gone through postal tuition throughout the year are invited again to a second round of training (Senior Batch). The senior batch participants who successfully complete the camp receive a prize of 5,000/- in the form of books and cash. On the basis of a number of selection tests through the camp, a team of the best six students is selected from the combined pool of junior and senior batch participants.

Stage IV Pre-departure Training Camp for IMO

The selected team of six students goes through another round of training and orientation for about 10 days prior to the departure for IMO.

Stage V International Mathematical Olympiad (IMO)

The six-members team selected at the end of the camp accompanied by a leader and a deputy leader represent the country at the IMO, held in July each year in a different member country of IMO. IMO consists of two 4 and 1/2-hour written tests held on two days. Travel to IMO venue and return takes about 2 weeks. India has been participating in IMO since 1989. Students of the Indian team who receive gold, silver and bronze medals at the IMO receive a cash prize of 5000/-, 4000/- and 3000/- respectively during the following year at a formal ceremony at the end of the training camp.

Ministry of Human Resource Development (MHRD) finances international travel of the 8-members Indian delegation, while NBHM (DAE) finances the entire in-country programme and other expenditure connected with international participation.

Awards and Recognition

- Participation Certificate to every student | Merit Certificate for students appearing in second level School Topper Medal to the class topper in each participating school (with more than 50 students).
- Top three students from each class are awarded Gold, Silver and Bronze medals.
- The top 500 winners (classwise) are endowed with cash prizes and scholarships, courtesy of the official sponsor.
- Every participant who scores 50% or more is awarded a Certificate of Participation. Toppers are awarded merit certificate.
- School/College of each medal winning participant is awarded the 'Intellect Trophy' for having groomed and nurtured the talent.

Career Prospects

Mathematical Olympiads were created to hunt out the talented students and to give them an opportunity to express their talent in creative and intuitive way. It reads well on a student's curriculum vitae to have participated in Mathematical Olympiads. Those who have participated in the Olympiads have a good chance of doing well in Engineering.

The top few students from Mathematical Olympiads are selected for admission to premier institutions. The career of the students takes up an upwards mobility curve after participating in this contest. The students become confident of their performance and rich with the feeling of self-worth after selection to represent at Mathematical Olympiads.

Number of Students Appearing

- Approximately 10,000 students appear for this examination at regional level, though the number is only eight when they represent the country at international level.
- Students from rural and urban areas alike participate in the regional competition. This competition is open for all the students from Class IX onwards.

Month and Frequency of Examination

International Mathematical Olympiads is held once every year in July. The qualifying rounds of Indian National Mathematics Olympiads is held in different months of the years starting from first Sunday of December.

Craze of the Examination

Mathematical Olympiads, whether they are held at national level or international level, attract interest not only from the mathematicians but also from the student community, the parents and society at large. Even those students who may not be participating in the contest are interested in it.

The students who participate in the contest start preparing for it at least three months prior to the date of examination. The craze for the examination is not limited only to a certain region, the verve engulfs the entire country. Those who are selected are viewed as the promising stars of future who will contribute to the progress of science and technology in years to come.

General Eligibility

The students must be from Class IX and must be at least 14 years of age to participate in this contest both at national and international levels. They must pass the different rounds of selection in order to be eligible.

At the first round, all students who are in Class IX are eligible to appear for the qualifying test. In the subsequent rounds, however, only those whom make it in the test would be eligible to carry forth.

Skill Sets Required

In Mathematics, the skill most required is ability to think fast, be spatially gifted and compute complex data without committing error. The students must develop the habit of calculating data mentally and simplify the methods in order to arrive at the correct answer with minimum effort. Another very important part is concentration; lack of concentration can cause the candidates to commit errors, so sufficient effort must be put into develop the power of concentration.

Difficulty Level of Problems

The problems of Olympiads, chosen from various areas of secondary school Mathematics, require exceptional mathematical ability and mathematical knowledge on the part of the contestants. Generally, there are six to seven questions asked and they are of extremely high level. To solve these questions, the students need to be thorough with their basics and have good knowledge of all the principles of Mathematics. Since two or three principles from different mathematical streams must be applied to solve the questions, the problems assume a sophistication which requires an agile and quick thinking mind with plenty of common sense to answer them.

How to Prepare for Different Subjects in Stipulated Time

The best way of dealing with this contest is to cultivate scientific thinking and to develop power of concentration. The students must try to deal with two or three mathematical principles simultaneously so that they become adept at solving complex problems which require application of higher mental faculty. There are many books in the market but Arihant books are the best as they help in development of scientific temper along with providing reading material and solution.

General Mental Set up Required for the Examination

The candidates must have a positive frame of mind and they must be able to deal with stress that comes as a package with these examinations. They must have the desire to succeed and ability to work for long lasting success. The candidates must be well versed in all the fundamentals of the principles of Mathematics. They must be original thinker and must have a scientific temperament. The power of concentration of the candidates must be of exceptionally high order. The calculation skill of the candidates must be extremely good since all Maths sums are based on reasoning and calculation.

Do's and Don't on the Day of Examination

On the day of the examination, the candidates must take care about a few things which are listed below:

- The candidates must reach the venue of the examination at least half an hour before time.
- The candidates must carry their admit cards to the examination hall.
- The candidates must carry their own pens, pencils, erasers, sharpeners and must refrain from borrowing these articles from the other candidates.
- The candidates must abstain from talking to other candidates in the examination hall while the examination is being conducted.
- The candidates must hand over their answer sheets to the invigilator as soon as the stipulated time is over.

How this Book is Useful for You

This book by Arihant is the best in the market for purpose of preparation. It has many complex problems and tricky questions requiring sophisticated methods of calculation.

This book coaches the students to think in multidimensional way and apply the principles of Mathematics to solve the most complex problems in the simplest way. It also gives tremendous exposure to different types of problems which exist in the realm of Mathematics.

This book has matter for study by the students and solved problems which help in understanding of the subject itself. It also has previous years papers from Regional and International Olympiads along with their solutions. It is an excellent book to have which will prove to be the best friend to the contestants.

From Page 336 Inequalities Solutions

Solutions

Level 1

1. $(x^3 + 13a^2x) - (5ax^2 + 9a^3)$

$= (x - a)(x^2 - 4ax + 9a^2)$

$= (x - a)[(x - 2a)^2 + 5a^2]$

$= -\text{ve}$

if $x > a$

Hence, $(x^3 + 13a^2x) > (5ax^2 + 9a^3)$

2. $(x^n + y^n) - (x^{n-1}y + xy^{n-1})$

$\Rightarrow (x^n - x^{n-1}y) + (y^n - xy^{n-1})$

$\Rightarrow x^{n-1}(x - y) - y^{n-1}(x - y)$

$\Rightarrow (x^{n-1} - y^{n-1})(x - y)$

$(x - y)^2(x^{n-2} + x^{n-3}y + \ldots + y^{n-2})$

$(x - y)^2 > 0$

$\because$ x and y are +ve.

$(x^{n-2} + x^{n-3}y + \ldots + y^{n-2}) > 0$

$\therefore\ x^n + y^n > x^{n-1}y + xy^{n-1}$

3. $\because$ The given inequality is symmetrical in a, b, c.

Without loss of generality assume $a \ge b \ge c$ we have

$$a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b)$$

$$= (a - b)\{a^2(a - c) - b^2(b - c)\} + c^2(c - a)(c - b)$$

$$= (a - b)\{(a^3 - b^3) - c(a^2 - b^2)\} + c^2(a - c)(b - c)$$

$$= (a - b)\{(a - b)(a^2 + ab + b^2) - c(a - b)(a + b)\} + c^2(a - c)(b - c)$$

$$= (a - b)^2\{a^2 + ab + b^2 - c(a + b)\} + c^2(a - c)(b - c)$$

$$= (a - b)^2\{a^2 + ab + b^2 - ca - bc\} + c^2(a - c)(b - c)$$

$$= (a - b)^2\{a(a - c) + b(b - c) + ab\} + c^2(a - c)(b - c)$$

which is non -ve as each term of RHS is non -ve.

$\because$ $a \ge b \ge c$ and a, b, c be non -ve

Thus, $a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \ge 0$

4. $a = (\sqrt{a})^2$

$b = (\sqrt{b})^2$

$c = (\sqrt{c})^2$

$\therefore$ Now, let $\sqrt{a} = u$

$\sqrt{b} = v$

$\sqrt{c} = w$

$u^2 + v^2 + w^2 > uv + vw + uw$

$a + b + c > \sqrt{ab} + \sqrt{bc} + \sqrt{ac}$

5. $a^8 + b^8 + c^8 \ge a^2b^2c^2(a^2 + b^2 + c^2)$

$(a^4)^2 + (b^4)^2 + (c^4)^2 \ge a^4b^4 + b^4c^4 + a^4c^4$ ...(i)

$$(a^2b^2)^2 + (b^2c^2)^2 + (a^2c^2)^2 \ge (a^2b^2)(b^2c^2) + (b^2c^2)(a^2c^2) + (a^2b^2)(a^2c^2)$$

$\therefore a^4b^4 + b^4c^4 + a^4c^4 \ge a^2b^2c^2(a^2 + b^2 + c^2)$ ...(ii)

From (i) and (ii), we get

$a^8 + b^8 + c^8 \ge a^2b^2c^2(a^2 + b^2 + c^2)$

6. Let $\dfrac{ab}{c} = u, \dfrac{bc}{a} = v, \dfrac{ca}{b} = w$

We know that

$u^4 + v^4 + w^4 > uvw(u + v + w)$ ...(i)

$$u^4 + v^4 + w^4 > abc\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right)$$

$$u^4 + v^4 + w^4 > abc\left(\frac{a^2b^2 + b^2c^2 + c^2a^2}{abc}\right)$$

$u^4 + v^4 + w^4 > a^2b^2 + b^2c^2 + c^2a^2$ ...(ii)

Now, we know that

$a^2b^2 + b^2c^2 + c^2a^2 > abc(a + b + c)$ ...(iii)

From (ii) and (iii), we have

$$u^4 + v^4 + w^4 > abc(a + b + c)$$

Substitute the value of u, v, w we get the desired result.

7. If $x = 0$, then result $\frac{0}{1+0} \le \frac{1}{2}$ i.e., $0 \le \frac{1}{2}$ is true.

For $x \ne 0$, x^2 is +ve.

Let $S = \frac{x^2}{1 + x^4}$ for $x \ne 0$

We can write $S = \frac{1}{\frac{1}{x^2} + x^2}$

$$\text{Max } S = \frac{1}{\min\left(\frac{1}{x^2} + x^2\right)} \qquad \left[\because x^2 + \frac{1}{x^2} \ge 2\right]$$

$$S \le \frac{1}{2}$$

$$\therefore \quad \frac{x^2}{1 + x^4} \le \frac{1}{2}$$

8. Let $\tan A = u$, $\tan B = v$, $\tan C = w$, we have

$$\frac{u^2 + v^2}{u + v} + \frac{v^2 + w^2}{v + w} + \frac{w^2 + u^2}{u + w} \ge u + v + w$$

$$\Rightarrow \quad u + v + w = \tan A + \tan B + \tan C$$

Now, when $A + B + C = \pi$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Hence proved.

9. Use fourth fundamental concept.

10. For $m > 1$

AM of mth power > mth power of AM

$$\therefore \quad \frac{1^m + 2^m + 3^m + \ldots + n^m}{n} > \left(\frac{1 + 2 + 3 \ldots + n}{n}\right)^m$$

$$\Rightarrow \quad \frac{1^m + 2^m + 3^m + \ldots + n^m}{n} > \left(\frac{n+1}{2}\right)^m$$

$$\Rightarrow \quad 1^m + 2^m + \ldots + n^m > n\left(\frac{n+1}{2}\right)^m$$

11. Now, AM of mth power > mth power of AM

$$\frac{1^m + 3^m + 5^m + \ldots + (2n-1)^m}{n} > \left\{\frac{1 + 3 + 5 + \ldots + (2n-1)}{n}\right\}^m$$

$$\Rightarrow \quad \frac{1^m + 3^m + 5^m + \ldots + (2n-1)^m}{n} > \left(\frac{n^2}{n}\right)^m$$

$$\Rightarrow \quad 1^m + 3^m + 5^m + \ldots + (2n-1)^m > n^{m+1}$$

12. $\forall x, y, z \ge 0$

$$x^2 + y^2 + z^2 = \frac{x^2 + y^2}{2} + \frac{y^2 + z^2}{2} + \frac{z^2 + x^2}{2}$$

$$\ge xy + yz + zx$$

Let $\quad x = \sqrt{ab}, y = \sqrt{bc}, z = \sqrt{ca}$

Put the value of x, y, z, we get

$$ab + bc + ca \ge a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$$

13. $a, b, c, d > 0$ (given)

$$\text{Then } (a + c)(b + d) = ab + ad + bc + cd$$

$$\ge ab + 2\sqrt{adbc} + cd$$

$$= (\sqrt{ab} + \sqrt{cd})^2$$

$$\text{or} \quad \sqrt{(a + c)(b + d)} \ge \sqrt{ab} + \sqrt{cd}$$

14. $3xy^2 - (x^3 + 2y^3) = xy^2 + 2xy^2 - x^3 - 2y^3$

$$= (xy^2 - x^3) + (2xy^2 - 2y^3)$$

$$= x(y^2 - x^2) + 2y^2(x - y)$$

$$= (x - y)[-x(y + x) + 2y^2]$$

$$= (x - y)[2y^2 - yx - x^2]$$

$$= (x - y)(2y + x)(y - x)$$

$$= -(x - y)^2(x + 2y) < 0$$

If $\quad x \ne y$

Hence, $\quad 3xy^2 < (x^3 + 2y^3)$

i.e., $\quad x^3 + 2y^3 > 3xy^2$

15. We have

$$a^2 \ge a^2 - (b - c)^2 \quad \ldots(i)$$

$$b^2 \ge b^2 - (c - a)^2 \quad \ldots(ii)$$

$$c^2 \ge c^2 - (a - b)^2 \quad \ldots(iii)$$

Multiplying (i), (ii) and (iii), we get

$$a^2b^2c^2 \ge (a + b - c)^2(a + c - b)^2(b + c - a)^2$$

$$\text{or} \quad abc \ge (a + b - c)(a + c - b)(b + c - a)$$

17. Let $a = x^3$, $b = y^3$, $c = z^3$

It is sufficient to prove that

$$x^3 + y^3 + z^3 - 3xyz \ge 0$$

for any non -ve x, y, z

We know that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

we also know that

$$x^2 + y^2 + z^2 - xy - yz - zx \ge 0$$

$\therefore\ x^3 + y^3 + z^3 - 3xyz \ge 0$

Hence proved.

18. Let a and b be two real +ve numbers. If $p > 0$, then $a^p - b^p > 0$ for $a > b$.

If $p < 0$, then $a^p - b^p < 0$ for $a > b$

$\therefore$ We may assert $(a^p - b^p)(a^q - b^q) \ge 0$, if p and q are of same sign.

We have

$$a^{p+q} + b^{p+q} \ge a^p b^q + a^q b^p$$

$$a^{p+q} + c^{p+q} \ge a^p c^p + a^q c^p$$

$$a^{p+q} + l^{p+q} \ge a^p l^q + a^q l^p$$

$$b^{p+q} + c^{p+q} \ge b^p c^q + b^q c^p$$

Adding these inequalities termwise, we get

$$(n-1)(a^{p+q} + b^{p+q} + \dots + l^{p+q}) \ge \Sigma a^p b^q$$

where a and b attain all the values from the series $a, b, c, \dots, l$.

Adding Σa^{p+q} to both number of this inequality, we get

$$n(a^{p+q} + b^{p+q} + \dots + l^{p+q}) \ge (a^p + b^p + \dots + l^p)(a^q + b^q + \dots + l^q)$$

19. (i) AM of two +ve numbers is not less than their GM

Indeed $\dfrac{a+b}{2} - \sqrt{ab}$

$$= \frac{1}{2}(a + b - 2\sqrt{ab})$$

$$= \frac{1}{8}(\sqrt{a} - \sqrt{b})^2 \ge 0$$

(ii) To prove that

$$\frac{a+b}{2} - \sqrt{ab} \le \frac{1}{8}\cdot\frac{(a-b)^2}{2}\ (a > b)$$

It is sufficient to prove

$$\left(\sqrt{a} - \frac{\sqrt{b}}{2}\right)^2 \le \frac{1}{8}\cdot\frac{(a-b)^2}{b}$$

It is necessary to prove following

$$\left(\frac{\sqrt{a}+\sqrt{b}}{8b}\right)^2 \ge \frac{1}{2}$$

We have, $\left(\dfrac{\sqrt{a}+\sqrt{b}}{8b}\right)^2 = \dfrac{1}{8}\left(1 + \sqrt{\dfrac{a}{b}}\right)^2 \ge \dfrac{1}{2}$

$\because\ \dfrac{a}{b} > 1$

Similarly, we can prove second inequality.

20. Let $S = \left(x + \dfrac{1}{x}\right)^2 + \left(y + \dfrac{1}{y}\right)^2$

Opening the brackets, we have

$$S = \left(x^2 + \frac{1}{x^2} + 2 + y^2 + \frac{1}{y^2} + 2\right)$$

$$S = x^2 + y^2 + \frac{y^2 + x^2}{x^2 y^2} + 4 \qquad \left[\begin{array}{r}\because\ \text{if } x + y = a \\ x^2 + y^2 \ge \dfrac{a^2}{2} \\ xy \le \dfrac{a^2}{4}\end{array}\right]$$

$$\therefore\ S \ge \frac{a^2}{2} + \frac{\frac{a^2}{2}}{\frac{a^4}{16}} + 4$$

$$S \ge \frac{a^2}{2} + \frac{a^2}{a^4} \times 8 + 4$$

$$S \ge \frac{a^2}{2} + \frac{8}{a^2} + 4$$

$$S \ge \frac{a^4 + 16 + 8a^2}{2a^2} = \frac{(a^2+4)^2}{2a^2}$$

$$S \ge \left[\frac{1}{2}\left(a + \frac{4}{a}\right)^2\right]$$

Hence proved.

21. We have $a^2 + b^2 - 2ab = (a - b)^2 \ge 0$

Hence, $\quad a^2 - ab + b^2 \ge ab$

$$a^3 + b^3 \ge ab(a + b)$$

Consequently

$$3a^3 + 3b^3 \ge 3a^2 b + 3ab^2$$

Adding $a^3 + b^3$, we get

$$4a^3 + 4b^3 \ge (a + b)^3$$

and so, $\quad \dfrac{a^3 + b^3}{2} \ge \left(\dfrac{a+b}{2}\right)^3$

22. Let $Z = (\cos a_1 + \cos a_2 + \dots + \cos a_n)^2$

$$+ (\sin a_1 + \sin a_2 + \dots + \sin a_n)^2$$

$$= (\cos^2 a_1 + \cos^2 a_2 + \dots + \cos^2 a_n)$$

$$+ (\sin^2 a_1 + \sin^2 a_2 + \dots + \sin^2 a_n)$$

$$+ 2(\cos a_1 \cos a_2 + \cos a_1 \cos a_3 + \dots$$

$$+ \cos a_{n-1} \cos a_n) + 2(\sin a_1 \sin a_2$$

$$+ \sin a_1 \sin a_3 + \dots + \sin a_{n-1} \sin a_n)$$

$$= (1 + 1 + \dots n \text{ times}) + 2[(\cos(a_1 - a_2)$$

$$+ \cos(a_1 - a_3) + \dots + \cos(a_{n-1} - a_n)]$$

$\because$ $\cos(a_1 - a_2) \le 1$

$\cos(a_1 - a_3) \le 1$

....

....

....

$\cos(a_{n-1} - a_n) \le 1$

$\therefore$ $\cos(a_1 - a_2) + \cos(a_1 - a_3)$

$$+ \dots + \cos(a_{n-1} - a_n)$$

$$\le 1 + 1 + 1 + \dots \frac{n(n-1)}{2} \text{ times}$$

$$= \frac{n(n-1)}{2}$$

$\therefore$ $Z \le n + 2 \cdot \frac{n(n-1)}{2} = n^2$

$\therefore$ $Z \le n^2$

23. $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n^2}$

$$> \frac{1}{n} + \left(\frac{1}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \dots + \frac{1}{n^2}\right)$$

$(n^2 - n \text{ times})$

$$\Rightarrow \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n^2}$$

$$> \frac{1}{n} + \frac{(n^2 - n)}{n^2} = \frac{1}{n} + 1 - \frac{1}{n} = 1$$

24. We have,

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n} + \frac{1}{3n+1}$$

$$= \left(\frac{1}{n+1} + \frac{1}{3n+1}\right) + \left(\frac{1}{n+2} + \frac{1}{3n}\right)$$

$$+ \dots + \frac{1}{2n+1}$$

$$= \frac{4n+2}{(n+1)(3n+1)} + \frac{4n+2}{(n+2)3n} + \dots + \frac{4n+2}{2(2n+1)^2}$$

$$> (4n+2)\left[\frac{n}{(2n+1)^2} + \frac{1}{2(2n+1)^2}\right] = 1$$

25. $\because$ AM > GM

$\therefore$ $\frac{b+c}{2} > \sqrt{bc}$ or $\left(\frac{b+c}{2}\right)^2 > bc$

or $\frac{b+c}{4} > \frac{bc}{b+c}$

Similarly, $\frac{c+a}{4} > \frac{ca}{c+a}$ and $\frac{a+b}{4} > \frac{ab}{a+b}$

Adding these results, we get

$$\frac{b+c}{4} + \frac{c+a}{4} + \frac{a+b}{4} > \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}$$

$$\frac{1}{2}(a+b+c) > \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}$$

or $\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} < \frac{1}{2}(a+b+c)$

26. Now,

$$(a^4 + b^4 + c^4 + d^4) + a(b^3 + c^3 + d^3)$$

$$+ b(a^3 + c^3 + d^3) + c(a^3 + b^3 + d^3)$$

$$+ d(a^3 + b^3 + c^3) > a^4 + b^4 + c^4 + d^4$$

$$+ 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2)$$

or $(a^4 + b^4 + c^4 + d^4) + (ab^3 + a^3b)$

$$+ (ac^3 + ca^3) + (ad^3 + da^3) + (bc^3 + b^3c)$$

$$+ (bd^3 + b^3d) + (cd^3 + c^3d)$$

$$> (a^4 + b^4 + c^4 + d^4) + 2a^2b^2$$

$$+ 2a^2c^2 + 2a^2d^2 + 2b^2c^2 + 2b^2d^2 + 2c^2d^2 \quad \dots(i)$$

$\because$ AM > GM

$\therefore$ $\frac{ab^3 + a^3b}{2} > \sqrt{ab^3 \times a^3b}$

or $(ab^3 + a^3b) > 2a^2b^2$

Similarly, $(ac^3 + a^3c) > 2a^2c^2$

$(ad^3 + a^3d) > 2a^2d^2$

and $(bc^3 + b^3c) > 2b^2c^2$

and $(bd^3 + b^3d) > 2b^2d^2$

and $(cd^3 + c^3d) > 2c^2d^2$

Adding these results, we get

$$(ab^3 + a^3b) + (ac^3 + a^3c) + (ad^3 + a^3d)$$
$$+ (bc^3 + b^3c) + (bd^3 + b^3d) + (cd^3 + c^3d)$$
$$> 2a^2b^2 + 2a^2c^2 + 2a^2d^2 + 2b^2c^2 + 2b^2d^2 + 2c^2d^2$$

Adding $(a^4 + b^4 + c^4 + d^4)$ to both sides, we get (i)

27. We are to prove that

$$(a_1b_1 + a_2b_2 + a_3b_3)\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3}\right) > (a_1 + a_2 + a_3)^2$$

or
$$a_1^2 + \frac{a_1a_2b}{b_2} + \frac{a_1b_1a_3}{b_3} + \frac{a_1a_2b_2}{b_1} + a_2^2 + \frac{a_2b_2a_3}{b_3} + \frac{a_1a_3b_3}{b_1} + \frac{a_2a_3b_3}{b_2} + a_3^2$$
$$> a_1^2 + d_2^2 + a_3^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3$$

or
$$(a_1^2 + a_2^2 + a_3^2) + \left(\frac{a_1a_2b_1}{b_2} + \frac{a_1a_2b_2}{b_1}\right) + \left(\frac{a_1a_3b_1}{b_3} + \frac{a_1a_3b_3}{b_1}\right) + \left(\frac{a_2a_3b_2}{b_3} + \frac{a_2a_3b_3}{b_2}\right)$$
$$> (a_1^2 + a_2^2 + a_3^2) + 2a_1a_2 + 2a_1a_3 + 2a_2a_3 \quad \ldots\text{(i)}$$

$\because$ AM > GM

$$\therefore \frac{1}{2}\left(\frac{a_1a_2b_1}{b_2} + \frac{a_1a_2b_2}{b_1}\right) > \sqrt{\left(\frac{a_1a_2b_1}{b_2} \times \frac{a_1a_2b_2}{b_1}\right)}$$

or
$$\frac{a_1a_2b_1}{b_2} + \frac{a_1a_2b_2}{b_1} > 2a_1a_2 \quad \ldots\text{(ii)}$$

Similarly,

$$\frac{a_1a_3b_1}{b_3} + \frac{a_1a_3b_3}{b_1} > 2a_1a_3 \quad \ldots\text{(iii)}$$

$$\frac{a_2a_3b_2}{b_3} + \frac{a_2a_3b_3}{b_2} > 2a_2a_3 \quad \ldots\text{(iv)}$$

Adding (ii), (iii) and (iv)

$$\left(\frac{a_1a_2b_1}{b_2} + \frac{a_1a_2b_2}{b_1}\right) + \left(\frac{a_1a_3b_3}{b_2}\right) + \left(\frac{a_2a_3b_2}{b_3} + \frac{a_2a_3b_3}{b_2}\right) > 2a_1a_2 + 2a_1a_3 + 2a_2a_3$$

Adding $(a_1^2 + a_2^2 + a_3^2)$ to both sides we get (i).

28. $\because$ AM > GM

$$\therefore \frac{1}{4}\left[\frac{3}{b+c+d} + \frac{3}{c+d+a} + \frac{3}{d+a+b} + \frac{3}{a+b+c}\right]$$
$$> \left[\frac{3}{b+c+d} \times \frac{3}{c+d+a} \times \frac{3}{d+a+b} \times \frac{3}{a+b+c}\right]^{1/4} \quad \ldots\text{(i)}$$

Also, $\frac{1}{4}\left[\frac{b+c+d}{3} + \frac{c+d+a}{3} + \frac{d+a+b}{3} + \frac{a+b+c}{3}\right]^{1/4}$

or
$$\frac{1}{4}\left[\frac{3(a+b+c+d)}{3}\right] > \left[\frac{b+c+d}{3} \times \frac{c+d+a}{3} \times \frac{d+a+b}{3} \times \frac{a+b+c}{3}\right]^{1/4}$$

or
$$\frac{a+b+c+d}{4} > \left[\frac{b+c+d}{3} \times \frac{c+d+a}{3} \times \frac{d+a+b}{3} \times \frac{a+b+c}{3}\right]^{1/4} \quad \ldots\text{(ii)}$$

Multiplying (i) and (ii), we get

$$\frac{1}{16}\left[\frac{3}{b+c+d} + \frac{3}{c+d+a} + \frac{3}{d+a+b} + \frac{3}{a+b+c}\right]$$

$$(a+b+c+d) > 1$$

or
$$\frac{3}{b+c+d} + \frac{3}{c+d+a} + \frac{3}{d+a+b} + \frac{3}{a+b+c} > \frac{16}{a+b+c+d}$$

29. We have to prove that

$$\frac{bcd}{a^2} + \frac{cda}{b^2} + \frac{dab}{c^2} + \frac{abc}{d^2} > a+b+c+d$$

or
$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} > \left(\frac{1}{bcd} + \frac{1}{cda} + \frac{1}{abd} + \frac{1}{abc}\right)$$

(Dividing both sides by $abcd$)

$\because$ AM> GM we have

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}>3\left(\frac{1}{a^3b^3c^3}\right)^{1/3}$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{d^3}>3\left(\frac{1}{a^3b^3d^3}\right)^{1/3}$$

$$\frac{1}{a^3}+\frac{1}{c^3}+\frac{1}{d^3}>3\left(\frac{1}{a^3c^3d^3}\right)^{1/3}$$

$$\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}>3\left(\frac{1}{b^3c^3d^3}\right)^{1/3}$$

Adding these inequalities, we get

$$3\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}\right)$$
$$>3\left(\frac{1}{abc}+\frac{1}{abd}+\frac{1}{acd}+\frac{1}{bcd}\right)$$

$$\Rightarrow \quad \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}$$
$$>\frac{1}{abc}+\frac{1}{abd}+\frac{1}{acd}+\frac{1}{bcd}$$

30. Make use of the following identity

$$(a+b)(b+c)(a+c)=(ab+ac+bc)(a+b+c)-abc$$

But $\dfrac{a+b+c}{3}\geq\sqrt[3]{abc}\ ;\dfrac{ab+ac+bc}{3}\geq\sqrt[3]{a^2b^2c^2}$

$\therefore \quad (a+b+c)(ab+ac+bc)\geq 9\,abc$

Consequently, $(a+b)(a+c)(b+c)\geq 8abc$

31. Consider a quantities equal to $\frac{1}{a}$.

b quantities equal to $\frac{1}{b}$.

c quantities equal to $\frac{1}{c}$.

AM of these quantities will be

$$\frac{a\cdot\frac{1}{a}+b\cdot\frac{1}{b}+c\cdot\frac{1}{c}}{a+b+c}=\frac{3}{a+b+c}$$

GM is equal to

$$\sqrt[a+b+c]{\frac{1}{a^a}\cdot\frac{1}{b^b}\cdot\frac{1}{c^c}}$$

Consequently, $\dfrac{3}{a+b+c}\geq\sqrt[a+b+c]{\dfrac{1}{a^a}\cdot\dfrac{1}{b^b}\cdot\dfrac{1}{c^c}}$

i.e., $a^{\frac{a}{a+b+c}}\,b^{\frac{b}{a+b+c}}\,c^{\frac{c}{a+b+c}}\geq\frac{1}{3}(a+b+c)$

32. Let $\lambda=\dfrac{m}{n}$, $m>n$

We have

$$\sqrt[m]{\left(1+\alpha\frac{m}{n}\right)\left(1+\alpha\frac{m}{n}\right)\ldots\left(1+\alpha\frac{m}{n}\right)\cdot 1\cdot 1\ldots 1}$$

$$<\frac{\left(1+\alpha\frac{m}{n}\right)+\left(1+\alpha\frac{m}{n}\right)+\ldots+\left(1+\alpha\frac{m}{n}\right)+m-n}{m}$$

The factor $1+\alpha\dfrac{m}{n}$ of the radicand is taken n times, the factor 1 is taken as $m-n$ times.

Hence, $\left(1+\alpha\dfrac{m}{n}\right)^{\frac{n}{m}}<1+\alpha$

or $\quad (1+\alpha)^{m/n}>1+\alpha\dfrac{m}{n}$

33. Let $\lambda=\dfrac{m}{n}$

Assume that $m>n$ i.e., $\lambda>1$

we have

$$\sqrt[m]{\left(1-\alpha\frac{m}{n}\right)\left(1-\alpha\frac{m}{n}\right)\cdot\left(1-\alpha\frac{m}{n}\right)\cdot 1\cdot 1\cdot 1}$$

$$<\frac{\left(1-\alpha\frac{m}{n}\right)n+m-n}{m}$$

The factor $1-\alpha\dfrac{m}{n}$ of the radicand taken n times and factor 1 is taken $m-n$ times.

Hence, $\left(1-\alpha\dfrac{m}{n}\right)^{\frac{n}{m}}<1-\alpha<\dfrac{1}{1+\alpha}$

$$1-\alpha\frac{m}{n}<\frac{1}{(1+\alpha)^{\frac{m}{n}}}$$

$$(1+\alpha)^{\frac{m}{n}}<\frac{1}{1-\alpha\frac{m}{n}}$$

Assume that $m<n$ we have

$$\sqrt[n]{(1+\alpha)^m}=\sqrt[n]{(1+\alpha)(1+\alpha)\ldots(1+\alpha)\cdot 1\cdot 1\cdot 1}$$
$$<\frac{(1+\alpha)m+n-m}{n}=1+\alpha\cdot\frac{m}{n}<\frac{1}{1-\frac{\alpha m}{n}}$$

So in this case, also $(1+\alpha)^{\frac{m}{n}}<\dfrac{1}{1-\frac{\alpha m}{n}}$

34. Consider x quantities each equal to x; y quantities each equal to y and z quantities each equal to z.

$\because$ AM > GM

$$\frac{(x + x + \ldots \text{to } x \text{ terms}) + (y + y + \ldots y \text{ terms}) + (z + z + \ldots z \text{ terms})}{x + y + z}$$

$$> [(x \cdot x \ldots \text{to } x \text{ factors})(y \cdot y \ldots \text{to } y \text{ factors})(z \cdot z \ldots z \text{ factors})]^{\frac{1}{x+y+z}}$$

or $$\frac{(x \cdot x) + (y \cdot y) + (z . z)}{x + y + z} > [(x^x)(y^y)(z^z)]^{\frac{1}{x+y+z}}$$

or $$\left[\frac{x^2 + y^2 + z^2}{x + y + z}\right]^{x+y+z} > x^x \cdot y^y \cdot z^z$$

Again, consider x quantities each equal to $\frac{1}{x}$, y quantities each equal to $\frac{1}{y}$ and z quantities each equal to $\frac{1}{z}$.

$\because$ AM > GM

$\therefore$

$$\frac{\left(\frac{1}{x} + \frac{1}{x} + \ldots \text{to } x \text{ terms}\right) + \left(\frac{1}{y} + \frac{1}{y} + .. \text{to } y \text{ terms}\right) + \left(\frac{1}{z} + \frac{1}{z} + \ldots \text{to } z \text{ terms}\right)}{xyz}$$

$$> \left[\left(\frac{1}{x} \cdot \frac{1}{x} \ldots \text{to } x \text{ terms}\right)\left(\frac{1}{y} \cdot \frac{1}{y} \ldots \text{to } y \text{ terms}\right)\left(\frac{1}{z} \cdot \frac{1}{z} \ldots \text{to } z \text{ terms}\right)\right]^{\frac{1}{x+y+z}}$$

or $$\frac{\left(x \cdot \frac{1}{x}\right) + \left(y \cdot \frac{1}{y}\right) + \left(z \cdot \frac{1}{z}\right)}{x + y + z} > \left[\left(\frac{1}{x}\right)^x \left(\frac{1}{y}\right)^y \left(\frac{1}{z}\right)^z\right]^{\frac{1}{x+y+z}}$$

or $$\frac{1 + 1 + 1}{x + y + z} > \left(\frac{1}{x^x y^y z^z}\right)^{\frac{1}{x+y+z}}$$

or $$\frac{x + y + z}{3} < (x^x y^y z^z)^{\frac{1}{x+y+z}}$$

or $$\left(\frac{x + y + z}{3}\right)^{x+y+z} < x^x y^y z^z$$

or $$x^x y^y z^z > \left(\frac{x + y + z}{3}\right)^{x+y+z}$$

35. We have to prove that

$$\left(\frac{bc + ca + ab}{a + b + c}\right)^{a+b+c} > \sqrt{b^a c^a \cdot c^b a^b \cdot a^c b^c}$$

or $$\left(\frac{bc + ca + ab}{a + b + c}\right)^{a+b+c} > \sqrt{a^{b+c} \times b^{c+a} \times c^{a+b}} \quad \ldots \text{(i)}$$

Consider, $(b + c)$ quantities each equal to a, $(c + a)$ quantities each equal to b and $(a + b)$ quantities each equal to c.

$\because$ AM > GM

$$\therefore \frac{[\text{to } (b + c) \text{terms}] \times [b + b + \ldots \text{to } (c + a) \text{terms}] \times [c + c + \ldots \text{to } (a + b) \text{ terms}]}{(b + c) + (c + a) + (a + b)}$$

$$> [\{a \cdot a \ldots \text{to } (b + c) \text{ factors}\} \{b \cdot b \ldots \text{to } (c + a) \text{ factors}\} \times \{c \cdot c \ldots \text{to } (a + b) \text{ factors}]^{\frac{1}{(b+c)+(c+a)+(a+b)}}$$

or $$\frac{[(b + c)a + (c + a)b + (a + b)c]}{(b + c) + (c + a) + (a + b)} > [a^{b+c} \times b^{c+a} \times c^{a+b}]^{\frac{1}{(b+c)+(c+a)+(a+b)}}$$

or $$\frac{2(ab + bc + ca)}{2(a + b + c)} > [a^{b+c} \times b^{c+a} \times c^{a+b}]^{\frac{1}{2(a+b+c)}}$$

or $$\left(\frac{ab + bc + ca}{a + b + c}\right)^{a+b+c} > [a^{b+c} \times b^{c+a} \times c^{a+b}]^{1/2}$$

or $$\left(\frac{ab + bc + ca}{a + b + c}\right)^{a+b+c} > \sqrt{a^{b+c} \times b^{c+a} \times c^{a+b}}$$

36. Apply AM-GM inequality to the 9 numbers

$$\frac{2x}{3}, \frac{2x}{3}, \frac{2x}{3}, \frac{5y}{2}, \frac{5y}{2}, \frac{3z}{4}, \frac{3z}{4}, \frac{3z}{4}, \frac{3z}{4}$$

we get

$$\frac{1}{9}(2x+5y+3z) \geq \left[\left(\frac{2x}{3}\right)^3\left(\frac{5y}{2}\right)^2\left(\frac{3z}{4}\right)^4\right]^{\left(\frac{1}{9}\right)}$$

$$= \left[\left(\frac{2}{3}\right)^3\left(\frac{5}{2}\right)^2\left(\frac{3}{4}\right)^4 x^3y^2z^4\right]^{\frac{1}{9}}$$

$$= \left(\frac{5^2 \cdot 3}{2^7} \cdot 7\right)^{\frac{1}{9}} = \left(\frac{525}{2^7}\right)^{\frac{1}{9}}$$

37. If m does not lie between 0 and 1. Now, AM of mth power > mth power of AM

$$\therefore \quad \frac{b^m + c^m}{2} > \left(\frac{b+c}{2}\right)^m$$

Dividing both sides by $(b+c)^{m-1}$

$$\frac{b^m + c^m}{2(b+c)^{m-1}} > \frac{(b+c)^m}{2^m (b+c)^{m-1}}$$

or $$\frac{b^m + c^m}{(b+c)^{m-1}} > \frac{b+c}{2^{m-1}} \quad \ldots(\text{i})$$

Similarly, $$\frac{c^m + a^m}{(c+a)^{m-1}} > \frac{c+a}{2^{m-1}} \quad \ldots(\text{ii})$$

and $$\frac{a^m + b^m}{(a+b)^{m-1}} > \frac{a+b}{2^{m-1}} \quad \ldots(\text{iii})$$

Adding (i), (ii) and (iii), we get

$$\frac{b^m + c^m}{(b+c)^{m-1}} + \frac{c^m + a^m}{(c+a)^{m-1}} + \frac{a^m + b^m}{(a+b)^{m-1}}$$

$$> \frac{1}{2^{m-1}}[(b+c)+(c+a)+(a+b)]$$

$$= \frac{1}{2^{m-1}}[2(a+b+c)] = \frac{a+b+c}{2^{m-2}}$$

Hence proved.

38. AM of mth power > mth power of AM

$$\therefore \quad \frac{b^2 + c^2}{2} > \left(\frac{b+c}{2}\right)^2$$

or $$\frac{b^2 + c^2}{b+c} > \frac{b+c}{2} \quad \ldots(\text{i})$$

Similarly, $$\frac{c^2 + a^2}{c+a} > \frac{c+a}{2} \quad \ldots(\text{ii})$$

$$\frac{a^2 + b^2}{a+b} > \frac{a+b}{2} \quad \ldots(\text{iii})$$

Adding (i), (ii) and (iii), we get

$$\frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b}$$

$$> \frac{b+c}{2} + \frac{c+a}{2} + \frac{a+b}{2} = (a+b+c)$$

39. Let $z = (8-x)^3(x+6)^4$

$$= 3^3 4^4 \left(\frac{8-x}{3}\right)^3\left(\frac{x+6}{4}\right)^4 \quad \ldots(\text{i})$$

$\therefore$ z will be maximum when $\left(\frac{8-x}{3}\right)^3\left(\frac{x+6}{4}\right)^4$ is maximum.

But $\left(\frac{8-x}{3}\right)^3\left(\frac{x+6}{4}\right)^4$ is product of 7 factors, sum of which = 14

$\therefore \left(\frac{8-x}{3}\right)^3\left(\frac{x+6}{4}\right)^4$ will be maximum if all the factors are equal *i.e.*,

$$\frac{8-x}{3} = \frac{x+6}{4} = 2$$

$\therefore$ Maximum value of $z = 6^3 \cdot 8^4$

40. Let $z = (7-x)^4(2+x)^5$

$$= 4^4 5^5 \left(\frac{7-x}{4}\right)^4\left(\frac{2+x}{5}\right)^5 \quad \ldots(\text{i})$$

$\therefore$ z is maximum when $\left(\frac{7-x}{4}\right)^4\left(\frac{2+x}{5}\right)^5$ is maximum.

But it is the product of 9 factors, sum of which is 9.

$\therefore$ $\left(\frac{7-x}{4}\right)^4\left(\frac{2+x}{5}\right)^5$ will be maximum, if all factors are equal.

i.e., If $$\frac{7-x}{4} = \frac{2+x}{5}$$

$$= \frac{(7-x)+(2+x)}{4+5} = \frac{9}{9} = 1$$

Now, do yourself.

41. Let $z = x^{1/2} \cdot (1-x)^{1/3}$

$$z^6 = x^3(1-x)^2$$

or $$z^6 = 3^3 2^2 \left(\frac{x}{3}\right)^3\left(\frac{1-x}{2}\right)^2 \quad \ldots(\text{i})$$

$\therefore$ z is maximum when $\left(\frac{x}{3}\right)^3\left(\frac{1-x}{2}\right)^2$ is maximum.

Sum of the factors of $\left(\frac{x}{3}\right)^3\left(\frac{1-x}{2}\right)^2$

$$= 3\left(\frac{x}{3}\right) + 2\left(\frac{1-x}{2}\right)$$

$$= x + (1-x) = 1$$

$\therefore$ $\left(\frac{x}{3}\right)^3\left(\frac{1-x}{2}\right)^2$ will be maximum, if all factors are equal.

i.e., $\frac{x}{3} = \frac{1-x}{2} = \frac{x+(1-x)}{3+2} = \frac{1}{5}$

From Eq. (i), maximum value of

$$z^6 = 3^3 2^2 \left(\frac{1}{5}\right)^3\left(\frac{1}{5}\right)^2$$

$\therefore$ Maximum value is $\left(\frac{3}{5}\right)^{\frac{1}{2}}\left(\frac{2}{5}\right)^{\frac{1}{3}}$.

42. Let $y = 1 + x$

or $x = y - 1$

$$\therefore \quad \frac{(5+x)(2+x)}{1+x} = \frac{(5+y-1)(2+y-1)}{y}$$

$$= \frac{(4+y)(y+1)}{y}$$

$$= \frac{(4y+4+y^2+y)}{y} = y + 5 + \frac{4}{y}$$

$$= \left(\sqrt{y} - \frac{2}{\sqrt{y}}\right)^2 + 5 + 4$$

$$= \left(\sqrt{y} - \frac{2}{\sqrt{y}}\right)^2 + 9 \qquad \text{...(i)}$$

The given expression will have minimum value when $\left(\sqrt{y} - \frac{2}{\sqrt{y}}\right)^2 = 0$

$\therefore$ Minimum value of given expression = 9

43. For real number $a, b, x\ y$, we have inequality

$$ax + by \le \sqrt{a^2+b^2}\,\sqrt{x^2+y^2}$$

Hence, $60 = 5x + 12y \le \sqrt{5^2+12^2}\,\sqrt{x^2+y^2}$

$$= 13\sqrt{x^2+y^2}$$

The minimum value of $\sqrt{x^2+y^2}$ is $\frac{60}{13}$.

44. Dividing out gives

$$f(x) = 9x\sin x + \frac{4}{x\sin x}$$

$\because$ $x \sin x$ is non -ve in $0 < x < \pi$. Using AM-GM inequality, we get

$$f(x) \ge 2\sqrt{9x\sin x \cdot \frac{4}{x\sin x}} = 12$$

$\therefore$ Minimum possible value is 12.

It is attained when $x^2\sin x^2 = \frac{4}{9}$

$\because$ $\frac{\pi}{2} > 1$, $x \sin x$ being continuous does attain the value $\frac{2}{3}$ for some x in the interval $\left[0, \frac{\pi}{2}\right]$.

45. Let $u = \frac{1}{\sqrt{ab}}$

Then, $a^2 + b^2 = 1 \Rightarrow 2ab \le 1 \Rightarrow u \ge \sqrt{2}$

$$\therefore \quad \left(u + \sqrt{2} - \frac{\sqrt{2}}{u}\right) > 0$$

Hence, $(u - \sqrt{2})\left(u + \sqrt{2} - \frac{\sqrt{2}}{u}\right) \ge 0$

$$\Rightarrow \quad u^2 - 2 - \sqrt{2} + \frac{2}{u} \ge 0$$

$$\Rightarrow \quad u^2 + \frac{2}{u} \ge 2 + \sqrt{2}$$

$$\therefore \quad a + b + \frac{1}{ab} \ge 2\sqrt{ab} + \frac{1}{ab} \ge 2 + \sqrt{2}$$

Equality holds if and only if $a = b = \sqrt{2}$

46. Now, $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$

Let $x = (a^n + k^n)^{1/n}$

$y = (b^n + k^n)^{1/n}$

We get

$$\sqrt[n]{(a^n + k^n)} - \sqrt[n]{(b^n + k^n)}\,(a^{n-1} + a^{n-2}b + \dots + b^{n-1}) \le a^n - b^n \qquad \text{...(i)}$$

$[\because x \ge a, y \ge b, k \ge 0 \text{ and } a > b > 0]$

But $a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}$

$$= \frac{a^n - b^n}{a - b} \text{ is +ve}$$

$\therefore$ From (i)

$$\sqrt[n]{a^n + k^n} - \sqrt[n]{b^n + k^n} \le a - b$$

Hence proved.

47. $\because a, b, c$ are +ve rational numbers.

Using Weighted mean theorem for $a, a; b, b; c, c$ we get

$$\frac{(a \cdot a + b \cdot b + c \cdot c)}{a+b+c} > (a^a b^b c^c)^{\frac{1}{a+b+c}}$$

or $$\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c} > a^a b^b c^c \qquad \ldots\text{(i)}$$

Using same result for $a, \frac{1}{a}; b, \frac{1}{b}; c, \frac{1}{c}$ we get

$$\frac{a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{1}{c}}{a+b+c} > \left[\left(\frac{1}{a}\right)^a \left(\frac{1}{b}\right)^b \left(\frac{1}{c}\right)^c\right]^{\frac{1}{a+b+c}}$$

or $$\left(\frac{3}{a+b+c}\right) < \left(\frac{1}{a^a \cdot b^b \cdot c^c}\right)^{\frac{1}{a+b+c}}$$

or $$\left(\frac{3}{a+b+c}\right)^{a+b+c} > \frac{1}{a^a b^b c^c}$$

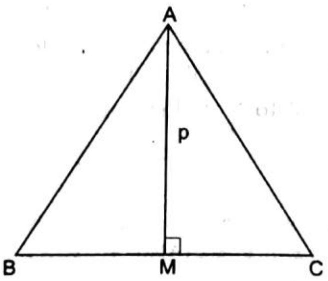
or $$\left(\frac{a+b+c}{3}\right)^{a+b+c} < a^a b^b c^c \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\left(\frac{a+b+c}{3}\right)^{a+b+c} < a^a b^b c^c < \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}$$

48. $$\frac{1}{2} bc \sin A = \frac{1}{2} a \cdot p$$

$\Rightarrow$ $$p = \frac{bc \sin A}{a}$$



$$= \frac{abc(\sin^2 B - \sin^2 C)\sin A}{a^2(\sin^2 B - \sin^2 C)}$$

$$= \frac{abc \sin (B - C) \sin^2 A}{a^2(\sin^2 B - \sin^2 C)}$$

$$= \frac{abc \sin (B-C)}{a^2\left(\frac{\sin^2 B}{\sin^2 A} - \frac{\sin^2 C}{\sin^2 A}\right)} = \frac{abc \sin (B-C)}{(b^2 - c^2)}$$

$$\leq \frac{abc}{b^2 - c^2} = \frac{a\left(\frac{c}{b}\right)}{1 - \left(\frac{c}{b}\right)^2} = \frac{ar}{1 - r^2}$$

$\therefore$ $$p \leq \frac{ar}{1 - r^2}$$

49. Let k be a natural number and n be the unique integer such that $(n-1)^2 \leq k < n^2$. Then, we see that

$$\sum_{r=1}^{k} \frac{x_r}{r} \leq \left(\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3}\right) + \left(\frac{x_4}{4} + \frac{x_5}{5} + \ldots + \frac{x_8}{8}\right)$$
$$+ \ldots + \left(\frac{x_{(n-1)^2}}{(n-1)^2} + \ldots + \frac{x_k}{k} + \ldots + \frac{x_{n^2-1}}{n^2-1}\right)$$
$$\leq \left(\frac{x_1}{1} + \frac{x_1}{1} + \frac{x_1}{1}\right) + \left(\frac{x_4}{4} + \frac{x_4}{4} + \ldots + \frac{x_4}{4}\right)$$
$$+ \ldots + \left(\frac{x_{(n-1)^2}}{(n-1)^2} + \ldots + \frac{x_{(n-1)^2}}{(n-1)^2}\right)$$
$$= \frac{3x_1}{1} + \frac{5x_2}{4} + \ldots + \frac{(2n-1)x_{(n-1)^2}}{(n-1)^2}$$
$$= \sum_{r=1}^{n-1} \frac{(2r+1)x_{r^2}}{r^2}$$
$$\leq \sum_{r=1}^{n-1} \frac{3r}{r^2} x_{r^2}$$
$$= 3 \sum_{r=1}^{n-1} \frac{x_{r^2}}{r} \leq 3,$$

where the last inequality follows from the given hypothesis.

50. The given inequality may be written in the form

$$|(x-y)(y-z)(z-x)| < xyz.$$

Since x, y, z are the sides of a triangle, we know that

$$|x - y| < z, |y - z| < x \text{ and } |z - x| < y.$$

Multiplying these, we obtain the required inequality.

51. We may write (ii) in the form

$$ab - a - b + 1 + cd - c - d + 1 = 5$$

Thus, we obtain the equation
$(a-1)(b-1) + (c-1)(d-1) = 5$. If $a - 1 \geq 2$, then (i) shows that

$$b-1\geq 2, c-1\geq 2 \text{ and } d-1\geq 2$$

so that $(a-1)(b-1)+(c-1)(d-1)\geq 8$.

If follows that $a-1=0$ or 1.

If $a-1=0$, then the contribution from $(a-1)(b-1)$ to the sum is zero for any choice of b. But then $(c-1)(d-1)=5$ implies that $c-1=1$ and $d-1=5$ by (i). Again (i) shows that $b-1=0$ or 1 since $b\leq c$. Taking $b-1=0, c-1=1$ and $d-1=5$ we get the solution $(a,b,c,d)=(1,1,2,6)$.

Similarly, $b-1=1, c-1=1$ and $d-1=5$ gives $(a,b,c,d)=(1,2,2,6)$.

In the other case $a-1=1$, we see that $b-1=2$ is not possible for then $c-1\geq 2$ and $d-1\geq 2$. Thus $b-1=1$ and this gives $(c-1)(d-1)=4$. It follows that $c-1=1$, $d-1=4$ or $c-1=2, d-1=2$. Considering each of these, we get two more solutions $(a,b,c,d)=(2,2,2,5),(2,2,3,3)$.

It is easy to verify all these four quadruples are indeed solutions to our problem.

52. We have

$$n^2<n^2+1<n^2+2<n^2+3<\ldots.<n^2+n$$

Hence, we see that

$$\frac{1}{n^2+1}+\frac{2}{n^2+2}+\ldots+\frac{n}{n^2+n}>\frac{1}{n^2+n}+\frac{2}{n^2+n}+\ldots+\frac{n}{n^2+n}$$

$$=\frac{1}{n^2+n}(1+2+3+\ldots+n)=\frac{1}{2}$$

Similarly, we see that

$$\frac{1}{n^2+1}+\frac{2}{n^2+2}+\ldots+\frac{n}{n^2+n}$$

$$<\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n}{n^2}=\frac{1}{n^2}(1+2+3+\ldots+n)$$

$$=\frac{1}{2}+\frac{1}{2n}$$

53. (i) Fix any pair $\{a_j,b_j\}$. We have $a_1<a_2<\ldots<a_{j-1}<a_j$ and $b_j>b_{j+1}>\ldots>b_5$. Thus there are $j-1$ numbers smaller than a_j and $5-j$ numbers smaller than b_j. Together they account for $j-1+5-j=4$ distinct numbers smaller than a_j as well as b_j. Hence, the larger of a_j and b_j is at least 6.

(ii) The first part shows that the larger numbers in the pairs $\{a_j,b_j\}$, $1\leq j\leq 5$, are 6, 7, 8, 9, 10 and the smaller numbers are 1, 2, 3, 4, 5. This implies that

$$|a_1-b_1|+|a_2-b_2|+|a_3-b_3|+|a_4-b_4|+|a_5-b_5|$$
$$=10+9+8+7+6-(1+2+3+4+5)=25$$

54. By AM-GM inequality, we have

$$a(1-a)\leq\left(\frac{a+1-a}{2}\right)^2=\frac{1}{4}$$

Similarly, we also have

$$b(1-b)\leq\frac{1}{4} \text{ and } c(1-c)\leq\frac{1}{4}$$

Multiplying these we obtain

$$abc\,(1-a)(1-b)(1-c)\leq\frac{1}{4^3}$$

We may rewrite this in the form

$$a(1-b)\cdot b\,(1-c)\cdot c(1-a)\leq\frac{1}{4^3}$$

Hence, one factor at least (among $a(1-b)$, $b(1-c), c(1-a)$) has to be less than or equal to $\frac{1}{4}$; otherwise lhs would exceed $\frac{1}{4^3}$.

Again, consider the sum

$$a(1-b)+b\,(1-c)+c\,(1-a).$$

This is equal to $a+b+c-ab-bc-ca$

We observe that

$$3(ab+bc+ca)\leq(a+b+c)^2$$

which, in fact, is equivalent to

$$(a-b)^2+(b-c)^2+(c-a)^2\geq 0$$

This leads to the inequality

$$a+b+c-ab-bc-ca\geq(a+b+c)-\frac{1}{3}(a+b+c)^2=1-\frac{1}{3}=\frac{2}{3}$$

Hence, one summand at least (among $a(1-b), b(1-c), c(1-a)$) has to be greater than or equal to $\frac{2}{9}$; (otherwise lhs would be less than $\frac{2}{3}$.)

55. (i) Since, x and y are positive, we have $y\leq x-x^3-y^3<x$. Also, $x-x^3\geq y+y^3>0$. So, $x(1-x^2)>0$

Hence, $x<1$. Thus, $y<x<1$, proving part (i).

(ii) Again, $\quad x^3+y^3\leq x-y$

So, $\quad x^2-xy+y^2\leq\dfrac{x-y}{x+y}$

That is

$$x^2 + y^2 \le \frac{x-y}{x+y} + xy = \frac{x - y + xy(x+y)}{x+y}$$

Here, $xy(x+y) < 1 \cdot y \cdot (1+1) = 2y$

So, $$x^2 + y^2 < \frac{x - y + 2y}{x+y} = \frac{x+y}{x+y} = 1$$

This proves (ii)

56. Suppose the equation $x^2 + x + 4\lambda = 0$ has no real roots. Then $1 - 16\lambda < 0$. This implies that $1 - 16(a^3 + a^2bc) < 0$, $1 - 16(b^3 + ab^2c) < 0$,

$$1 - 16(c^3 + abc^2) < 0$$

Observe that

$$1 - 16(a^3 + a^2bc) < 0$$

$$\Rightarrow \quad 1 - 16a^2(a + bc) < 0$$

$$\Rightarrow \quad 1 - 16a^2(1 - b - c + bc) < 0$$

$$\Rightarrow \quad 1 - 16a^2(1-b)(1-c) < 0$$

$$\Rightarrow \quad \frac{1}{16} < a^2(1-b)(1-c)$$

Similarly, we may obtain

$$\frac{1}{16} < b^2(1-c)(1-a), \frac{1}{16} < c^2(1-a)(1-b)$$

Multiplying these three inequalities, we get

$$a^2b^2c^2(1-a)^2(1-b)^2(1-c)^2 > \frac{1}{16^3}$$

However, $0 < a < 1$ implies that $a(1-a) \le 1/4$. Hence,

$$a^2b^2c^2(1-a)^2(1-b)^2(1-c)^2$$

$$= (a(1-a))^2(b(1-b))^2(c(1-c))^2 \le \frac{1}{16^3}$$

a contradiction. We conclude that the given equation has real roots.

57. Using $|a - b| \ge |c|$, we obtain $(a-b)^2 \ge c^2$ which is equivalent to $(a - b - c)(a - b + c) \ge 0$.

Similarly, $(b - c - a)(b - c + a) \ge 0$ and $(c - a - b)(c - a + b) \ge 0$. Multiplying these inequalities, we get

$$-(a + b - c)^2(b + c - a)^2(c + a - b)^2 \ge 0.$$

This forces that lhs is equal to zero. Hence, it follows that either $a + b = c$ or $b + c = a$ or $c = a = b$.

58. We use the trivial inequalities $a^2 + 1 \ge 2a$, $b^2 + 1 \ge 2b$ and $c^2 + 1 \ge 2c$. Hence, we obtain

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \ge \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b}$$

$$\frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \ge 3$$

Adding 6 both sides, this is equivalent to

$$(2a + 2b + 2c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \ge 9$$

Taking $x = b + c$, $y = c + a$, $z = a + b$, this is equivalent to

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge 9$$

This is a consequence of AM-GM inequality.

Aliter The substitutions $b + c = x$, $c + a = y$, $a + b = z$ leads to

$$\Sigma\frac{2a}{b+c} = \Sigma\frac{y + z - x}{x} = \Sigma\left(\frac{x}{y} + \frac{y}{x}\right) - 3 \ge 6 - 3 = 3$$

59. We gave $(2.2)^2 = 4.84 < 5$, so that $\sqrt{5} > 2.2$

Hence, $\sqrt[4]{5} > \sqrt{2.2} > 1.4$, as $(1.4)^2 = 1.96 < 2.2$, therefore $\sqrt[3]{5} > \sqrt[4]{5} > 1.4$

Adding, we get

$$\sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5} > 2.2 + 1.4 + 1.4 = 5$$

We observe that

$\sqrt{3} < 3$, $\sqrt[3]{8} = 2$ and $\sqrt[4]{8} < \sqrt[3]{8} = 2$. Thus,

$$\sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8} < 3 + 2 + 2 = 7 < 8$$

Suppose $n \ge 9$. Then $n^2 \ge 9n$, so that $n \ge 3\sqrt{n}$.

This gives $\sqrt{n} \le n/3$.

Therefore, $\sqrt[4]{n} < \sqrt[3]{n} < \sqrt{n} \le n/3$.

We thus obtain

$$\sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} < (n/3) + (n/3) + (n/3) = n$$

60. Let us look at the problem of finding two positive integers a, b such that $1 < a < b < a^2$, $a - 1$ divides $b - 1$ and $a^2 - 1$ divides $b^2 - 1$.

Thus, we have

$$b - 1 = k(a-1) \text{ and } b^2 - 1 = l(a^2 - 1)$$

Eliminating b from these equations, we get

$$(k^2 - l)a = k^2 - 2k + l.$$

Thus, it follows that

$$a = \frac{k^2 - 2k + l}{k^2 - l} = 1 - \frac{2(k-l)}{k^2 - l}$$

We need a to be an integer. Choose $k^2 - l = 2$ so that $a = 1 + l - k = k^2 - k - 1$ and $b = k(a-1) + 1 = k^3 - k^2 - 2k + 1$. We want $a > 1$ which is assured if we choose $k \geq 3$. Now, $a < b$ is equivalent to $(k^2 - 1)(k-2) > 0$ which again is assured once $k \geq 3$. It is easy to see that $b < a^2$ is equivalent to $k(k^3 - 3k^2 + 4) > 0$ and this is also true for all $k \geq 3$. Thus we define

$$a_n = (n+2)^2 - (n+2) - 1 = n^2 + 3n + 1,$$

$$b_n = (n+2)^3 - (n+2)^2 - 2(n+2) + 1$$
$$= n^3 + 5n^2 + 6n + 1,$$

for all $n \geq 1$. Then we see that

$$1 < a_n < b_n < b_n^2,$$

for all $n \geq 1$. Moreover

$$a_n - 1 = n(n+3),\ b_n - 1 = n(n+3)(n+2)$$

and $a_n^2 - 1 = n(n+3)(n+1)(n+2)$,

$$b_n^2 - 1 = n(n+3)(n+2)(n+1)(n^2 + 4n + 2)$$

Thus, we have a pair of desired sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

Level 2

1. Since, $a^2 + b^2 + c^2 = 1$, the inequalities to be proved may be written in the form

$$-\frac{1}{2}(a^2 + b^2 + c^2) \leq ab + bc + ca \leq a^2 + b^2 + c^2$$

or $-(a^2 + b^2 + c^2) \leq 2(ab + bc + ca)$

$$\leq 2(a^2 + b^2 + c^2).$$

These inequalities are indeed true since

$$2(ab + bc + ca) + (a^2 + b^2 + c^2) = (a + b + c)^2 \geq 0$$

and $2(a^2 + b^2 + c^2) - 2(ab + bc + ca)$

$$= (a-b)^2 + (a-c)^2 + (b-c)^2 \geq 0$$

2. Let $a = mc + r$ and $b = nc + s$, where m, n, r and s are integers and $0 \leq r < c$ and $0 \leq s < c$. As the coming proof will indicate, a certain relationship must be established; for conciseness of proof, we will establish that first

If $s \geq r$, then $2s + 1 \geq r + s + 1$

So, $\left[\frac{2s+1}{c}\right] \geq \left[\frac{r+s+1}{c}\right]$;

if $s < r$, then $2r \geq r + s + 1$ and

$$\left[\frac{2r}{c}\right] \geq \left[\frac{r+s+1}{c}\right];$$

in either case,

$$\left[\frac{2r}{c}\right] + \left[\frac{2s+1}{c}\right] \geq \left[\frac{r+s+1}{c}\right]$$

Furthermore, since $\left[\frac{r}{c}\right] = \left[\frac{s}{c}\right] = 0$, we can also say that

$$\left[\frac{2r}{c}\right] + \left[\frac{2s+1}{c}\right] \geq \left[\frac{r}{c}\right] + \left[\frac{s}{c}\right] + \left[\frac{r+s+1}{c}\right]$$

Now, the proof

$$\left[\frac{2a}{c}\right] + \left[\frac{2b+1}{c}\right] = \left[2m + \frac{2r}{c}\right] + \left[2n + \frac{2s+1}{c}\right]$$

$$= 2m + 2n + \left[\frac{2r}{c}\right] + \left[\frac{2s+1}{c}\right]$$

$$\geq m + \left[\frac{r}{c}\right] + n + \left[\frac{s}{c}\right] + m + n + \left[\frac{r+s+1}{c}\right]$$

$$= \left[m + \frac{r}{c}\right] + \left[n + \frac{s}{c}\right] + \left[m + n + \frac{r+s+1}{c}\right]$$

$$= \left[\frac{a}{c}\right] + \left[\frac{b}{c}\right] + \left[\frac{a+b+1}{c}\right]$$

3. Without loss of generality, we may assume that $a \geq b \geq c$.

$$a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4}(a+b+c)^3$$

$$\Leftrightarrow 4(a^3 + b^3 + c^3 + 6abc)$$
$$\geq (a^3 + b^3 + c^3) + 3ab(a+b) + 3bc(b+c) + 3ca(c+a) + 6abc$$

$$\Leftrightarrow (a^3 + b^3 + c^3 + 6abc) \geq ab(a+b) + bc(b+c) + ca(c+a)$$

We have, $a - c \geq b - c$

$$\Rightarrow a(a-c) \geq (b-c)$$

$$\Rightarrow a(a-c)(a-b) \geq b(b-c)(a-b)$$

$$\Rightarrow a(a-b)(a-c) + b(b-c)(b-a) \geq 0$$

Also, $c - a \leq 0$

and $c - b \leq 0$

Thus, $c(c-a)(c-b) \geq 0$

Adding these two inequalities, we get

$$a(a-b)(a-c) + b(b-c)(b-a) + (c-a)(c-b) \geq 0$$

Expanding the left hand side of this inequality, we get

$$a^3 + b^3 + c^3 - ab\,(a+b) - bc\,(b+c) - ca(c+a) + 3abc \ge 0$$

Thus, $ab(a+b) + bc(b+c) + ca(c+a)$

$$\le a^3 + b^3 + c^3 + 3abc \le a^2 + b^3 + c^3 + 6abc$$

The equality holds if and only if $a(a-b)(a-c) + b(b-c)(b-a) = 0$ and $c(c-a)(c-b) = 0$. The second equality holds if either $c = 0$ or $c - a = 0$ or $c - b = 0$. If $c = 0$, from the first equality, we get $a^2(a-b) + b^2(b-a) = 0 \Rightarrow (a-b)(a^2+b^2) = 0$ $\Rightarrow a = b$. If $c - a = 0$, then since $a \ge b \ge c$, it follows that $a = b = c$ and substituting in the equality we see that $a = b = c = 0$. If $c = b$, then $a(a-b)(a-b) = 0 \Rightarrow a = 0$ or $a = b$. If $a = b$, we see as before that $a = b = c = 0$.

This completes the proof.

4. By the power means inequality we have

$$\sqrt[n]{\frac{1+(n+1)^{n+1}}{n+2}} = \sqrt[n]{\frac{1+(n+1)^n + \ldots + (n+1)^{n+1}}{n+2}} > \sqrt[n-1]{\frac{1+(n+1)^{n-1} + \ldots + (n+1)^{n-1}}{n+2}}$$

If we take away one of the $(n+1)^{n-1}$ terms in this average, the average will obviously decrease, so

$$\sqrt[n]{\frac{1+(n+1)^{n+1}}{n+2}} > \sqrt[n-1]{\frac{1+(n+1)^{n-1} + \ldots + (n+1)^{n-1}}{n+1}} > \sqrt[n-1]{\frac{1+(n+1)^{n-1} + \ldots + n^{n-1}}{n+1}}$$

$$= \sqrt[n-1]{\frac{1+n\cdot n^{n-1}}{n+1}} = \sqrt[n-1]{\frac{1+n^n}{n+1}}$$

and the proof is complete.

5. Writing $y + z = a, z + x = b$ and $x + y = c$, we have

$x = b + c - a/2$, etc.,

thus

$$yz = \frac{a^2 - (b-c)^2}{4} = \frac{a^2 - b^2 - c^2 + 2bc}{4}, \text{ etc.}$$

and hence, in cyclic sum notation,

$$\Sigma yz = \frac{1}{4}\,\Sigma\,(2bc - a^2)$$

Now, assume $a \ge b \ge c$ without loss of generality; then

$$2bc - a^2 \le 2ca - b^2 \le 2ab - c^2$$

and therefore, by Chebyshev's inequality and the AM-GM inequality.

$$\frac{4}{9}(\Sigma yz)\left(\Sigma \frac{1}{(y+z)^2}\right) = \frac{1}{3}\,\Sigma\,(2bc - a^2)\frac{1}{3}\,\Sigma\,\frac{1}{a^2}$$

$$\ge \frac{1}{3}\,\Sigma\left((2bc - a^2)\cdot\frac{1}{a^2}\right) = \frac{1}{3}\left(\Sigma\frac{2bc}{a^2}\right) - 1$$

$$\ge \left(\Pi\frac{2bc}{a^2}\right)^{1/3} - 1 = 2 - 1 = 1$$

The inequality follows.

6. By the AM-GM inequality,

$$(a+b)^2 + (a+b+4c)^2 = (a+b)^2 + (a+2c+b+2c)^2 \ge (2\sqrt{ab})^2 + (2\sqrt{2ac} + 2\sqrt{2bc})^2 = 4ab + 8ac + 8bc + 16c\sqrt{ab}$$

Therefore,

$$\frac{(a+b)^2 + (a+b+4c)^2}{abc}\cdot(a+b+c)$$

$$\ge \frac{4ab + 8ac + 8bc + 16c\sqrt{ab}}{abc} \times (a+b+c)$$

$$= \left(\frac{4}{c} + \frac{8}{b} + \frac{8}{a} + \frac{16}{\sqrt{ab}}\right)(a+b+c)$$

$$= 8\left(\frac{1}{2c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{ab}}\right)\left(\frac{a}{2} + \frac{a}{2} + \frac{b}{2} + \frac{b}{2} + c\right)$$

$$\ge 8\left(5\sqrt[5]{\frac{1}{2a^2b^2c}}\right)\left(5\frac{\sqrt[5]{a^2b^2c}}{2^4}\right) = 100$$

Again, by the AM-GM inequality. Hence, the largest constant k is 100. For $k = 100$, equality holds if and only if $a = b = 2c > 0$

7. Using the weighted AM-GM inequality three times, we have the following

$$\frac{c\cdot a + a\cdot b + b\cdot c}{c+a+b} \ge (a^c b^a c^b)^{\frac{1}{a+b+c}}$$

$$\frac{b\cdot a + c\cdot b + a\cdot c}{b+c+a} \ge (a^b b^c c^a)^{\frac{1}{a+b+c}}$$

$$\frac{a\cdot a+b\cdot b+c\cdot c}{a+b+c}\geq(a^a b^b c^c)^{\frac{1}{a+b+c}}$$

Adding these inequalities together gives

$$1=a+b+c=\frac{(a+b+c)^2}{a+b+c}$$

$$=\frac{a^2+b^2+c^2+2ab+2ac+2bc}{a+b+c}$$

$$=a^a b^b c^c+a^b b^c c^a+a^c b^a c^b$$

8. We have, $\left(\frac{a}{5}\right)^2+\left(\frac{b}{5}\right)^2+\left(\frac{c}{5}\right)^2-2$

$$\left[\frac{ax}{30}+\frac{by}{30}+\frac{cz}{30}\right]+\left(\frac{x}{6}\right)^2+\left(\frac{y}{6}\right)^2+\left(\frac{z}{6}\right)^2$$

$$=1-2+1=0$$

$$=\left(\frac{a}{5}-\frac{x}{6}\right)^2=\left(\frac{b}{5}-\frac{y}{6}\right)^2+\left(\frac{c}{5}-\frac{z}{6}\right)^2;$$

thus $\frac{a}{5}=\frac{x}{6}$, so $a=kx$ $\left[\text{where } k=\frac{5}{6}\right]$

and $b=ky$ and $c=kz$.

The answer is then k, or $\frac{5}{6}$. In a geometric setting, if the given information is applied to triangles ABC and XYZ (even replacing the 25, 36 and 30 by r^2, s^2 and r, s respectively), the triangles must then be similar.

9. We note that in the

$S=x^k+x^{k-1}+x^{k-2}+\ldots+x+1$ if $x>1$,

then the first term is numerically the greatest, but if $x<1$, then the last term is greatest. It follows that

$$(k+1)x^k>S>k+1, \text{ if } x>1;$$

$$(k+1)x^k<S<k+1, \text{ if } x<1.$$

If both sides of these inequalities are multiplied by $x-1$, it is found that for $x\neq 1$

$(k+1)x^k(x-1)>x^{k+1}-1>(k+1)(x-1)$.

Assume now that $x=\frac{p}{(p-1)}$; then we find

$$\frac{(k+1)p^k}{(p-1)^{k+1}}>\frac{p^{k+1}-(p-1)^{k+1}}{(p-1)^{k+1}}>\frac{k+1-(p-1)^k}{(p-1)^{k+1}}$$

Analogously, if we assume that $x=\frac{p+1}{p}$, we obtain

$$\frac{(k+1)(p+1)^k}{p^{k+1}}>\frac{(p+1)^{k+1}-p^{k+1}}{p^{k+1}}>\frac{(k+1)p^k}{p^{k+1}}$$

If follows that

$$(p+1)^{k+1}-p^{k+1}>(k+1)p^k>p^{k+1}-(p-1)^{k+1}$$

or letting p successively have the values $1, 2, 3, \ldots, n$

$$2^{k+1}-1^{k+1}>(k+1)1^k>1^{k+1}-0,$$

$$3^{k+1}-2^{k+1}>(k+1)2^k>2^{k+1}-1^{k+1},$$

$$4^{k+1}-3^{k+1}>(k+1)3^k>3^{k+1}-2^{k+1},$$

....

$$(n+1)^{k+1}-n^{k+1}>(k+1)n^k>n^{k+1}-(n-1)^{k+1}$$

If these inequalities are added together, the following inequalities result;

$$(n+1)^{k+1}-1>(k+1)(1^k+2^k+3^k+\ldots+n^k)>n^{k+1},$$

or dividing through these inequalities by $k+1$,

$$\left[\left(1+\frac{1}{n}\right)^{k+1}-\frac{1}{n^{k+1}}\right]\frac{1}{k+1}n^{k+1}$$

$$>1^k+2^k+3^k+\ldots+n^k>\frac{1}{k+1}n^{k+1}.$$

This is essentially the set of inequalities sought.

10. From $a\geq b\geq c>0$, we have

$$\frac{a+b}{c}\geq 2,\ 0<\frac{b+c}{a}\leq 2 \text{ and } \frac{a+c}{b}\geq 1$$

Now, we get $\frac{a^2-b^2}{c}\geq 2(a-b)$, because $a\geq b$;

$\frac{c^2-b^2}{a}\geq 2(c-b)$, because $c\leq b$; and

$\frac{a^2-c^2}{b}\geq a-c$, because $a\geq c$.

After addition of these inequalities, we have

$$\frac{a^2-b^2}{a}+\frac{c^2-b^2}{a}+\frac{a^2-c^2}{b}$$

$$\geq 2(a-b)+2(c-b)+(a-c)$$

that is,

$$\frac{a^2-b^2}{c}+\frac{c^2-b^2}{a}+\frac{a^2-c^2}{b}\geq 3a-4b+c$$

The equality holds if and only if $a=b=c>0$

11. $S=\frac{1}{1001}+\ldots+\frac{1}{3001}=\left(\frac{1}{1001}+\frac{1}{3001}\right)$

$$+\ldots+\left(\frac{1}{2000}+\frac{1}{2002}\right)+\frac{1}{2001}$$

For any n, we have

$$n^2 - 4002\,n + 2001^2 \geq 0 \text{ and}$$

hence $\qquad n(4002 - n) \leq 2001^2$

$$\therefore S > 4002\left[\underbrace{\frac{1}{2001^2} + \ldots + \frac{1}{2001^2}}_{1000 \text{ terms}}\right]$$

$$+ \frac{1}{2001} = 4002 \cdot \frac{1000}{2001^2} + \frac{1}{2001}$$

$$= \frac{2000 + 1}{2001} = 1$$

Again, taking terms hundred at a time

$$S = \frac{1}{1001} + \ldots + \frac{1}{3001} < 100$$

$$\left(\frac{1}{1001} + \frac{1}{1101} + \ldots + \frac{1}{2901}\right)$$

$$< 100\left(\frac{1}{1000} + \frac{1}{1100} + \ldots + \frac{1}{2900}\right)$$

$$< \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{29} < 5\left(\frac{1}{10} + \frac{1}{15} + \frac{1}{20} + \frac{1}{25}\right)$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1\frac{17}{60} < 1\frac{1}{3}$$

Hence, $\quad 1 < S < 1\frac{1}{3}$

12. From what has been given we have, for $1 \leq i \leq n - 1$,

$$a_i - a_{i+1} \leq |a_i - a_{i+1}| < 1 < a_{i+1}$$

$$\Rightarrow \quad a_i < 2a_{i+1}$$

Hence, each term in the sum

$$S = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_{n-1}}{a_n}$$

is less than 2. Suppose that k terms in this sum are ≥ 1 and the remaining $n - 1 - k$ terms are < 1. Then

$$S \leq 2k + (n - 1 - k)\cdot 1 = n - 1 + k$$

Next, $a_n - a_1 = (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})$

each term on the right hand side is less than 1. If $a_{i+1} - a_i > 0$, then $a_{i+1} > a_i$, or $a_{i+1} < 1$. Hence, there are only $n - 1 - k$ positive terms in this sum for $a_n - a_1$. Thus

$$a_n - a_1 \leq (n - 1 - k) \times 1 + k \times 0$$

$$= n - 1 - k < (n - 1 - k)\,a_1$$

i.e., $\quad a_n < (n - k)\,a_1, \dfrac{a_n}{a_1} < n - k$

Hence, the given sum

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

$$< n - 1 + k + n - k = 2n - 1$$

This completes the proof.

13. Proof by induction. Given inequality is true for $n = 1$. Assume it to be true as such and show that when n is replaced by $(n + 1)$ too it is true. Rewriting, the inequality is

$$\frac{1}{n+1}\left\{1 + \frac{1}{3} + \ldots + \frac{1}{2n-1}\right\}$$

$$\geq \frac{1}{n}\left\{\frac{1}{2} + \ldots + \frac{1}{2n}\right\}$$

$$= \frac{n+1}{n} \cdot \frac{1}{n+1}\left\{\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\right\}$$

$$= \left(1 + \frac{1}{n}\right) \cdot \frac{1}{n+1}\left\{\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\right\}$$

which is equivalent to

$$\frac{1}{n+1}\left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{5} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}\right\}$$

$$\geq \frac{1}{n(n+1)}\left\{\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\right\}$$

i.e., $\left\{1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{5} + \ldots + \dfrac{1}{2n-1} - \dfrac{1}{2n}\right\}$

$$\geq \frac{1}{n}\left\{\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\right\}$$

(As $n \to \infty$, note that LHS converges to log 2 while RHS converges to $\lim\limits_{n \to \infty} \frac{1}{2} \cdot \frac{\log n}{n} = 0$.)

Assuming the last inequality we are to prove

$$1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$+ \frac{1}{2n+1} - \frac{1}{2n+2} \geq \frac{1}{n+1}\left\{\frac{1}{2} + \ldots + \frac{1}{2n+2}\right\}$$

lhs (by induction hypothesis)

$$\geq \frac{1}{n}\left\{\frac{1}{2} + \ldots + \frac{1}{2n}\right\} + \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$= \frac{n+1}{n(n+1)}\left\{\frac{1}{2} + \ldots + \frac{1}{2n}\right\} + \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$= \left(1 + \frac{1}{n}\right)\left(\frac{1}{n+1}\right)\left\{\frac{1}{2} + \ldots + \frac{1}{2n}\right\}$$

$$+ \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$\geq \frac{1}{n+1}\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)+\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)$$

$$\frac{1}{n(n+1)}+\frac{1}{2n+1}-\frac{1}{2n+2}$$

If suffices to show that

$$\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)\frac{1}{n(n+1)}+\frac{1}{2n+1}-\frac{1}{2n+2}$$

$$\geq \frac{1}{n+1}\cdot\frac{1}{2n+2}$$

$$\text{lhs} \geq \frac{1}{2n(n+1)}+\frac{1}{2n+1}-\frac{1}{2n+2}$$

$$=\frac{1}{2n(n+1)}+\frac{1}{(2n+1)(2n+2)}$$

$$=\frac{1}{2n(n+1)}+\frac{1}{2(n+1)(2n+2)}$$

$$=\frac{1}{2(n+1)}\left[\frac{1}{n}+\frac{1}{2n+1}\right]$$

$$=\frac{3n+1}{2(n+1)n(2n+1)}=\frac{3n+1}{2n(n+1)(2n+1)}$$

$$\geq \frac{1}{n+1}\cdot\frac{1}{2n+2}$$

14. Let us consider the sum $1+1/4+1/9+\ldots+1/n^2$ with n smaller than 2^{k+1} and also the sum $1+1/2^2+1/3^2+\ldots+1/(2^{k+1}-1)^2$.

On grouping the terms

$$1+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}\right)+\ldots+$$

$$\left(\frac{1}{(2^k)^2}+\frac{1}{(2^k+1)^2}+\ldots+\frac{1}{(2^{k+1}-1)^2}\right)$$

$$<1+\left(\frac{1}{2^2}+\frac{1}{2^2}\right)+\left(\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}\right)+\ldots+$$

$$\left(\frac{1}{(2^k)^2}+\frac{1}{(2^k)^2}+\ldots+\frac{1}{(2^k)^2}\right)$$

$$=1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^k}=\frac{1-\frac{1}{2^{k+1}}}{1-\frac{1}{2}}=2-\frac{1}{2^k}<2$$

which is what we intended to prove.

Remark

In a completely similar manner we can show that if α is a number greater than 1, then

$$1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\ldots+\frac{1}{n^\alpha}<\frac{2^{\alpha-1}}{2^{\alpha-1}-1}$$

for any n

Thus, for any $\alpha>1$ the sum

$$1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\ldots+\frac{1}{n^\alpha}$$

remains bounded for arbitrarily large n it follows that for $\alpha \leq 1$ the sum $1+1/2^\alpha+1/3^\alpha+\ldots+1/n^\alpha$ can be made arbitrarily large by taking a sufficiently large value of n.

15. Since, λ is a root of the equation $x^3+ax^2+bx+c=0$, we have

$$\lambda^3=-a\lambda^2-b\lambda-c$$

This implies that

$$\lambda^4=-a\lambda^3-b\lambda^2-c\lambda$$

$$=(1-a)\lambda^3+(a-b)\lambda^2+(b-c)\lambda+c$$

where we have used again

$$-\lambda^3-a\lambda^2-b\lambda-c=0$$

Suppose $|\lambda|\geq 1$. Then we obtain

$$|\lambda|^4 \leq (1-a)|\lambda|^3+(a-b)|\lambda|^2+(b-c)|\lambda|+c$$

$$\leq (1-a)|\lambda|^3+(a-b)|\lambda|^3+(b-c)|\lambda|^3+c|\lambda|^3$$

$$\leq |\lambda|^3$$

This shows that $|\lambda|\leq 1$. Hence, the only possibility in this case is $|\lambda|=1$. We conclude that $|\lambda|\leq 1$ is always true.

16. Note that the inequality is symmetric in a, b, c so that we may assume that $a\geq b\geq c$. Since $abc=1$, it follows that $a\geq 1$ and $c\leq 1$. Using $b=1/ac$, we get

$$a^{b+c}b^{c+a}c^{a+b}=\frac{a^{b+c}c^{a+b}}{a^{c+a}+c^{c+a}}$$

$$=\frac{c^{b-c}}{a^{a-b}}\leq 1$$

because $c\leq 1, b\geq c, a\geq 1$ and $a\geq b$.

17. We have, from the AM-GM inequality, that

$$xy\leq\left(\frac{x+y}{2}\right)^2=1$$

Thus, we obtain $0<xy\leq 1$. We write

$$x^3y^3(x^3+y^3)=(xy)^3(x+y)(x^2-xy+y^2)$$

$$=2(xy)^3((x+y)^2-3xy)$$

$$=2(xy)^3(4-3xy)$$

Thus, we need to prove that

$$(xy)^3(4-3xy)\leq 1$$

Putting $z=xy$, this inequality reduces to

$$z^3(4-3z)\leq 1$$

for $0 < z \le 1$. We can prove this in different ways. We can put the inequality in the form

$$3z^4 - 4z^3 + 1 \ge 0$$

Here, the expression in the LHS factors to $(z-1)^2(3z^2 + 2z + 1)$ and $(3z^2 + 2z + 1)$ is positive since its discriminant $D = -8 < 0$. Or applying the AM-GM inequality to the positive reals $4 - 3z, z, z, z$ we obtain

$$z^3(4 - 3z) \le \left(\frac{4 - 3z + 3z}{4}\right)^4 \le 1$$

18. It is easy to observe that there is a triangle with sides $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$. Using Heron's formula, we get

$$16[ABC]^2 = (a + b + c)(a + b - c)(b + c - a)(c + a - b)$$

and $$16[A_1B_1C_1]^2 = \frac{3}{16}(a + b + c)(-a + b + 3c)(-b + c + 3a)(-c + a + 3b)$$

Since a, b, c are the sides of a triangle, there are positive real numbers p, q, r such that $a = q + r, b = r + p, c = p + q$. Using these relations, we obtain

$$\frac{[ABC]^2}{[A_1B_1C_1]^2} = \frac{16pqr}{3(2p + q)(2q + r)(2r + p)}$$

Thus, it is sufficient to prove that

$$(2p + q)(2q + r)(2r + p) \ge 27pqr$$

for positive real numbers p, q, r. Using AM-GM inequality, we get

$$2p + q \ge 3\,(p^2q)^{1/3}, 2q + r \ge 3\,(q^2r)^{1/3},$$
$$2r + p \ge 3(r^2p)^{1/3}$$

Multiplying these, we obtain the desired result. We also observe that equality holds if and only if $p = q = r$. This is equivalent to the statement that ABC is equilateral.

19. We have

$$\frac{77}{17} < \frac{\beta}{\alpha} < \frac{197}{43}$$

That is,

$$4 + \frac{9}{17} < \frac{\beta}{\alpha} < 4 + \frac{25}{43}$$

Thus, $4 < \frac{\beta}{\alpha} < 5$. Since, α and β are positive integers, we may write $\beta = 4\alpha + x$, where $0 < x < \alpha$.

Now, we get

$$4 + \frac{9}{17} < 4 + \frac{x}{\alpha} < 4 + \frac{25}{43}$$

So, $\frac{9}{17} < \frac{x}{\alpha} < \frac{25}{43}$; that is, $\frac{43x}{25} < \alpha < \frac{17x}{9}$

We find the smallest value of x for which α becomes a well-defined integer. For $x = 1, 2, 3$ the bounds of α are respectively

$$\left(1\frac{18}{25}, 1\frac{8}{9}\right), \left(3\frac{11}{25}, 3\frac{7}{9}\right), \left(5\frac{4}{9}, 5\frac{2}{3}\right)$$

None of these pairs contain an integer between them.

For $x = 4$, we have

$$\frac{43x}{25} = 6\frac{12}{25} \text{ and } \frac{17x}{9} = 7\frac{5}{9}.$$

Hence, in this case

$$\alpha = 7 \text{ and } \beta = 4\alpha + x = 28 + 4 = 32$$

This is also the least possible value, because, if $x \ge 5$, then $\alpha > \frac{43x}{25} \ge \frac{43}{5} > 8$ and so $\beta > 37$. Hence, the minimum possible value of β is 32.

20. (i) Let $n \ge 4011^2$ and $m \in N$ be such that $m^2 \le n < (m+1)^2$. Then

$$\left(1 + \frac{1}{2005}\right)n - (m+1)^2$$
$$\ge \left(1 + \frac{1}{2005}\right)m^2 - (m+1)^2$$
$$= \frac{m^2}{2005} - 2m - 1$$
$$= \frac{1}{2005}(m^2 - 4010m - 2005)$$
$$= \frac{1}{2005}[(m - 2005)^2 - 2005^2 - 2005]$$
$$\ge \frac{1}{2005}[(4011 - 2005)^2 - 2005^2 - 2005]$$
$$= \frac{1}{2005}(2006^2 - 2005^2 - 2005)$$
$$= \frac{1}{2005}(4011 - 2005) = \frac{2006}{2005} > 0$$

Thus, we get $n < (m+1)^2 < \left(1 + \frac{1}{2005}\right)n$ and $l^2 = (m+1)^2$ is the desired square.

(ii) We show that $M = 4010^2 + 1$ is the required least number. Suppose $n \ge M$. Write $n = 4010^2 + k$, where k is a positive integer. Note that we may assume $n < 4011^2$ by part (i). Now,

$$\left(1+\frac{1}{2005}\right)n - 4011^2$$
$$= \left(1+\frac{1}{2005}\right)(4010^2 + k) - 4011^2$$
$$= 4010^2 + 2\cdot 4010 + k + \frac{k}{2005} - 4011^2$$
$$= (4010+1)^2 + (k-1) + \frac{k}{2005} - 4011^2$$
$$= (k-1) + \frac{k}{2005} > 0$$

Thus, we obtain

$$4010^2 < n < 4011^2 < \left(1+\frac{1}{2005}\right)n.$$

We check that $M = 4010^2$ will not work.

For suppose $n = 4010^2$. Then

$$\left(1+\frac{1}{2005}\right)4010^2 = 4010^2 + 2\cdot 4010$$
$$= 4011^2 - 1 < 4011^2$$

Thus, there is no square integer between n and $\left(1+\frac{1}{2005}\right)n$.

This proves (ii).

21. We begin with the observation that

$$x^2 + xy + y^2$$
$$= \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2 \le \frac{3}{4}(x+y)^2,$$

and similar bounds for

$$y^2 + yz + z^2, z^2 + zx + x^2$$

Thus,

$$3(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2)$$
$$\ge \frac{81}{64}(x+y)^2 (y+z)^2 (z+x)^2$$

Thus, it is sufficient to prove that

$$(x+y+z)(xy+yz+zx)$$
$$\le \frac{9}{8}(x+y)(y+z)(z+x)$$

Equivalently, we need to prove that

$$8(x+y+z)(xy+yz+zx)$$
$$\le 9(x+y)(y+z)(z+x)$$

However, we note that

$$(x+y)(y+z)(z+x)$$
$$= (x+y+z)(yz+zx+xy) - xyz$$

Thus, the required inequality takes the form

$$(x+y)(y+z)(z+x) \ge 8xyz$$

This follows from AM-GM inequalities

$$x+y \ge 2\sqrt{xy}, y+z \ge 2\sqrt{yx}, z+x \ge \sqrt{zx}$$

Aliter

Let us introduce $x+y=c, y+z=a$ and $z+x=b$. Then a, b, c are the sides of a triangle. If $s = (a+b+c)/2$, then it is easy to calculate $x = s-a, y = s-b, z = s-c$ and $x+y+z = s$. We also observe that

$$x^2 + xy + y^2 = (x+y)^2 - xy$$
$$= c^2 - \frac{1}{4}(c+a-b)(c+b-a)$$
$$= \frac{3}{4}c^2 + \frac{1}{4}(a-b)^2 \ge \frac{3}{4}c^2$$

Moreover,

$$xy + yz + zx = (s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)$$

Thus, it is sufficient to prove that

$$s\,\Sigma\,(s-a)(s-b) \le \frac{9}{8}abc$$

But $\Sigma(s-a)(s-b) = r(4R+r)$, where r, R are respectively the inradius, the circumradius of the triangle whose sides are a, b, c and $abc = 4Rrs$. Thus, the inequality reduces to

$$r(4R+r) \le \frac{9}{2}Rr$$

This is simply $2r \le R$. This follows from $IO^2 = R(R-2r)$, where I is the incentre and O the circumcentre.

Aliter

If we set $x = \lambda a, y = \lambda b, z = \lambda c$, then the inequality changes to

$$(a+b+c)^2(ab+bc+ca)^2$$
$$\le 3(a^2+ab+b^2)(b^2+bc+c^2)$$
$$(c^2+ca+a^2)$$

This shows that we may assume $x+y+z=1$. Let $\alpha = xy+yz+zx$. We see that

$$x^2 + xy + y^2 = (x+y)^2 - xy$$
$$= (x+y)(1-z) - xy$$
$$= x+y-\alpha = 1-z-\alpha$$

Thus, $\quad \Pi(x^2+xy+y^2)$

$$= (1-\alpha-z)(1-\alpha-x)(1-\alpha-y)$$
$$= (1-\alpha)^3 - (1-\alpha)^2 + (1-\alpha)\alpha - xyz$$
$$= \alpha^2 - \alpha^3 - xyz$$

Thus, we need to prove that $\alpha^2 \le 3(\alpha^2 - \alpha^3 - xyz)$. This reduces to

$$3xyz \le \alpha^2 (2 - 3\alpha)$$

However,

$$3\alpha = 3\,(xy + yz + zx) \le (x + y + z)^2 = 1,$$

so that $2 - 3\alpha \ge 1$. Thus, it suffices to prove that $3xyz \le \alpha^2$. But

$$\alpha^2 - 3xyz = (xy + yz + zx)^2 - 3xyz\,(x + y + z)$$

$$= \sum_{\text{cyclic}} x^2y^2 - xyz(x + y + z)$$

$$= \frac{1}{2} \sum_{\text{cyclic}} (xy - yz)^2 \ge 0$$

22. The given inequality may be written in the form

$$7c^2 - 6(a + b)c - (a^2 + b^2 - 6ab) < 0.$$

Putting $x = 7c^2, y = -6\,(a + b)c$,

$$z = -(a^2 + b^2 - 6ab),$$

we have to prove that $x + y + z < 0$. Observe that x, y, z are not all equal $(x > 0, y < 0)$. Using the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)$$
$$[(x - y)^2 + (y - z)^2 + (z - x)^2].$$

we infer that it is sufficient to prove

$$x^3 + y^3 + z^3 - 3xyz < 0$$

Substituting the values of x, y, z we see that this is equivalent to

$$343c^6 - 216(a + b)^3c^3 - (a^2 + b^2 - 6ab)^3$$
$$- 126c^3 (a + b)(a^2 + b^2 - 6ab) < 0$$

Using $c^3 = a^3 + b^3$, this reduces to

$$343(a^3 + b^3)^2 - 216(a + b)^3(a^2 + b^3)$$
$$- (a^2 + b^2 - 6ab)^3 - 126\,[(a^3 + b^3)$$
$$(a + b)(a^2 + b^2 - 6ab)] < 0$$

This may be simplified (after some tedious calculations) to

$$-a^2b^2(129a^2 - 254ab + 129b^2) < 0$$

But $129a^2 - 254ab + 129b^2$

$$= 129(a - b)^2 + 4ab > 0$$

Hence, the result follows.

Remark

The best constant θ in the inequality $a^2 + b^2 - c^2 \ge \theta\,(c - a)\,(c - b)$, where a, b, c are positive reals such that $a^3 + b^3 = c^3$, is $\theta = 2\,(1 + 2^{1/3} + 2^{-1/3})$. Here again, there were some beautiful solutions given by students.

1. We have

$$a^3 = c^3 - b^3 = (c - b)(c^2 + cb + b^2),$$

which is same as $\dfrac{a^2}{c - b} = \dfrac{c^2 + cb + b^2}{a}$

Similarly, we get

$$\frac{b^2}{c - a} = \frac{c^2 + ca + a^2}{b}$$

We observe that

$$\frac{a^2}{c - b} + \frac{b^2}{c - a} = \frac{c(a^2 + b^2) - a^3 - b^3}{(c - a)(c - b)}$$

$$= \frac{c(a^2 + b^2 - c^2)}{(c - a)(c - b)}$$

This shows that

$$\frac{a^2 + b^2 - c^2}{(c - a)(c - b)} = \frac{c^2 + cb + b^2}{ca} + \frac{c^2 + ca + a^2}{cb}$$

Thus, it is sufficient to prove that

$$\frac{c^2 + cb + b^2}{ca} + \frac{c^2 + ca + a^2}{cb} \ge 6$$

However, we have

$c^2 + b^2 \ge 2cb$ and $c^2 + a^2 \ge 2ca$.

Hence, $\dfrac{c^2 + cb + b^2}{ca} + \dfrac{c^2 + ca + a^2}{cb}$

$$\ge 3\left(\frac{b}{a} + \frac{a}{b}\right) \ge 3 \times 2 = 6$$

We have used AM-GM inequality.

2. Let us set $x = a/c$ and $y = b/c$. Then $x^3 + y^3 = 1$ and the inequality to be proved is

$$x^2 + y^2 - 1 > 6\,(1 - x)(1 - y)$$

This reduces to

$$(x + y)^2 + 6\,(x + y) - 8xy - 7 > 0 \qquad \text{...(i)}$$

But $1 = x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

which gives $xy = [(x + y)^3 - 1]/3(x + y)$.

Substituting this in (i) and introducing $x + y = t$, the inequality takes the form

$$t^2 + 6t - \frac{8}{3} \cdot \frac{(t^3 - 1)}{t} - 7 > 0 \qquad \text{...(ii)}$$

This may be simplified to

$$-5t^3 + 18t^2 - 2t + 8 > 0$$

Equivalently

$$-(5t - 8)(t - 1)^2 > 0$$

Thus, we need to prove that $5t < 8$. Observe that $(x + y)^3 > x^3 + y^3 = 1$, so that $t > 1$. We also have

$$\left(\frac{x + y}{2}\right) \le \frac{x^3 + y^3}{2} = \frac{1}{2}$$

This shows that $t^3 \le 4$. Thus,

$$\left(\frac{5t}{8}\right)^3 \le \frac{125 \times 4}{512} = \frac{500}{512} < 1.$$

Hence, $5t < 8$, which proves the given inequality.

3. We write $b^3 = c^3 - a^3$ and $a^3 = c^3 - b^3$ so that

$$c - a = \frac{b^3}{c^2 - ca + a^2}, \quad c - b = \frac{a^3}{c^2 - cb + b^2}$$

Thus, the inequality reduces to

$$a^2 + b^2 - c^2 > 6 \cdot \frac{a^3b^3}{(c^2 - ca + a^2)(c^2 - cb + b^2)}$$

This simplifies (after some lengthy calculations) to

$$-c^6 - (a + b)c^5 - abc^4 + (a^3 + b^3)c^3$$
$$+ (a^4 + a^3b + a^2b^2 + ab^3 + b^4)c^2$$
$$(a^2b + ab^2 + a^3 + b^3)abc + (a^4b^2 - 6a^3b^3 + a^2b^4) > 0$$

Substituting

$$c^3 = a^3 + b^3, c^4 = c(a^3 + b^3),$$
$$c^5 = c^2(a^3 + b^3), c^6 = (a^3 + b^3)^2$$

the inequality further reduces to

$$a^2b^2(a^2 + b^2 + c^2 + ac + bc - 6ab) > 0$$

Thus, we need to prove that

$$a^2 + b^2 + c^2 + ac + bc - 6ab > 0$$

Since, $a^2 + b^2 \ge 2ab$, it is enough to prove that $c^2 + c(a + b) - 4ab > 0$. Multiplying this by c and using $a^3 + b^3 = c^3$, we need to prove that

$$a^3 + b^3 + c^2a + c^2b > 4abc$$

Using AM-GM inequality to these 4 terms and using $c > a$, $c > b$, we get

$$a^3 + b^3 + c^2a + c^2b > 4(a^3b^3c^2ac^2b)^{1/4}$$
$$= 4abc$$

which proves the inequality.

Unit 4 Combinatorics

Unit 4

Combinatorics

Basic Rule of Counting

In discrete mathematics and everyday walk of life, we always face the problem of counting the things or finding the number of ways in which we can perform a certain activity.

Let us have some examples

1. Suppose there are 3 junior colleges attached to senior colleges 5 junior colleges attached to schools and 6 separate junior colleges. Then, how many options are there to Mr. X for taking admission to his son in a junior college ?
2. Suppose there are 10 locks and corresponding 10 keys. A child insert the keys into the locks at random. In how many different ways can a child do this?

   In how many cases all the key will go wrong ?
3. If we write 1 to 100000, how many times the digit 7 will be written?
4. In how many different ways, the letters of 'WWW DOT COM' can be arranged in a row? Out of these arrangements, how many of them have vowels in symmetric positions?
5. How many different 6 digits numbers can be formed by using six digits 4, 4, 2, 2, 2, 5?

These examples varies from very simple childish problems to harder one.

Combinatorics (Combinatorial Analysis) is the branch of mathematics which develops the rules and find solutions to such problems.

There are two basic rules.

1. The Rule of Addition
2. The Rule of Multiplication

Before discussing the rule, let us take the above example 1 of Mr. X

As there are 3 junior colleges attached to senior college, 5 junior colleges attached to school and 6 separate junior colleges.

So, Mr. X has $3 + 5 + 6 = 14$ options while seeking the admission to his son in a junior college. This is the rule of addition.

The Rule of Addition

If a collection of objects consists of r_1 distinct objects of type 1, r_2 distinct objects of type 2, ..., r_n distinct objects of type n, then the total number of options to pick an object from the collection is

$$r_1 + r_2 + r_3 + \dots + r_n.$$

This rule can be put in set notation also.

Let us $|S|$ denote the number of elements in the set S, then

If $S_1, S_2, \ldots, S_n$ are pair-wise disjoint sets, then

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = |S_1| + |S_2| + \ldots + |S_n|$$

n sets must be pair-wise disjoint.

If this condition is violated, then there are chances of misuse of rule.

Let us take the example 2 in which a child is playing with 10 locks and 10 keys. His experiment of inserting the keys into the locks consists of 10 successive activities.

First activity To insert the first key into one of the available locks.

Second activity To insert the second key into one of the available locks.

Now, while performing the first activity, *i.e.*, to insert the first key k_1 he has 10 available options *viz.*, 10 locks $L_1, L_2, \ldots, L_{10}$. First activity can be performed in 10 different ways. When second activity of the experiment starts, there are 9 locks available to insert the second key. Hence, second activity can be performed in 9 different ways.

The total number of ways of performing the first and second activity is $10 \times 9 = 90$ continuing, in the same way the 10 successive activities can be performed in $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800$ different ways.

This is the rule of multiplication.

Proof of the Rule of Addition (By Principle of Induction)

Corresponding to a +ve integer 'n', consider the statement

$P(n)$: If $S_1, S_2, \ldots, S_n$ are pair-wise disjoint sets then

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = |S_1| + |S_2| + \ldots + |S_n|$$

If $n = 1$, then there is only one set

$$S_1 = \{a_1, a_2, \ldots, a_k\}$$

and one object can be chosen in k ways, so $P(1)$ is true.

If

$$S_1 = \{a_1, a_2, \ldots, a_k\}$$
$$S_2 = \{b_1, b_2, \ldots, b_l\}$$

are two disjoint sets

i.e., $a_i \neq b_j$, then

$$S_1 \cup S_2 = \{a_1, a_2, \ldots a_k, b_1, b_2, \ldots, b_l\}$$
$$|S_1 \cup S_2| = k + l = |S_1| + |S_2|$$

So, $P(2)$ is also true.

Assume that $P(n)$ is true.

Now, consider $n + 1$ sets $S_1, S_2, \ldots, S_n, S_{n+1}$, so that any element of S_{n+1} is not previously listed.

Then,

$$\begin{aligned} &|S_1 \cup S_2 \cup \ldots \cup S_n \cup S_{n+1}| \\ &= |(S_1 \cup S_2 \cup \ldots \cup S_n) \cup S_{n+1}| \\ &= |S_1 \cup S_2 \cup \ldots \cup S_n| + |S_{n+1}| \\ &= |S_1| + |S_2| \ldots + |S_n| + |S_{n+1}| \qquad \text{(since by hypothesis)} \end{aligned}$$

Thus, $P(n + 1)$ is true whenever $P(n)$ is true.

Hence, by the principle of induction $P(n)$ is true for every +ve integer 'n'.

The Rule of Multiplication

If an experiment consists of n successive activities, where the first activity can be performed in r_1 different ways, second activity performed in r_2 different ways, ... and nth activity can be performed in r_n different ways.

Then, the total number of ways in which experiment can be performed is $r_1, r_2, \ldots, r_n$

In set notation, this rule is stated as.
If $S_1, S_2, \ldots, S_n$ are n sets, then

$$|S_1 \times S_2 \times \ldots \times S_n| = |S_1||S_2|\ldots|S_n|$$

Here again it is to be remembered that the n successive activities must be different from each other, otherwise the application of the rule results in wrong answer.

Proof of the Rule of Multiplication (By Induction Method)

Corresponding to +ve integer $n \geq 2$

Consider the statement $\quad P(n): |S_1 \times S_2 \times \ldots \times S_n| = |S_1||S_2|\ldots|S_n|$

If $\quad S_1 = \{a_1, a_2, \ldots, a_k\}$

$\quad S_2 = \{b_1, b_2, \ldots, b_l\}$

Then, $S_1 \times S_2$ consists of kl ordered pairs

$$(a_1, b_1), \ldots, (a_1, b_l), \ldots, (a_k, b_1), \ldots, (a_k b_l)$$
$$|S_1 \times S_2| = kl = |S_1||S_2|$$

This shows that $P(2)$ is true.

Assume that $P(n)$ is true *i.e.,*

$$|S_1 \times S_2 \times \ldots \times S_n| = |S_1||S_2|\ldots|S_n|$$

Then, for next +ve integer $n+1$, we have

$$\begin{aligned}|S_1 \times S_2 \times \ldots \times S_n \times S_{n+1}| &= |(S_1 \times S_2 \times \ldots \times S_n) \times S_{n+1}| \\ &= |S_1 \times S_2 \times \ldots \times S_n||S_{n+1}| \\ &= |S_1||S_2|\ldots|S_n||S_{n+1}| \quad \text{(since by hypothesis)}\end{aligned}$$

Thus, $P(n+1)$ is true whenever $P(n)$ is true.

Hence, by the principle of induction, $P(n)$ is true for every integer $n \geq 2$.

Example 1 *Given 8 different Physics books, 7 different Chemistry books and 5 different Mathematics books. How many ways are there to select one book ?*

Solution We have to choose one book.

It can be

A Physics book = 8 choices

A Chemistry book = 7 choices

A Mathematics book = 5 choices

Also, all the books are different.

$\therefore$ By addition principle

Total number of choices = 8 + 7 + 5 = 20

Example 2 *How many 4 digit numbers (with possible repetition) can be formed with no digit less than 4 ?*

Solution The digits used in forming a 4 digit number are 4, 5, 6, 7, 8, 9 *i.e.,* 6 digits.

$\therefore$ Out of 4 places | ● | ● | ● | ● | the left most place can be filled in 6 ways.

Second left place can be filled in 6 ways.

Third left place can be filled in 6 ways.

Right most place can be filled in 6 ways.

So, by using multiplication rule, the required number of 4 digit number is

$$6 \times 6 \times 6 \times 6 = 1296$$

Example 3 *How many non-empty collections are possible by using 5 P's and 6 Q's?*

Solution We have 5 P's and 6 Q's, so any non-empty collection will contain 1 to 11 items.

Let p denotes the number of P's and q denotes the number of Q's in the collection, then

$$p = 0, 1, 2, 3, 4, 5$$

and $$q = 0, 1, 2, 3, 4, 5, 6$$

The number of pairs of the type (p, q) is, then $6 \times 7 = 42$

Out of these 42 pairs, the one pair *viz.*, (0, 0) gives empty collection.

Hence, the number of non-empty collections is $42 - 1 = 41$

Example 4 *Four digit numbers are formed by using the digits* 1, 2, 3, 4, 5 *with possible repetitions. Find how many of them are divisible by* 4?

Solution $\because$ 4 digit number is divisible by 4, the unit's place must be either 2 or 4.

We consider the following two cases.

Case I Unit's place is 2

$\because$ The number is divisible by 4. Ten's place must be 1 or 3 or 5.

$\therefore$ There are 3 choices for ten's place. Now there are 5 choices for hundred's place and 5 for thousand's place.

In this case, the number of 4 digit number is

$$1 \times 3 \times 5 \times 5 = 75$$

Case II Suppose unit's place is 4

$\because$ The number is divisible by 4

$\therefore$ The ten's place must be 2 or 4. Thus, there are 2 choices for ten's place, then there are 5 choices for hundred's place and 5 for thousand's place. Hence, in this case, the number of 4 digit numbers is $1 \times 2 \times 5 \times 5 = 50$. By addition rule, the required number of 4 digit number is

$$75 + 50 = 125$$

Example 5 *Three digits numbers are formed by using the digits* 1, 2, 3, 4, *find how many of them are there when*

(*i*) *repetition of digits is permitted*

(*ii*) *repetition of digits is not permitted*

(*iii*) *repetition of digits is not permitted but contain the digit* 3

(*iv*) *repetition of digits is permitted and contain the digit* 3.

Solution We shall read the digits from left to right.

(i) In this case, there are 4 choices for each of three places.
By multiplication rule, the number of 3 digit number is $4 \times 4 \times 4 \times = 64$

(ii) In this case, there are 4 choices for the first place, 3 for second and 2 for third. By multiplication rule, the number of 3 digit number is $4 \times 3 \times 2 = 24$

(iii) Here, digit 3 will appear either in the first, second or third place but not elsewhere more.

If 3 appears in the first place, then second place has 3 choices and third place has 2 choices.

By multiplication rule, the number is $3 \times 2 = 6$

If 3 appears in second place, then first place has 3 choices and third place has 2 choices. By multiplication rule, the number is $3 \times 2 = 6$.

If 3 appears in third place, then first place has 3 choices and second place has 2 choices. By multiplication rule, the number is $3 \times 2 = 6$.

Now, the above 3 cases are pair wise disjoint

$\therefore$ By addition rule, the required number of 3 digit numbers is $6 + 6 + 6 = 18$.

(d) Here the digit 3 will appear once, twice or thrice. We have 3 cases.

Case I First appearance of 3 is in the first place. Then, second place has 4 choices and 3rd place has 4 choices. By multiplication rule, the number of 3 digit numbers is $4 \times 4 = 16$

Case II First appearance of 3 is in the second place.

Then first place has 3 choices and third place has 4 choices.

By multiplication rule, the number of 3 digits numbers is $3 \times 4 = 12$

Case III First appearance of 3 is in the third place. First place has 3 choices and second place has 3 choices. By multiplication rule, the number of 3 digit numbers is $3 \times 3 = 9$

By addition rule, the total number of 3 digit numbers is $16 + 12 + 9 = 37$

Example 6 *How many ways are there to pick a sequence of two different letters of the alphabets from the word BOAT and from MATHEMATICS. How many ways are there to pick first a vowel and then a consonant from each of these words ?*

Solution We are to choose 2 different letters from the 4 letters B, O, A, T. First letter can be chosen in 4 ways since it can be any one of B, O, A, T.

The second letter can be chosen in 3 ways, since it has to be different from the first, so it can be any one of the remaining 3 letters.

By multiplication principle, total number of way to pick a 2 letter sequence is

$$4 \times 3 = 12$$

Similarly, from the 8 different letters M, A, T, H, E, I, C and S, a 2 letters sequence can be picked in $8 \times 7 = 56$ ways.

Now, in the word *BOAT*, there are 2 vowels O and A and B, T as consonants.

Hence, first a vowel and then a consonant can be chosen in 4 ways.

Similarly, number of ways in the second case is $3 \times 5 = 15$ (3 vowels, 5 consonants).

Example 7 *A new flag has to be designed with 6 vertical stripes using some or all of the colours yellow, green, blue and red. In how many ways can this be done so that no two adjacent stripes have same colour.*

Solution Let a, b, c, d, e, f denote the 6 vertical stripes in order from the left.

Then, a can be of any one of the 4 colours, b can be any one of the other 3 colours, c can be any one of the 3 colours.(as colour used in a can be used in c)

Similarly, there are 3 possible colours for each of d, e, f.

Hence, by multiplication principle, there are 4×3^5 ways of designing the flag.

Principle of Inclusion-Exclusion

Use of Venn Diagrams

In some counting problems, it is required to count the subsets of outcomes that possess or do not possess the combination of various properties.

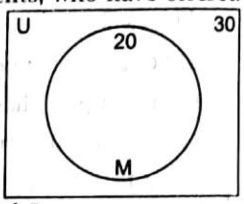
The 'Venn Diagrams' are very much useful in such cases to describe the situation pictorially.

In the forthcoming discussions throughout U stands for the universal set which contains all the sets under discussion. Also we assume that U contains N elements, *i.e.*, $|U| = N$.

Let us consider the first and the simplest case when the problems of counting takes care of only one property.

For example : Suppose out of 50 students in a class, 20 of them offer Mathematics as a major subject.

Here, universal set consists of 50 students and the set M consists of 20 students, who have offered Mathematics as a major subject, so that $N = |U| = 50$ and $|M| = 20$



It can be represented by a venn diagram also.

From the above venn diagram, we can easily conclude that there are 30 students who have not offered Mathematics as a major subject.

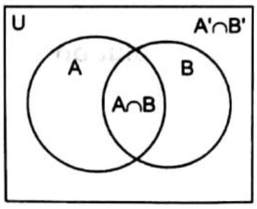
If M' denotes the complement of M in the universal set U, then $|M'| = 30$

i.e., $$|M'| = N - |M|$$

If a problem involves 2 properties say x_1 and x_2, then we consider two sets A and B

$$A = \{x : x \text{ satisfy property } x_1\}$$
$$B = \{y : y \text{ satisfy property } x_2\}$$

It can happen that some elements in the universe will satisfy both the properties and some will satisfy none of the two properties. This situation can be represented by venn diagram.



To find the number of elements which satisfy either property x_1 or x_2 or both.

We add $|A|$ and $|B|$

But in this sum of the elements of $A \cap B$ are counted twice, so we make the correction by subtracting $|A \cap B|$

Thus, $$|A \cup B| = |A| + |B| - |A \cap B|$$

Now, we find the number of elements not satisfying both the properties.

i.e., $$|A' \cap B'|$$

From De-Morgan's law, we have

$$(A \cup B)' = A' \cap B'$$

$\therefore$ $$|A' \cup B'| = |(A \cup B)'|$$

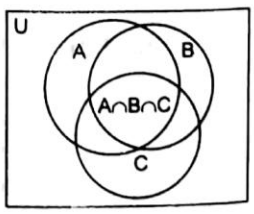
$\therefore$ $$|A' \cap B'| = N - |A \cup B|$$

$\therefore$ $$|A' \cap B'| = N - (|A| + |B| - |A \cap B|)$$

$\therefore$ $$|A' \cap B'| = N - |A| - |B| + |A \cap B|$$

Inclusion–Exclusion Principle for Three Sets

If A, B, C are three sets of elements satisfying the properties x_1, x_2, x_3 respectively. By using Venn Diagram, it can be shown as below.



Now, we find $|A \cup B \cup C|$

$$|A \cup B \cup C| = |(A \cup B) \cup C|$$

$$0 = |A \cup B| + |C| - |(A \cup B) \cap C|$$

$$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$$

$\therefore$ $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cup B| - (|A \cap C| + |B \cap C| - |A \cap C \cap B \cap C|)$$

$\therefore \qquad |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

From this it follows that $\qquad |A' \cap B' \cap C'| = N - |A \cup B \cup C|$

$\therefore \qquad |A' \cap B' \cap C'| = N - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| + |B \cap C| - |A \cap B \cap C|$

Example 1 *How many integers between* 1 *and* 567 *are divisible by either* 3 *or* 5 ?

Solution Let $Z = \{1, 2, 3, \ldots, 567\}$

$$P = \{x \in Z \,/\, 3 \text{ divides } x\}$$
$$Q = \{x \in Z \,/\, 5 \text{ divides } x\}$$

We have to find $|P \cup Q|$

Now since $\qquad 567 = 189 \times 3$,

The set P of multiples of 3 in Z contains 189 numbers *i.e.,*

$$|P| = 189$$

Similarly, since $567 = 113 \times 5 + 2$

The set Q of multiples of 5 in Z contains 113 number. *i.e.,*

$$|Q| = 113$$
$$567 = 37 \times 15 + 12$$

The set $P \cap Q$ of multiples of both 3 and 5 *i.e.,* multiples of 15 in Z contains 37 numbers *i.e.,*

$$|P \cap Q| = 37$$

Hence, $\qquad |P \cup Q| = 189 + 113 - 37 = 265$

Example 2 105 *students appeared in an examination. Out of which* 80 *students pass in English,* 75 *students pass in Mathematics and* 60 *students pass in both subjects. How many students fail in both subjects ?*

Solution Let Z = the set of students appeared in the examination.

E = the set of students pass in English

and M = the set of students pass in Mathematics

We are given

$$n(Z) = 105$$
$$n(E) = 80$$
$$n(M) = 75 \quad \text{and} \quad n(E \cap M) = 60$$

$\because \qquad n(E \cup M) = n(E) + (M) - n(E \cap M)$

$\therefore \qquad n(E \cup M) = 80 + 75 - 60 = 95$

Required number $= n(Z) - n(E \cup M) = 105 - 95 = 10$

So 10 students fail in both subjects.

Example 3 *How many integers between* 999 *and* 9999 *either begin or end with* 3 ?

Solution Let Z = the set of 4 digit numbers.

$$P = \{x \in Z \,/\, x \text{ begins with } 3\}$$
$$Q = \{x \in Z \,/\, x \text{ ends with } 3\}$$

We have to find $|P \cup Q|$.

If a 4 digit number begins with 3, then each of its remaining 3 digits can be chosen in 10 ways and so by multiplication principle

$$|P| = 10^3$$

Similarly if a 4 digit number ends with 3, then its leading digit being non-zero can be chosen is 9 ways and each of its remaining two digits can be chosen in 10 ways, so by multiplication principle

$$|Q| = 9 \times 10 \times 10 = 900$$

Finally, if a 4 digit number begins and ends with 3, then each of its remaining 2 digits can be chosen in 10 ways, so by multiplication principle.

$$|P \cap Q| = 10 \times 10 = 100$$

$$\therefore \quad |P \cup Q| = 1000 + 900 - 100 = 1800$$

Example 4 *How many arrangements of the digits* 0, 1, 2, ..., 9 *are there that do not end with* 8 *and do not begin with* 3 ?

Solution The total number of arrangements of 10 digits 0, 1, 2, ... ,9 in a row are 10!. This is universal set.

Let $A = \{\text{arrangement ending with } 8\}$

$B = \{\text{arrangements beginning with } 3\}$

We have to find $|A' \cap B'|$

If an arrangement ends with 8, they are 9! in number.

If an arrangement begins with 3, they are also 9! in number, so $|A| = 9!$, $|B| = 9!$

and $|A \cap B| = 8!$

Now,
$$|A' \cap B'| = N - |A| - |B| + |A \cap B|$$
$$= 10! - 9! - 9! + 8!$$
$$= 8!(90 - 9 - 9 + 1) = (73)(8!) = 2943360$$

Example 5 *How many integers between* 1 *and* 567 *are divisible by either* 3 *or* 5 *or* 7?

Solution Let
$$Z = \{1, 2, \ldots\ldots, 567\}$$
$$A = \{x \in Z / 3 \text{ divides } x\}$$
$$B = \{x \in Z / 5 \text{ divides } x\}$$
$$C = \{x \in Z / 7 \text{ divides } x\}$$

Now, $|Z| = 567$

If $[x]$ denotes the integral part of x, then

$$|A| = [567/3] = 189$$
$$|B| = [567/5] = 113$$
$$|C| = [567/7] = 81$$
$$|A \cap B| = [567/15] = 37$$
$$|B \cap A| = [567/35] = 16$$
$$|C \cap A| = [567/21] = 27$$
$$|A \cap B \cap C| = [567/105] = 5$$

Hence,

$$|A' \cap B' \cap C'| = |Z| - |A| - |B| - |C| + |A \cap B| + |B \cap C| + |C \cap A| - |A \cap B \cap C|$$
$$= 567 - 189 - 113 - 81 + 37 + 16 + 27 - 5 = 308$$

Example 6 *Three identical blue balls, four identical red balls and 5 identical white balls are to be arranged in a row, find the number of ways that this can be done, if all the balls with the same colour do not form a single block.*

Solution Total number of arrangements is $n = 12!/3!\,4!\,5!$.

Let A, B, C denote the set of arrangements with blue balls together, red ball together and white balls together. Then, by Inclusion-Exclusion principle, required number of arrangements is

$$|A' \cap B' \cap C'| = n - \left[\frac{10!}{4!\,5!} + \frac{9!}{3!\,4!} + \frac{8!}{3!\,4!}\right] + \left[\frac{7!}{5!} + \frac{5!}{3!} + \frac{6!}{4!}\right] - 3!$$

Example 7 *Find the number of numbers from* 1 *to* 1000 *which are neither divisible by* 2 *nor by* 3 *nor by* 5.

Solution From 1 to N, the number of integers which are divisible by a fixed integer k.

$$1 < 1 \le N \text{ is equal to } \left[\frac{N}{k}\right]$$

where $[x]$ denotes greatest integer $\le x$

Let us define the sets

A : Numbers which are divisible by 2.

B : Number which are divisible by 3.

C : Numbers which are divisible by 5.

Then, $$n(A) = \left[\frac{1000}{2}\right];\ n(B) = \left[\frac{1000}{3}\right];$$

$$n(C) = \left[\frac{1000}{5}\right]$$

$n(A \cap B)$ = number of numbers which are divisible by both 2 and 3 (*i.e.*, by 6)

$= \left[\frac{1000}{6}\right]$

$$n(B \cap C) = \left[\frac{1000}{15}\right];\ n(C \cap A) = \left[\frac{1000}{10}\right]$$

$n(A \cap B \cap C)$ = number of numbers which are divisible by 2, 3 and 5 $= \left[\frac{1000}{30}\right]$

Now, $n(A \cup B \cup C)$ = number of numbers which are divisible at least one of the three numbers 2, 3 or 5.

Then,

$$n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$= \left[\frac{1000}{2}\right] + \left[\frac{1000}{3}\right] + \left[\frac{1000}{5}\right] - \left[\frac{1000}{6}\right] - \left[\frac{1000}{15}\right] - \left[\frac{1000}{10}\right] + \left[\frac{1000}{30}\right]$$

$$= 500 + 333 + 200 - 166 - 66 - 100 + 33$$

$$= 734$$

$\therefore$ The required number $= 1000 - 734$

$= 266$

General Form of Principle of Inclusion and Exclusion

Theorem *Let $A_1, A_2, \ldots, A_m$ be subsets of a finite set U.*

Let
$$S_r = \Sigma |A_{i1} \cap A_{i2} \cap \ldots \cap A_{ir}|$$
where the sum is taken over the $\binom{m}{r}$ choices of sets of r integers $i_1, \ldots i_r$ such that
$$1 \le i_1 < i_2 < \ldots < i_r \le m$$
Then

(i) $|A_1 \cup A_2 \cup \ldots \cup A_m| = S_1 - S_2 + \ldots + (-1)^{m-1} S_m$

(ii) $|A'_1 \cap A'_2 \cap \ldots \cap A'_m| = |U| - S_1 + S_2 \ldots + (-1)^m S_m$

Dearrangements

Consider n distinct objects $a_i, 1 \le i \le n$, arranged in a row in the following order $a_1, a_2, \ldots, a_n$. Then, a dearrangement of these objects is a permutation in which no object is in its original position *i.e,* a_1 is not in the first place, a_2 is not in the second place, ..., a_n is not in the nth place.

Theorem *The number D_n of dearrangements of n distinct objects is given by*
$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right]$$

Proof Let the given objects be denoted by the integers 1, 2, ..., n.

Suppose that these are arranged in their natural order.

Let U be the set of all permutations of these integers.

Let A_i denote the set of those permutation in each of which the integer i is in the ith place. Then, it is clear that
$$D_n = |A_1' \cap A_2' \cap \ldots \cap A_n'|$$
Now, for each $i = 1, 2, \ldots n$, $|A_i| = (n-1)!$ because after putting the integer i in the ith place, the remaining $n - 1$ integers can be arranged the remaining $(n-1)$ places in $(n-1)!$ ways.

So,
$$S_1 = \Sigma |A_i| = n \times (n-1)!$$
For $1 \le i < j \le n$, we have
$$|A_i \cap A_j| = (n-2)!$$
because after putting the integers i, j in their respective original places, the remaining $n - 2$ integers can be arranged in the remaining $(n-2)$ places in $(n-2)!$ ways.

$\because$ There are $\binom{n}{2}$ pairs $A_i\ A_j$, we have
$$S_2 = \Sigma |A_i \cap A_j| = \binom{n}{2}(n-2)!$$
Similarly, for any set $T = \{i_1, \ldots, i_r\}$ of r integers such that $1 \le i_1 < i_2 < \ldots < i_r \le n$.

We have
$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}| = (n-r)!$$
There are $\binom{n}{r}$ different r sets T.

Hence,
$$S_r = \Sigma |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}|$$
$$= \binom{n}{r}(n-r)!, S_n = 1$$

Remark

- $\binom{n}{r} = {}^nC_r$

Hence, by principle of inclusion-exclusion, the number of dearrangements is

$$D_n = |U| - S_1 + S_2 - S_3 + \ldots + (-1)^n S_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^n$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right]$$

- If r things goes to wrong place out of n things, then $(n-r)$ things goes to original place (here $r < n$)

If D_n = Number of ways, if all n things goes to wrong place.

D_r = Number of ways, if r things goes to wrong place.

If r goes to wrong places out of n, then $(n-r)$ goes to correct places.

Then, $$D_n = {}^nC_{n-r}\, D_r$$

If at least p of them are in wrong places.

Then, $$D_n = \sum_{r=p}^{n} {}^nC_{n-r}\, D_r$$

where $$D_r = r!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!}\right]$$

Example 1 *A new employee checks the hats of n people visiting a restaurant, forgetting to put claim check numbers on the hats, when customer return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat ?*

Solution The probability that no one receives the correct hat is $\frac{D_n}{n!}$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!}$$

$$\because \quad e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!} + \ldots$$

e^{-1} is a good approximation to $\frac{D_n}{n!}$, when n is large *i.e.,*

$\frac{D_n}{n!}$ is approximately equal to $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Example 2 *While at racetrack Jayesh bets on each of the ten horses in a race to come according to how they are favoured. In how many ways can they reach the finish line so that he loses all his bets? What is the probability that he wins at least one bet ?*

Solution We actually want to know in how many ways we can arrange the numbers 1, 2, ..., 10, so that 1 is not in first place (its natural position, 2 is not in second place and so on)

Number is $$D_{10} = 10!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{1}{10!}\right) = 10!\,\frac{1}{e}$$

The probability that he wins at least 1 bet

$$= 1 - \frac{10!\,1/e}{10!}$$

$$= 1 - \frac{1}{e} = 0.632$$

Example 3 *A person writes letters to 6 friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that*

(i) at least 2 of them are in the wrong envelopes.

(ii) all the letters in wrong envelopes.

Solution (i) The number of ways in which at least 2 of them are in wrong places

$$= \sum_{r=2}^{6} {}^nC_{n-r}\, D_r$$

$$= {}^nC_{n-2}D_2 + {}^nC_{n-3}D_3 + {}^nC_{n-4}D_4 + {}^nC_{n-5}D_5 + {}^nC_{n-6}D_6$$

Here, $n = 6$

$$= {}^6C_4 2!\left(1-\frac{1}{1!}+\frac{1}{2!}\right) + {}^6C_3 \cdot 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right)$$

$$+ {}^6C_2 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) + {}^6C_1 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$$

$$+ {}^6C_0 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$$= 15 + 40 + 135 + 264 + 265 = 719$$

(ii) The number of ways in which all letters be placed in wrong envelopes

$$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$$= 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-120+\frac{1}{720}\right) = 360 - 120 + 30 - 6 + 1$$

$$= 265$$

Example 4 *(i) For $n \geq 2$, show that $D_n = (n-1)(D_{n-1} + D_{n-2})$*

(ii) show that $D_n = nD_{n-1} + (-1)^n$, $n \geq 2$

(iii) find $D_4, \ldots, D_9$.

Solution (i) The set of the D_n dearrangement of $1, 2, \ldots, n$ is the disjoint union of the following $n-1$ sets A_i. Fix an integer i with $2 \leq i \leq n$. Let A_i be the set of those dearrangements $x = a_1 a_2 \ldots a_n$ for which $a_1 = i$

Now, the dearrangements in A_i are of 2 types

(a) $a_1 = i$ and $a_i = 1$ (b) $a_1 = i$ and $a_i \neq 1$

For example, for $n = 4$, A_3 contains 1 dearrangement 3, 4, 1, 2 of type (a) in which 1 and 3 have been interchanged and 2 dearrangements 3, 1, 4, 2 and 3, 4, 2, 1 of type (b) in which the first element is 3 but 1 is not in the third place.

The number of dearrangements of type (a) is D_{n-2}, since in this case 1 and i have changed places and the remaining $n-2$ integers are deranged. Also, the number of dearrangements of type (b) is D_{n-1}, since in this case i is put in the first place and 1 is not allowed to occupy the place of i and the remaining $n-1$ integers including 1 are deranged. Thus

$$|A_i| = D_{n-2} + D_{n-1}, \text{ for } i = 2, \ldots, n$$

Hence,

$$D_n = (n-1)|A_2| = (n-1)[D_{n-2} + D_{n-1}]$$

(ii) Rewrite the formula in (i), thus

$$D_n - nD_{n-1} = -D_{n-1} + (n-1)D_{n-2}$$

Then, successively, we get

$$\begin{aligned} D_n - D_{n-1} &= (-1)[D_{n-1} - (n-1)D_{n-2}] \\ &= (-1)^2[D_{n-2} - (n-2)(D_{n-3}] \\ &= (-1)^{n-2}[D_2 - (2)D_1] \end{aligned}$$

Thus, $\quad D_n - nD_{n-1} = (-1)^{n-2}[1-(2)(0)] = [-1]^n$

$\because \quad D_1 = 0, D_2 = 1$

(iii) $\because D_2 = 1, D_3 = 3$, using (ii) we get

$$D_4 = 9, D_5 = 44, D_6 = 265,$$
$$D_7 = 1854, D_8 = 14833, D_9 = 133496$$

Example 5 *On a rainy day n people go to a party in a hotel. Each of them leaves his umbrella at the property counter. Find the number of ways in which the umbrellas are handed back to them after the party in such a manner that no person recieves his own umbrella.*

Solution Let us name the persons as $A_1, A_2, \ldots, A_n$ and label their umbrellas as $a_1, a_2, \ldots, a_n$ respectively.

Let D_n denote the number of ways in which the umbrellas can be returned so that no person recieves his own umbrella.

To find D_n, we shall obtain a recurrence relation connecting D_n, D_{n-1}, D_{n-2}.

Step 1. There are $n-1$ possible choices for A_1 to recieve a wrong umbrella. Suppose A_1 is given the umbrella a_2

Step 2. There are 2 different strategies for disposing off umbrella a_1.

(i) Give umbrella a_1 to A_2. We are left with $n-2$ persons $A_3, A_4, \ldots, A_n$ and their umbrellas $a_3, a_4, \ldots, a_n$. The desired task of handing over umbrellas can be accomplished in D_{n-2} ways.

(ii) Do not give umbrella a_1 to A_2. We have $n-1$ persons $A_2, A_3, \ldots, A_n$ and $n-1$ ur 'brellas $a_1, a_3, \ldots, a^n$. The number of ways of giving away the umbrellas so that A_2 does not recieve a_1, A_3 does not recieved $a_3, \ldots, A_n$ does not recieve a_n is D_{n-1}. From the two mutually exclusive and exhaustive cases (i) and (ii), the total number of ways $D_{n-1} + D_{n-2}$.

Step 3. Since for each of the $n-1$ ways of giving a wrong umbrella to A_1, there are $D_{n-1} + D_{n-2}$ ways of giving the remaining umbrellas to the remaining persons so that no one recieves his own umbrella.

$$\therefore \quad D_n = (n-1)(D_{n-1} + D_{n-2})$$

Step 4. In order to obtain an explicit expression for D_n, we proceed as follows

$$\begin{aligned} D_n - nD_{n-n} &= (n-1)D_{n-2} - D_{n-1} \\ &= (-1)[D_{n-1} - (n-1)D_{n-2}] = (-1)^2[D_{n-2} - (n-2)D_{n-3}] \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ &= (-1)^{n-2}(D_2 - 2D_1) \end{aligned}$$

Now, $D_1 = 0, D_2 = 1$

$\therefore$ $D_n - nD_{n-1} = (-1)^{n-2} = (-1)^n$

Dividing throughout by $n!$, we have

$$\frac{D_n}{n!} - \frac{D_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!}$$

Replacing n successively by $n-1, n-2, \ldots, 2$ and adding corresponding sides, we have

$$\frac{D_n}{n!} - \frac{D_1}{1!} = \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^{n-2}}{(n-2)!} + \ldots\ldots \frac{(-1)^2}{2!}$$

$\because$ $D_1 = 0$, we can rearrange the above relation as.

$$D_n = (n!)\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right]$$

Permutations-Combinations

Theorem 1 *The number of r permutations of a set S containing n different objects is denoted by $P(n, r)$ or ^nP_r and is given by*

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

At times the word permutation is used for arrangement. $P(n, r)$ is also called the number of permutations of n distinct objects taken r at a time. Note that $P(n, r)$ is also equal to

(a) Number of r digits numbers, which can be made by using n distinct digits and repetition of a digit is not allowed.

(b) Number of ways in which r persons can stay $\{n \geq r\}$ in r hotels (each in a different hotel).

(c) Number of ordered samples of size r without repetitions taken from the set $\{a_1, a_2, \ldots a_n\}$.

(d) Number of ways of making r person stand in a line taken from n persons.

(e) Number of one-one functions, which can be defined from a set containing r elements to a another set containing n elements $(n \geq r)$.

(f) $P(n, n) = \frac{n!}{0!} = n!$

= Number of ways arranging n distinct objects taken all at a time.

Theorem 2 (a) $P(n, r) = n\,P(n-1, r-1)$

(b) $P(n, r) = P(n-1, r) + rP[(n-1), (r-1)]$

Proof (a) LHS $= \frac{n!}{(n-r)!}$; RHS $= \frac{n(n-1)!}{(n-1-r+1)!} = \frac{n!}{(n-r)!}$

To prove it physically, we note that LHS is number of r digit numbers which can be made by using n distinct digits $d_1, d_2, \ldots, d_n$, when repetition of a digit is not allowed, fill up the first position by d_1 and fill up remaining $(r-1)$ places by the digits $d_2, d_3, \ldots, d_n$ also (n ways).

Thus, it is evident that

$$P(n, r) = n \cdot P(n-1, r-1)$$

(b) RHS $= P(n-1, r) + r \cdot P(n-1, r-1)$

$$= \frac{(n-1)!}{(n-1-r)!} + \frac{r(n-1)!}{(n-1-r+1)!}$$

$$= \frac{(n-1)!}{(n-1-r)!} + r \cdot \frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-1-r)!}\left(1 + \frac{r}{n-r}\right) = \frac{n!}{(n-r)!} = P(n, r)$$

To prove it physically, the LHS again has the same meaning as in (a). The RHS can be explained as follows.

In forming r digits numbers, the digit d_1 is either used or not used. If it is not used, then the number of numbers which can be formed = $P[(n-1), r]$

If the digit d_1 is used, it can be used at r possible positions and in every position of d_1, the number of numbers which can be made must be $P(n-1, r-1)$. Number of r digit numbers which can be made from n distinct digits = Number of numbers which can be used without d_1 + Number of such numbers which can be used with d_1

$$\Rightarrow \qquad P(n, r) = P(n-1, r) + r \cdot P(n-1, r-1)$$

Theorem 3 *The number of r combinations of an element set S is denoted by* $\binom{n}{r}$ *or* ${}^nC_r$ *or* $C(n, r)$ *and is given by*

$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Permutations with repetitions

Concept So far we have considered permutations of distinct or unrepeated objects. The effect of allowing repetition is to decrease the number of permutations.

For example : 2 distinct letters a_1 and a_2 can be arranged in one way as a_1a_2. Before deriving the general formula for the number of permutations with repetitions, consider the problem of arranging the 7 letters a, a, b, b, b, c, c in a row.

Here, we are to fill 7 places –, –, –, –, –, –, –

with the given letters. First for placing the a's, 2 of these places can be chosen in ${}^7C_2$ ways.

Note that for each such choice we can place the letters a, a in the chosen places in one way only. Hence, the a's can be placed in ${}^7C_2$ ways.

Then, there remain 7 – 2 = 5 places and the b's can be placed in 3 of these 5 places in ${}^5C_3$ ways.

Then, there remain 5 – 3 = 2 places, in which the c's can be put in ${}^2C_2$ ways, so by the multiplication principle, the number of permutations is

$${}^7C_2 \times {}^5C_3 \times {}^2C_2 = \frac{7!}{2!5!} \times \frac{5!}{3!2!} \times 1 = \frac{7!}{2!3!2!}$$

Alternatively, we can find the number, say m of the 7 permutations of the given letters as follows.

Replace the 2 identical a's by the distinct letters a_1, a_2. Similarly, replace the b's by b_1, b_2, b_3 and c's by c_1, c_2. Then, we get 7 different letters which can be arranged in a row in 7! ways.

Now, these 7! permutations arise out of the required m permutations in the following ways.

Consider any one of the required permutations say $t = bacabbc$.

On replacing the letters as above, we get $t_1 = b_1a_1c_1a_2b_2b_3c_2$, as one possible permutation of the distinct latters.

Now, we can arrange a_1, a_2 in their positions in 2! ways. This gives the following 2 permutations.

$$t_1 = b_1a_1c_1a_2b_2b_3c_2$$
$$t_2 = b_1a_2c_1a_1b_2b_3c_2$$

Similarly, b_1, b_2, b_3 can be arranged in their positions in 3! ways.

c_1, c_2 can be arranged in their positions in 2! ways, so the permutation t give rise to in all $2! \times 3! \times 2!$ different permutations of the 7 different letters.

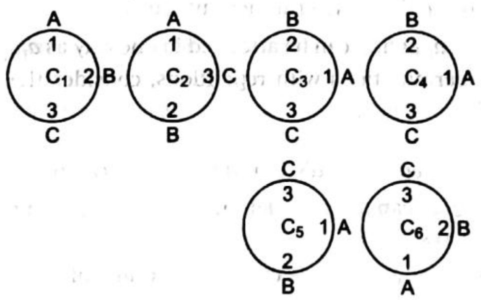
Hence, the required m permutations give $m \times 2! \times 3! \times 2!$ permutations.

This happens because every permutation of the 7 different letters can be obtained by first choosing places to put the a's, b's and c's and then rearranging the letters in those places in a particular order.

Hence,
$$7! = m \times 2! \times 3! \times 2!$$
i.e., as before
$$m = \frac{7!}{2!3!2!}$$

Theorem 4 *Suppose there are n objects of which n_1 are identical of first type, n_2 are identical of second type, ..., n_k are identical of kth type so that*
$$n = n_1 + n_2 + \dots + n_k$$
Then, the number of permutations of these n objects, taken all at a time is denoted by $P(n; n_1, n_2, \dots, n_k)$ and is given by.
$$P(n, n_1, \dots, n_k) = \binom{n}{n_1}\binom{n-n_1}{n_2} \dots \dots \binom{n - n_1 - n_2 - \dots - n_{k-1}}{n_k}$$
$$= \frac{n!}{n_1! n_2! \dots n_k!}$$

Circular Permutations

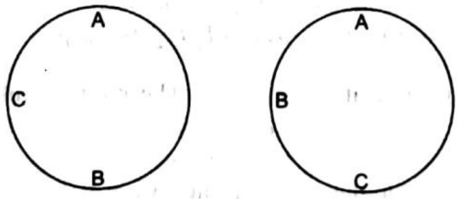


Suppose in a circular order, there are n numbered seats numbered 1 through n and we want to make n persons sit at these seats, then the number of ways must be same as number of ways in a line *i.e.*, the number of ways must be $n!$.

We will illustrate this for $n = 3$. (The persons are denoted by A, B, C).

Observe carefully that all the 6 cases $C_1, C_2, \dots, C_6$ are different.

Now, if the number 1, 2, 3 are removed we can easily note that C_1, C_3 and C_5 denote the same cases, since all the cases are equivalent to A, B, C, C_2, C_4 and C_6 are equivalent to only one case.



Thus, 3 persons on a circular order can sit in 2 ways only namely which can be proved theoretically as follows. Keep A fixed and arranged remaining 2 persons in a line which can be done in 2! ways.

In the general case, we easily conclude that n persons in a circular order can stand in $(n-1)!$ ways.

We further observe that corresponding to any clockwise arrangement of distinct objects in a circular order, there is an anticlockwise arrangement.

By the principle of homogeniety, the number of anticlockwise arrangements must be same as number of clockwise arrangements. Now as there are $(n-1)!$ arrangements in all, the number of clockwise or anticlockwise arrangement

$$= \frac{1}{2}(n-1)!$$

Theorem 5 *The number of distinct circular permutation of n different objects is* $(n-1)!$

Corollary *Let $n \geq 3$ be a +ve integer. The number of different circular necklaces that can be made from n different beads is* $\frac{1}{2}(n-1)!$.

Example 1 *How many numbers each lying between* 1000 *and* 10000 *can be formed with the digits* 0, 1, 2, 3, 4, 5, *no digit being repeated?*

Solution Here, number of digits (objects $= n = 6$).

Numbers lying between 1000 and 10000 are of 4 digits.

Hence, number of places to be filled up $= r = 4$. Hence, number of different arrangements of 6 given

But in these ${}^6P_4$ numbers, some number begin with 0 and hence these numbers are actually of 3 digits not of our purpose.

Now, we find number of numbers beginning with zero. In this case, number digits remain to be utilised in $(1, 2, 3, 4, 5) = 5$ and number of places to be filled up $= r = 3$

Hence, number of such numbers

$$= {}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2} = 60$$

Number of numbers of 4 digits formed by 0, 1, 2, 3, 4, 5 is $360 - 60 = 300$

Example 2 *Find the number of numbers between* 300 *and* 3000 *that can be formed with the digits* 0, 1, 2, 3, 4 *and* 5, *no digit being repeated.*

Solution We first notice that numbers between 300 and 3000 are of 3 and 4 digits. We first find the number of number's of 3 digits.

But we require only these numbers of 3 digits which begin with 3, 4 or 5 for those numbers. Number of places to be filled up $= r = 2$ and number of digits remain to be used $= n = 5$.

Hence, number of numbers of 3 digits begining with 3 $= {}^5P_2 = \frac{5!}{3!} = 20$

Number of numbers of 3 digit begining with 5 $= {}^5P_2 = 20$

Hence, number of number's of 3 digits begining with 3, 4 or 5 $= 20 + 20 + 20$

$$= 60 \qquad \ldots(i)$$

Also, for the number of 4 digits less than 3000, we require only these numbers which begin with 1 or 2.

In this case, number of places to be filled up $= r = 3$.

Number of digits remain to be used $= n = 5$.

Hence, number of numbers of 4 digit begining with $2 = {}^5P_3 = 60$

Hence, number of numbers of 4 digits begining with $2 = 60 + 60 = 120$ ...(ii)

Total number of numbers lying between 300 and 3000 and formed with the digits 0, 1, 2, 3, 4,

$$5 = 60 + 120 = 180 \qquad \text{[from Eqs. (i) and (ii)]}$$

Example 3 *How many numbers between* 400 *and* 1000 *can be formed with the digits* 2, 3, 4, 5, 6, 0? *Number lying between* 400 *and* 1000 *are of* 3 *digits only.*

Solution Now, we first find the number of numbers of 3 digits and greater than 400. These number begins with 4, 5 or 6. For these numbers of places to be filled up $r = 2$ and number of digits remain to be used $= n = 5$

Hence, number of numbers of 3 digits beginning with $4 = {}^5P_2 = \frac{5!}{3!} = 20$

Number of numbers of 3 digit beginning with $5 = {}^5P_2 = 20$

Number of numbers of 3 digit beginning with $6 = {}^5P_2 = 20$

Hence, number of numbers 3 digits beginning with 4, 5 or 6

$$= 20 + 20 + 20 = 60$$

$\therefore$ 60 numbers are lying between 400 and 1000 formed with digits 2, 3, 4, 5, 6, 0.

Example 4 *How many different numbers of* 6 *digits can be formed with the digits* 4, 5, 6, 7, 8, 9? *How many of them are divisible by* 6. *How many of them are not divisible by* 5?

Solution Number of digits $= 6 = n$

For the numbers of 6 digits number of places to be filled up $= r = 6$. Hence, total number of numbers of 6 digits formed with the digits 4, 5, 6, 7, 8, 9. is

$$ {}^6P_6 = \frac{6!}{0!} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720 \qquad \text{...(i)}$$

Also of these numbers of numbers which end with 5 must be divisible by 5. Now, for the number of numbers of 6 digits divisible by 5.

Number of digits remain to be utilised $= n = 5$.

Hence, number of number's of 6 digits divisible by 5 and formed with the digits 4, 5, 6, 7, 8, 9 is

$$ {}^5P_5 = \frac{5!}{0!} = 120 \qquad \text{...(ii)}$$

Now, number of numbers not divisible by 5 = total number of numbers − number of numbers divisible by 5

$$= 720 - 120 = 600$$

Example 5 *How many numbers of* 6 *digits can be formed with the digits* 1, 2, 3, 4, 5, 6, *in which* 5 *always occur in ten's place?*

Solution Number of digits = 6

Number of places to be filled up = r = 6

But as 5 always occur in ten place, so that number of places to be filled up = r = 5.

Number of (objects) digits remain to be utilised = n = 5

Hence, number of numbers of 6 digits formed with 1, 2, 3, 4, 5, 6, such that 5 always occurs in the ten's place is

$$= {}^5P_5 = \frac{5!}{0!} = 120$$

Example 6 *Find the sum of all the numbers greater than* 10000 *formed with the digits* 0, 2, 4, 6 *and* 8. *No digit being repeated.*

Solution Numbers formed with 0, 2, 4, 6 and 8 and greater than 1000 are of 5 digits.

Keeping 0 at units (or tens or hundreds or thousands place).

Number of places remain to be filled up = r = 4 Number of digits remain to be utilised = n = 4. Hence, number of numbers in which 0 comes at unit (tens or hundred or thousands place)

$$= {}^4P_4 = \frac{4!}{0!} = 24$$

i.e., 0 may come at unit (or tens or hundred or thousands place) = 24 number of times. Also, keeping 2 (or 4 or 6 or 8) at unit (or tens or hundreds or thousands) place.

Number of places to be filled up = r = 4

Number of digits remain (0, 4, 6, 8) to be utilised = n = 4

Hence, number of numbers of 4 digits such that 2 comes at unit place = ${}^4P_4$ = 24. But out of these 24 numbers, some numbers begins with 0.

To find the number of such numbers, let us keep 0 at 10000 place and 2 at unit place, then number of places remain to be filled up r = 3

Number of digits remain to be utilised

$$= n = 3$$

Hence, number of numbers of 5 digits in which 2(or 4 or 6 or 8) comes at unit (or tens or hundred or thousands) place = 24 − 6 = 18

Also, keeping 2 (or 4 or 8) at 10000 place, number of places remain to filled up = r = 4 Number of digits remain to be utilised = 4. Hence, number of numbers in which 2(or 4 or 6 or 8) comes at 10000 place = ${}^4P_4 = \frac{4!}{0!}$ = 24. Hence, 2(4,6 or 8) comes at unit (or tens or hundred or thousands) place in 18 number of times.

But 2(4, 6 or 8) comes at 10000 place in 24 number of times.

Hence, sum of the numbers at unit place

$$= [0 \times 24 + (2 + 4 + 6 + 8) \times 18] \times 1 \quad \ldots\text{(i)}$$

Sum of numbers at tens place

$$= [0 \times 24 + (2 + 4 + 6 + 8) \times 18] \times 10 \quad \ldots\text{(ii)}$$

Sum of numbers at hundreds place

$$= [0 \times 24 \times (2 + 4 + 6 + 8) \times 18] \times 100 \quad \ldots\text{(iii)}$$

sum of numbers at thousands place

$$= [0 \times 24 + (2 + 4 + 6 + 8) \times 18] \times 100 \quad \ldots\text{(iv)}$$

sum of numbers at ten thousands place

$$= [0 \times 24 + (2 + 4 + 6 + 8) \times 24] \times 10000 \quad \ldots\text{(v)}$$

Adding Eqs. (i), (ii), (iii), (iv) and (v) altogether, we get 5199960.

Hence, sum of all numbers greater than 10000 formed with digits 0, 2, 4, 6, 8 is 5199960.

Example 7 *How many number of 4 digits can be formed with the digits* 1, 2, 3, 4, 5, *when repetition of digits is allowed.*

Solution In the number of 4 digits 4 places are to be filled up by 5 digits.

Now, 1st place can be filled up in 5 ways (out of any one of 1, 2, 3, 4, 5).

Also, 2nd place can be filled up in 5 ways.

Hence, first two places can be filled up in $5 \times 5 = 5^2$ ways.

Also, 3rd place can be filled up in 5 ways .

Hence, first three places can be filled up in $5^2 \times 5 = 5^3$ ways.

Lastly 4th place can be filled up in 5 ways.

Hence, all the 4 places can be filled up in $5 \times 5^3 = 5^4$ ways.

Hence, number of numbers of 4 digits formed with the digit 1, 2, 3, 4, 5, when repetitions of digits is allowed $= 5^4 = 625$

Example 8 *In how many ways can a ten question multiple choice exam be answered if there are* 4 *choices* a, b, c *and* d, *to each question? If no two consecutive questions can be answered the same. In how many ways can* 10 *questions can be answered ?*

Solution In the first case, when there is no restriction, number of questions = 10

Number of choices = 4

1st question can be answered in 4 ways *a* or *b* or *c* or *d*. 2nd question can be answered can be in 4 ways.

Hence, first 2 questions can be answered in $4 \times 4 = 4^2$ ways.

Also 3rd question can be answered in 4 ways, so first three questions can be answered in 4^3 ways.

Finally, all ten questions can be answered in 4^{10} ways.

In the second case, it is given that no two consecutive question can be answered the same.

First question can be answered in 4 ways and second in 3 ways, third question also in 3 ways.

So, first 3 questions can be answered in

$$4 \times 3 \times 3 = 4 \times 3^2 \text{ ways.}$$

4th question can also be answered in 3 ways and so on.

Hence, total number of ways to answer all the ten questions is 4×3^9.

Example 9 *In how many ways can n things be given to p persons, when, each person can get any number of things ? ($n > p$).*

Solution First thing can be given to p person in p ways.

Similarly, 2nd thing and 3rd thing can be given to p persons in p ways respectively.

So, all the n things can be given to p persons in p^n ways.

Concept I Any number of things out of the given things occur together.

For example, we have to find the number of arrangements of 9 persons including 4 boys, when it is given that 4 boys will sit together.

Concept II When any number of things (or no two of a certain given things) comes together, to find the number of permutations, when any number of a kind of things do not come altogether. We find the total number of permutations without any restriction ...(i)

Then, we find the number of permutations when all the things of that kind are kept together ...(ii)

Subtracting (ii) from (i), we get the number of permutations, such that all the given number of things of a kind do not come together.

For example, there are 9 students of which 5 are boys and 4 are girls and we have to find the number of arrangements of all the 9 students in such a way that all the 4 girls do not come together (2 may come together, 3 may come together but not all the 4 girls come together).

Then, we first find the total number of arrangements without any restriction for number of students (things) to be arranged $= n = 9$

Number of places to be filled up $= r = 9$

Hence, number of arrangements of all the 9 students without restriction is

$$= {}^9P_3 = \frac{9!}{0!} = 9!$$

Then, keeping all the 4 girls together.

$i < B_1, B_2, B_3, B_4, B_5, (C_1, C_2, C_3, C_4,)$ and counting then as one (i) we see that number of students $= 6\ (5+1)$ and number of places to be filled up $= r = 6$.

(5 places for boys and 1 place for girls)

Hence, number of arrangements of these 6 is ${}^6P_6 = 6!$. But all the 4 girls among themselves can be arranged in ${}^4P_4 = 4!$ ways.

Hence, number of arrangements of 9 students when all the 4 girls are together is $9! - 6! \times 4!$

Concept III GAP METHOD

But to find the number of permutations of some number of things such that no two of the things of a kind come together.

We put all the things on which there is no restriction in a line. Then, we count the places between every pair of things including the places to the left of the Ist thing and to the right of the 2nd thing if l is the number of things on which there is no restriction and m is the number of things such that no two of m things come together, then we place things in a line in ${}^lP_l = l!$ ways.

For m things there are $(l + 1)$ places and hence required number of arrangements

$$= {}^{l+1}P_m\ l!$$

For example, there are 9 students of which 5 are boys and 4 are girls.

We have to find the number of arrangements of all the 9 students that no 2 girls sit together.

Then, we first place 5 boys in a line in ${}^5P_5 = 5!$ ways for 4 girls such that no 2 girls sit together there 6 places.

Hence, we made the arrangements of 4 girls in 6 places in ${}^6P_4$ ways. So that required number of arrangements is $5!\,{}^6P_4$.

Concept IV Number of permutations of n things taken all at a time out of which p things all alike (of one kind), q things all alike (of 2nd kind and rest are of different is

$$\frac{{}^nP_n}{p!q!} = \frac{n!}{p!q!}$$

As for example, we have to find the number of numbers of 8 digits formed with the digits 1, 2, 3, 3, 4, 5, 5, 5.

Number of things $= n = 8$.

Out of which two are alike (3, 3) and three are alike (5, 5, 5).

Number of places to be fixed up = 8.

Number of numbers of 8 digits formed with the digits 1, 2, 3, 4, 5, 5, 5 is

$$= \frac{8!}{2!3!}$$

Example 1 *There are 2 books each of 3 volumes and 2 books each of 2 volumes. In how many ways can the 10 books be arranged on a long table so that volume of the volume of same books are not separated?*

Solution Let A and B two books having volumes A_1, A_2, A_3 and B_1, B_2, B_3 respectively.

Let C and D be the other two books C_1, C_2 and D_1, D_2 be their volumes.

There are 4 books. These 4 books, on 4 places can be arranged in ${}^4P_4$ ways, *i.e.*, 4! ways.

Also three volumes (A_1, A_2, A_3) can be arranged among themselves in ${}^3P_3$, *i.e.*, in 3! ways.

Three volumes B_1, B_2, B_3 of the books B can be arranged among themselves in 3! ways.

Again, (C_1, C_2) two volumes of the book C can be arranged in 2! ways and (D_1, D_2) two volumes of the book D can be arranged in 2! ways.

Total number of arrangements of book $= 4!\ 3!\ 3!\ 2!\ 2! = 3456$

Example 2 *In a dinner party there, are 10 Indians, 5 Americans and 5 English men. In how many ways can they be arranged in row so that all persons of same rationality sit together?*

Solution Let $I_1, I_2, \ldots, I_{10}$ be 10 Indians.

$A_1, A_2, \ldots, A_5$ be 5 Americans.

$E_1, E_2, \ldots, E_5$ be 5 English men.

Now, there are 20 persons.

For the arrangement of these 3 sections.

No. of places to be filled up $= r = 3$.

Number of sections (thing) $= n = 3$.

Arrangement of these 20 persons is ${}^3P_3 = 3!$

Arrangement of 10 Indians among themselves $= {}^{10}P_{10} = 10!$

Arrangement of 5 American $= {}^5P_5 = 5!$

Arrangement of 5 English men $= 5!$

Total number of arrangements $= 3!\,10!\,5!\,5!$

Example 3 *There are 6 balls of different colours (black, white, red, green, violet, yellow). In how many ways can the 6 balls be arranged in a row so that black and white balls may never come together?*

Solution There is no restriction on red, green, violet and yellow balls. We first place the four balls in a line for the arrangements of there 4 balls.

Number of place to be filled up $= r = 4$

$$\checkmark R \checkmark G \checkmark V \checkmark Y$$

Number of balls $n = 4$

Hence, four balls red, green, violet and yellow can be arranged in a line in $^4P_4 = \frac{4!}{0!} = 4!$ ways.

Now, for black and white balls, these are five (5) places and hence two balls (black and white) can be arranged in $^5P_2 = \frac{5!}{3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} = 20$

Required number of arrangements of the 6 balls such that white and black balls do not come together

$$= 4! \times 20 = 24 \times 20 = 480$$

Example 4 *In a class of students, there are 4 girls and 6 boys. In how many ways can they sit in a row that no 2 girls sit together?*

Solution 6 boys can be arranged in a line in $^6P_6 = \frac{6!}{0!} = 6!$ ways.

$$\checkmark B_1 \checkmark B_2 \checkmark B_3, \checkmark B_4 \checkmark B_5 \checkmark B_6$$

Also, number of places for 4 girls when no 2 girls sit together.

$= 7$(marked with ✓)

Number of arrangements of the 4 girls is

$$= {}^7P_4 = \frac{7!}{3!}$$

Required number of arrangements in this case $= 6! \times \frac{7!}{3!} = \frac{6!7!}{3!}$

Example 5 3 *women and* 5 *men are to sit in a row. Find, in how many ways they can be arranged so that no* 2 *women sit next to each other?*

Solution There is no restriction on men, so we first place 5 men in a line.

Here, $n = 5$ (number of men)

Number of place to be filled up $= r = 5$.

5 men can be arranged in $^5P_5 = \frac{5!}{0!} = 5!$ ways

Also for the 3 women, there are 6 places and 3 women can take their sits in $^6P_3$

$$= \frac{6!}{3!} \text{ ways.}$$

Required number of arrangements $= 5! \times \frac{6!}{3!} = 14400$

Example 6 *In how many ways 16 rupees and 12 paise coins be arranged in a line, so that no 2 paise coins may occupy consecutive positions?*

Solution We first arrange 16 rupees coin in a line.

For $n=16$ and $r=16$

All the 16 coin are identical.

Number of arrangements of 16 coins $= \dfrac{^{16}P_{16}}{16!} = \dfrac{16!}{16!}$

Also since no two paise coins come together *i.e.*, no two paise coin occupy consecutive places so that number of places for 12 coins are = 17

Hence, 12 paise coins can be arranged in $\dfrac{^{17}P_{12}}{12!}$ ways.

(Since all the 12 paise coins are also identical)

Required number of arrangements $= \dfrac{16!}{16!} \times \dfrac{^{17}P_{12}}{12!} = 1 \times \dfrac{17!}{5!12!}$

$$= \frac{12! \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17}{12! \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 6188$$

Example 7 *Show that the number of ways in which n books may be arranged on a shelf so that 2 particular books shall not be together is $(n-2)(n-1)!$.*

Solution Total number of arrangements of the n books $= {^nP_n} = n!$

Keeping 2 particular books together, number of book is $(n-1)$

These $(n-1)$ books can be arranged in ${^{n-1}P_{n-1}} = (n-1)!$ ways.

But two particular books which are kept together can be arranged among themselves in ${^2P_2} = 2!$ ways.

Number of arrangements of n books when 2 particular books are together is

$$(n-1)!\,2! = 2(n-1)!$$

Required number of arrangements of n books when 2 particular books are not kept together

$$= n! - 2(n-1)!$$
$$= n(n-1)! - 2(n-1)!$$
$$= (n-2)(n-1)!$$

Example 8 *Find the number of ways of arranging the letters a, a, a, a, b, b, b, c, c, c, d, e, c, f in a row, if the letter c are separated from one another.*

Solution Total number of letters is $=15(5a's + 3b's + 3c's + 1d + 2e's + 1f)$ Number of arrangements of 12 letters excluding $3c's$ on which, there is no restriction is

$$\frac{^{12}P_{12}}{5!\,3!\,2!} = \frac{12!}{5!\,3!\,2!}$$

Now since no 2 c's come together

so that number of places for c is 13.

Hence, arrangements of 3 c's $= \dfrac{^{13}P_3}{3!} = \dfrac{1}{3!} \times \dfrac{13!}{10!} = \dfrac{13!}{3!\,10!}$

Required number of arrangements of these 15 letters, when no 2 *c's* are together

$$=\frac{13!}{3!10!}\times\frac{12!}{3!10!}$$

Example 9 *In how many ways can the letters of the word 'SALOON' be arranged, if the consonant and vowels must occupy alternate place?*

Solution Number of places, $r=6$.

Case I When vowels comes at first place. In this case 3 vowels (A, O, O) occupy 3 places 1st, 3rd and 5th.

Number of arrangements of vowels $=\frac{^3P_3}{2!}=\frac{3!}{2!}$

Since consonants also come alternately so that number of places provided for the 3 consonants $(S, L, N)=r=3$

Number of arrangements of consonants $= {}^3P_3=3!$

Required number of arrangements $=\frac{3!}{2!}\times 3!=\frac{6.6}{2}=18$

Case II When consonant comes at 1st place. If a consonant comes at the first place, then (R, L, N) is $r=3$ (1st, 3rd, 5th)

Number of arrangements of consonants alternately is ${}^3P_3\,(n=3, r=3)$.

$$=\frac{3!}{0!}=3!$$

Since vowels also come alternately, so that number of places for the $(n=3)$ vowels $(A, 0, 0)$, 2 alike $(0,0)$ is $r=3$.

Number of arrangements of vowels on the alternate place is $\frac{^3P_3}{2!}=\frac{3!}{2!}$.

Required number of arrangement $=3!\times\frac{3!}{2!}=18$

Required number of ways of arrangement of the letters of the word 'SALOON' so that consonants and vowels must occupy alternate places

$$=18+18=36$$

Example 10 *How many different words can be formed with the letters of the word 'MATHEMATICS'?*

Solution In the word 'MATHEMATICS'

Number of letters $=n=11$

2 A's are identical.

2 T's are identical.

2 M's are identical.

Number of places to be filled up $=r=11$.

Number of different arrangements or different words that can be formed with the letters of the word 'MATHEMATICS' is $\frac{^{11}P_{11}}{2!\,2!\,2!}=\frac{11!}{2!\,2!\,2!}$

Example 11 *How many different signals can be made by hoisting 6 differently coloured flags one above the other, when any number of them may be hoisted at once ?*

Solution There are 6 differently coloured flags and any number of them may be taken at a time.

Hence, number of things, $n=6$

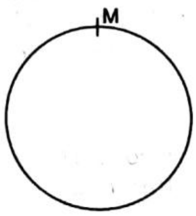
But number of places to be filled up vary from $r=1$ to 6. *i.e.*, $r=1$ or 2 or 3 or 4 or 5 or 6.

Hence, required number of signals (arrangements)

$$= {}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6$$
$$= \frac{6!}{5!} + \frac{6!}{4!} + \frac{6!}{3!} + \frac{6!}{2!} + \frac{6!}{1!} + \frac{6!}{0!}$$
$$= 6 + 30 + 120 + 360 + 720 + 720 = 1956$$

Example 12 (i) *In how many ways 7 men can sit round a table ?*
(ii) *In how many ways we can place 7 apples in a circle ?*

Solution (i) Let $M_1, M_2, M_3, \ldots, M_7$ be 7 persons keeping one person M_1 fixed, remaining 6 persons can take their places in ${}^6P_6 = 6!$ ways.



Here $n=6, r=6$.
(Also, there is a difference in clockwise and anticlockwise arrangements).
Hence, number of ways in which 7 men can sit round a table in 6! ways.

(ii) Keeping one apple fixed $(n=6)$ 6 apples can be arranged in ${}^6P_6$ ways

$$= 6! = 720 \text{ ways.}$$

But there is no difference in clockwise and anticlockwise arrangements so that required number of arrangements

$$= \frac{6!}{2} = 360$$

Example 13 *4 gentleman and 4 ladies are invited to a certain party, find the number of ways of seating them around a table so that only ladies are seated on the 2 sides of each gentleman.*

Solution Let $G_1, \ldots, G_4$ be 4 gentleman
and $L_1, \ldots, L_4$ be 4 ladies

Keeping one gentleman G_1 fixed, remaining 3 gentleman G_2, G_3, G_4 can take their seats in ${}^3P_3 = 3!$ ways

Then number of seats for ladies (one lady between gentleman or only ladies are seated on either side of each gentleman) is 4.

Hence, 4 ladies can take their seats in ${}^4P_4 = 4!$ ways

Required number of arrangements $= 3! \times 4! = 144$

Example 14 *5 boys and 5 girls form a line with the boys and girls alternating. In how many different ways could they form a circle such that the boys and girls are alternate ?*

Solution Keeping any one say boy fixed, then remaining 4 boys can take their seats in 4! ways.

Now, boys and girls are alternating number of places for 5 girls is 5.

Hence, 5 ways can be arranged in $^5P_5$ ways

$$= 5! \text{ ways.}$$

Required number of arrangements of 5 boys and 5 girls in a circle so that boys and girls are alternating

$$= 4!\,5! = 2880$$

Example 15 *A gentleman invites a party of $m+n$ friends to a dinner and places m at one round table and n at another. Find the number of arrangements.*

Solution We first divide $m+n$ friends into 2 parts, one of part consist m and other n in

$$^{m+n}C_m \cdot {^nC_n} = \frac{(m+n)!}{m!\,n!} \text{ way}$$

Number of ways in which 2 groups (of m and n respectively) can go at 2 round tables in $2! = 2$ ways.

For the number of arrangements of m friends at a round table keeping one fixed and there remaining $(m-1)$ friends can take this seats in $(m-1)!$.

Other n persons can take their seats at other round table in $(n-1)!$ ways.

Required number of ways

$$= 2\frac{(m+n)!}{m!\,n!} \cdot (m-1)!\,(n-1)!$$

$$= 2\frac{(m+n)!}{mn}$$

Problem Based on the Use of

(a) $^nC_r = \dfrac{n!}{r!(n-r)!}$ (b) $^{n-x}C_{r-x} = {^{n-x}C_r}$

(i) Number of different selections of n things taken r at a time such that any thing comes once in a combination is given by $^nC_r = \dfrac{n!}{r!(n-r)!}$

(ii) Number of different selection of n things taken r at a time such that x particular thing always occur, is given by $^{n-x}C_r$.

Setting x particular things aside, number of remaining things $= n - x$.

Now to get all the combinations in which x particular things always occur, we have to select $r - x$ things in every possible way from $n - x$ things.

Hence, number of different combinations of n things taken r at a time such that x particular things are included

$$= {^{n-x}C_{r-x}}$$

(iii) Number of different combinations of n things taken r at a time, such that x particular things always occur (included) is given by $^{n-x}C_{r-x}$.

(iv) Now, setting x particular things aside, number of remaining things $= n - x$.

Now to get all the combinations in which x particular things never occur, we have to select r things from the remaining $n-x$ things.

Number of combination of n things taken r at a time in which x particular things are excluded is ${}^{n-x}C_r$.

Example 1 *A man has 7 friends. He invites 3 of them to party. Find, how many parties to each of 3 different friends he can give* and how many times any particular friend will attend the parties?

Solution Number of friends $n = 7$

Number of places to be filled up in a party, $r = 3$

Number of different parties to each of 3 different friends can be given, is ${}^7C_3$

$$= \frac{7!}{3!\,4!} = 35$$

If a particular friend always attend the parties, then keeping one particular friend aside, number of remaining friends is $n - x = 7 - 1 = 6$.

We select only $(3 - 1 = 2)$ friends from the 6 friends, since any particular friend always attend the parties and hence

$$^6C_2 = \frac{6!}{2!\,4!} = \frac{6.5 \times 4!}{2!\,4!} = 15$$

Example 2 *A delegation of 6 members is to be sent abroad out of 12 members. In how many ways can the selection be made so that a particular member is included ?*

Solution Number of members, $n = 12$

Number of places to be filled up for a delegation $= r = 6$

If a member (particular) is included, then keeping the particular member aside, number of remaining member $= n - x = 12 - 1 = 11$ and number of places to be filled up $= r = 6 - 1 = 5$

Number of ways to send a delegation of 6 members out of 12 members if a particular number is included $= {}^{11}C_5$

$$= \frac{11!}{6!\,5!} = 22 \times 21 = 462$$

Example 3 *At an election 3 wards of a town are canvassed by 4, 5 and 8 men respectively. If there are 20 volunteer. In how many ways can they be alloted to different wards ?*

Solution We have to select 4 men out of 20 for a ward which we can select in ${}^{20}C_4$ ways.

Now, only $20 - 4 = 16$ men remain.

Now, we have to select 5 men from 16 men which we can select in ${}^{16}C_5$ ways.

Now, $16 - 5 = 11$ men remain and we have to select 8 men from these 11 which we can select in ${}^{11}C_8$ ways.

Number of ways in which 20 volunteers be alloted to 3 different wards

$$= {}^{20}C_4 \cdot {}^{16}C_5 \cdot {}^{11}C_8$$

Example 4 *A candidate is required to answer 6 out of 10 questions which are divided into 2 groups each containing 5 questions and he is not permitted to attempt more than 4 from each group. In how many ways can he make up his choice?*

Solution Let there be 2 groups A and B. Let group A contains 5 questions and group B contains 5 questions. Since a candidate is required to answer 6 questions and not more than 4 questions from any group.

Hence, a candidate can make up his choice in the following ways

(i) 4 questions from group A and 2 from group B.
(ii) 3 questions from group A and 3 from group B.
(iii) 2 questions from group A and 4 from group B.

Number of ways to answer the questions is

$$\begin{aligned} {}^5C_4 \cdot {}^5C_2 + {}^5C_3 \cdot {}^5C_3 + {}^5C_2 \cdot {}^5C_4 &= 5 \times 10 + 10 \times 10 + 10 \times 5 \\ &= 50 + 100 + 50 = 200 \end{aligned}$$

Example 5 *A person has 12 friends of whom 8 are relatives in how many ways can he invite 7 friends such that at least 5 of them may be relatives?*

Solution Since out of 12 friends, 8 are relatives .

There are $12 - 8 = 4$ which are only friend not relative.

He can invite 7 friends such that at least 5 of them may be relatives in following ways.

He can invite 5 relatives and 2 friends in ${}^8C_5 \cdot {}^4C_2$ ways.

He can invite 6 relatives and 1 friends in ${}^8C_6 \cdot {}^4C_1$ ways.

He can invite 7 relatives and no friends in ${}^8C_7 \cdot {}^4C_0$ ways.

Total number of ways to invite 7 friends

$$= {}^8C_5 \cdot {}^4C_2 + {}^8C_6 \cdot {}^4C_1 + {}^8C_7 \cdot {}^4C_0$$

$$= \frac{8!}{5!\,3!} \cdot \frac{4!}{2!\,2!} + \frac{8!}{6!\,2!} \cdot \frac{4!}{1!\,3!} + \frac{8!}{7!\,1!} \cdot 1$$

$$= 56 \times 6 + 28 \times 4 + 8 = 336 + 112 + 8 = 456$$

Example 6 *There are 3 sections in a paper each having 3 questions. A candidate has to solve any 5 questions choosing at least one question from each section. In how many ways can he make up his choice?*

Solution Let A, B, C be 3 sections each containing 5 questions.

He can answer any 5 questions selecting 1 from A, 1 from B and 3 from C in

$${}^5C_1 \cdot {}^5C_1 \cdot {}^5C_3 = 250 \text{ ways}$$

He can answer any 5 questions selecting 1 from A, 2 from B, and 2 from C in

$${}^5C_1 \cdot {}^5C_2 \cdot {}^5C_2 = 500 \text{ ways}$$

He can answer any 5 questions selecting 1 from A, 3 from B, 1 from C in ${}^5C_1 \cdot {}^5C_3 \cdot {}^5C_1 = 250$ ways

He can answer any 5 questions selecting 2 from A, 2 from B, 1 from C in ${}^5C_2 \cdot {}^5C_2 \cdot {}^5C_1 = 500$ ways

He can answer any 5 questions selecting 3 from A, 1 from B, 1 from C in ${}^5C_3 \cdot {}^5C_1 \cdot {}^5C_1 = 250$ ways

Total number of ways to answer any 5 questions

$$= 250 + 500 + 250 + 500 + 500 + 250 = 2250$$

Example 7 *An 8 oared boat is to be moved by a crew chosen from 11 men of whom 3 can steer but can't row and rest can't steer. In how many ways can the crew be arranged ?*

Solution We can select one persons to steer the boat from the 3 who can steer in $^3C_1$ ways.

We now require 2 particular men who can row on the bow side.

We now require 2 more persons on the bow side and these can be selected in, from the remaining 6 men in $^6C_2$ways.

The remaining 4 can be placed on the other side in $^4C_4$ways.

Number of selections $= {}^3C_1 \cdot {}^6C_2 \cdot {}^4C_4$.

We have 4 persons on the side A and 4 persons on the side B.

Now, number of arrangements of 4 persons in the side $A = 4!$

Number of arrangements of 4 persons in the side $B = 4!$

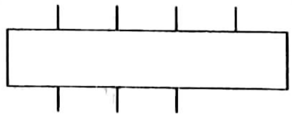
Required number of arrangements

$$= {}^3C_1 \cdot {}^6C_2 = 4!\ 4! = 25920$$

Example 8 *A table has 7 seats, 4 being on one side facing the window and 3 being on opposite side. In how many ways can 7 people be seated on table, if 3 people X, Y, Z must sit on the side facing the window?*

Solution Since the three people X, Y, Z must sit on the side facing the window (4 seats) so that we have to select 1 people for the side facing the window but of $7 - 3 = 4$ in $^4C_1$ ways.

Now, 3 people remain and we have to select all the remaining 3 people for other side in $^3C_3$ ways.



Number of selections $= {}^4C_1 \cdot {}^3C_3$

Now, we have 4 persons (including X, Y, Z) on the side facing the window and 3 persons on other side.

Now, number of arrangements of 4 people in the side facing the window $= 4!$ and number of arrangements of 3 people in the other side is $3!$.

Number of ways in which 7 people can be seated is

$$^4C_1 \cdot {}^3C_3 \cdot 4!\ 3! = 576$$

Example 9 *How many words can be formed out of 10 consonants and 4 vowels such that each contains 3 consonants and 2 vowels ?*

Solution We have to select 3 consonants out of 10 in $^{10}C_3$ ways.

2 vowels out of 4 in $^4C_2$ ways.

In this we have 5 letters $(3 + 2)$

which can be arranged in $5!$ ways. Required number of words

$$= {}^{10}C_3 \times {}^4C_2\ 5! = 720 \times 120$$
$$= 86400$$

Example 10 *Four visitor A, B, C, D arrised at a town which has 5 hotels. In how many ways can they disperse themselves among hotels ?*

(*i*) *If 4 hotels are used to accommodate them*

(*ii*) *If 3 hotels are used to accommodate them in such a way that A and B stay at the same hotel.*

Solution (i) Now, 4 hotels out of 5 can be selected in ${}^5C_4$ways.

Now, 4 visitor can stay in one set of 4 hotels (*i.e*, one visitor in one hotel) is ${}^4P_4 = 4!$
Hence, total number of way of accommodating the visitor is ${}^5C_4\ 4! = 120$

(ii) When A and B stay at the same hotel, then we require 3 hotels, one for (A, B), one for C and one for D. In this case, number of ways of accommodating the visitor is ${}^5C_3 \times 3! = 60$

Example 11 *10 persons amongst whom A, B, C are to speak at function. Find the number of ways in which it can the done, if A wants to speak before B and B wants the speak before C.*

Solution Here, places for A, B and C can be chosen in ${}^{10}C_3$ ways. Now, 7 persons remain to speak and these 7 persons can speak in 7! ways. Hence, number of way in which they can speak is ${}^{10}C_3 \times 7!$.

Example 12 *How many different word of 6 letter can be formed from the letters of the word 'ALLAHABAD'?*

Solution Words having 6 letters from the letters of the word ALLAHABAD can be made in the following case :

(i) 2 alike, 4 distinct (ii) 2 alike, 2 alike, 2 distinct

(iii) 3 alike, 2 alike, 1 other (iv) 3 alike, 3 distinct

(v) 4 alike, 2 distinct (vi) 4 alike, 2 alike.

In the word ALLAHABAD, the letters are *AAAA, LL,H,B,D.*

(i) Now, two alike can be chosen from $(AAAA)(LL)$ in ${}^2C_1 = 2$ ways. After this one pair and 3 (H, B, D) *i.e.*, 4 different letters from these 4 letters we have to choose 4 distinct which is selected in ${}^4C_4 = 1$.
Hence, number of selection of 2 alike and 4 distinct is $2 \times 1 = 2$.
Now, number of word of these 6 letters consisting of 2 alike and 4 distinct is

$$= 2 \times \frac{6!}{2!}$$

$$= 2 \times 360 = 720$$

(ii) Two alike and two alike can be chosen from $(AAAA), (LL)$ in ${}^2C_2 = 1$ way. After this H, B, D remain, from these 3 we have to select 2 different which we can select in ${}^3C_2 = 3$ ways.
Hence, number of selection in this case $= 1 \times 3 = 3$
Now, number of words of 6 letters consisting of 2 alike, 2 alike and 2 different

$$= 3 \times \frac{6!}{2! \cdot 2!} = 3 \times 180 = 540$$

(iii) 3 alike *i.e.*, 3A from 4 A's can be chosen in ${}^1C_1 = 1$ way, 2 alike *i.e.*, 2 L's from (LL) can be chosen in ${}^1C_1 = 1$ way. Now 1 distinct from the remaining 3 can be selected in ${}^3C_1 = 3$ ways.
Hence, number of selection in this case $= 1 \times 1 \times 3 = 3$

Now, number of words of these 6 letters consisting of 3 alike and 1 other is

$$= 3 \times \frac{6!}{3!\ 2!} = 3 \times 60 = 180$$

(iv) Now, 3 alike from *(AAAA)* can be chosen in ${}^1C_1 = 1$ way. Also 3 distinct from $4(LI + BD)$ can be chosen in ${}^4C_3 = 4$ ways.
Hence, number of selections in this case

$$= 1 \times 4 = 4$$

Now, number of words of these 6 letters

$$= 4 \times \frac{6!}{3!} = 480$$

(v) 4 alike can be chosen from *(AAAA)* in ${}^1C_1 = 1$ way.
Now, 2 distinct can be chosen from the remaining $4(L, H, B, D)$ in ${}^4C_2 = 6$ ways.
Hence, number of selection in this case

$$= 1 \times 6 = 6$$

Now, number of words of this 6

$$= 6 \times \frac{6!}{4!} = 6 \times 30 = 180$$

(vi) 4 alike and 2 alike can be chosen from (A, A, A, A), (L, L) in ${}^2C_2 = 1$ way.
Now, number of words of these $6 = 1 \times \frac{6!}{2! \cdot 4!} = 15$

Now, total number of words of 6 letters formed with the letters of the word ALLAHABAD

$$= 720 + 540 + 180 + 480 + 180 + 15 = 2115$$

Example 13 *How many quadrilateral can be formed by joining the vertices of a polygon of n sides?*

Solution Four sides are to be selected for a quadrilateral. Also, there are n vertices of a polygon of n sides and any 4 points out of n can be selected in ${}^nC_4$ ways.
Hence, required number of quadrilateral $= {}^nC_4$.

Example 14 *Find the number of diagonals that can be formed in a polygon of n sides.*

Solution We know there are n vertices in a polygon of n sides, we also know joining any two vertices of a polygon we have either a side or a diagonal of the polygon.
Now, number of ways in which any 2 points out of n points can be selected is ${}^nC_2$.
Hence, number of sides + number of diagonals $= {}^nC_2$.
But number of sides is equal to n and hence number of diagonals

$$= {}^nC_2 - n = \frac{n(n-1)}{2} - n$$

$$= \frac{n(n-3)}{2}$$

Example 15 *If m parallel lines are intersected by n other parallel lines, find the number of parallelogram then formed.*

Solution Now, forming a parallelogram, is actually selecting of 2 lines from the set of m parallel lines and 2 lines from the set of n parallel lines.

But number of selections of 4 lines and 2 lines from the set of m parallel lines and 2 lines from the set of n parallel lines is

$$= {}^mC_2 \cdot {}^nC_2$$

Hence, number of parallelogram thus, formed is ${}^mC_2 \cdot {}^nC_2 = mn(m-1)(n-1)$

Example 16 *There are 15 points in a plane, no two of them are collinear with the exception of 6, which are all the same straight line. Find the number of*

(i) straight lines formed *(ii) number of triangles*

Solution Supposing number three points of the 15 points are collinear.

Now, number of straight lines formed by the 15 points = ${}^{15}C_2$.

Also number of straight lines formed by the 6 points is = ${}^6C_2$.

But as 6 points are collinear and hence ${}^6C_2$ straight lines reduce to a single straight lines. Hence, number of straight lines formed by the 15 points such that 6 points are collinear = ${}^{15}C_2 - {}^6C_2 + 1$

$$= 105 - 15 + 1$$

$$= 91$$

Also, number of triangles formed by 15 points = ${}^{15}C_3$ if the 15 points be such that no three of the 15 points are collinear. But it is given that 6 points are collinear and no triangle is formed by taking any point out of these 6 points.

Hence, required number of triangles

$$= {}^{15}C_3 - {}^6C_3$$

$$= \frac{15}{3!12!} - \frac{6!}{3!\,3!}$$

$$= \frac{15 \cdot 14 \cdot 13 \cdot 12!}{3 \cdot 2 \cdot 12!} - \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$$

$$= 455 - 20$$

Concept Total number of combination or selections of different things taken some or all at a time

$$= {}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_n = 2^n - 1$$

In the problem of this type, all the things are different and also things to be selected are not fixed in number.

Example 1 *Given 5 different green dyes, 4 different blue dyes and 3 different red dyes. How many combination of dyes can be chosen taking at least one green and one blue dyes?*

Solution **Here, it is given that at least one green dyes must be selected. Hence one or more**
or all the 5 green dyes can be selected in

$${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5 - 1 = 31 \text{ ways.}$$

Similarly, one or more or all the 4 different blue dyes can be selected in

$${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$$

$$= 2^4 - 1 = 15 \text{ ways}$$

But it is not given that at least one red dye must be selected so that case of selection of zero (0) dye is also to be included.

And number of ways of selection of 0, 1, 2, 3, red dyes is

$$= {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3 = 8$$

Hence, total number of combination of taking at least one green and at least one blue dye out of 5 green, 4 blue and 3 red dyes

$$= 31 \times 15 \times 8 = 3720$$

Example 2 *A student is allowed to select at most n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select at least one book is 63. Find the value of n.*

Solution Since the students is allowed to select at most n books out of $(2n+1)$ books so that number of selections

$$= {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots + {}^{2n-1}C_n = 63 \quad \ldots(\text{i})$$

Now, $${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots {}^{2n+1}C_{2n+1} = 2^{2n+1}$$

or $$1 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n) + ({}^{2n+1}C_{2n+1-(n+1)} + {}^{2n+1}C_{2n+1-(n+2)} + {}^{2n+1}C_{2n+1-(n+2)} + \ldots + {}^{2n+1}C_{2n+1-2n}) = 2^{2n+1}$$

or $$2 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n) = 2^{2n+1}$$

or $$2 + 2 \cdot 63 = 2^{2n+1}$$

or $$2 + 126 = 2^{2n+1}$$

or $$2^{2n+1} = 128$$

$$2^{2n} = 64 = 2^6 \quad \text{or} \quad 2n = 6$$

$$n = 3$$

Example 3 *We like to form a bouquet from 11 different flowers so that it should contain not less than 3 flowers. Find the number of different ways of forming such a bouquet.*

Solution Since the bouquet should not contain less than 3 flowers so that to form a bouquet we are to select at least 3 flowers and at most 11 flowers out of 11 and hence, required number of ways

$$ {}^{11}C_3 + {}^{11}C_4 + {}^{11}C_5 + \ldots + {}^{11}C_{11}$$

$$= ({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2) + {}^{11}C_3 + {}^{11}C_4 + \ldots {}^{11}C_{11} - {}^{11}C_0 - {}^{11}C_1 - {}^{11}C_2$$

$$= 2^{11} - ({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2)$$

$$= 2^{11} - (1 + 11 + 55) = 1981$$

Distribution of Things into Persons or Groups, Sets, Lots, Packet, Parcels

Let total number of things be n. Now regarding distribution of these n things into groups or persons, we discuss some cases and each case discusses a type of problems.

Case I Number of ways of dividing the given different things into these groups containing respectively p, q, r things, is given by

$${}^{n}C_p \cdot {}^{n-p}C_q \cdot {}^{n-p-q}C_r = \frac{n!}{p!\, q!\, r!}$$

Here, the things p, q, r are distinct and

$$p + q + r = n$$

Case II If $n = m + m + m = 3m$. Then number of ways of dividing $n = 3m$ things into three persons, each containing m things or number of ways of dividing $n = 3m$ things equally among three persons or distributing $n = 3m$ things into 3 equal sets with permutation of sets is given by

$$^{3m}C_m \, ^{2m}C_m \, ^{m}C_m = \frac{(3m)!}{(m!)^3}$$

Case III If $n = m + m + m = 3m$, then number of ways of spliting or dividing $n = 3m$ things into three groups (equally) each containing m things is given by

$$= \frac{^{3m}C_m \cdot {}^{2m}C_m \cdot {}^{m}C_m}{3!}$$

$$= \frac{3m!}{3!\ (m!)^3}$$

Case IV The number of ways of distributing n different things into r different groups or lots is $= r^n$ Here, blank lots have been taken into account.

Case V The number of ways of distributing n different things into r different parcels no lots being blank is

$$r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n + \ldots + (-1)^{r-1}\, {}^rC_{r-1}$$

Case VI The number of ways of distributing n things all alike into r different groups is $^{n+r-1}C_{r-1}$ lots may be blank. No lots being blank and is

$$^{n-1}C_{r-1} \text{ or coefficient of } x^n \text{ in } (1 + x + x^2 + \ldots \infty)^r$$

Case VII Number of ways of arranging n different things into r different groups is

$$\frac{r(r+1)(r+2)\ldots(r+n-1)}{r(r+1)(r+2)\ldots(r+n-1)} \text{ or } n! \cdot {}^{n-1}C_{r-1}$$

according groups may *or* may not be blank.

Example 1 *In how many ways can 52 cards be distributed equally among four players ?*

Solution If we distribute 52 cards equally among 4 players, then $(13 \times 4 = 52)$ each player will receive 13 cards.

$$n = 13 + 13 + 13 + 13 = 52$$

Hence, number of ways of distribution

$$= {}^{52}C_{13} \cdot {}^{52-13}C_{13} \cdot {}^{52-13-13}C_{13} \cdot {}^{52-13-13-13}C_{13}$$

$$= {}^{52}C_{13} \cdot {}^{39}C_{13} \cdot {}^{26}C_{13} \cdot {}^{13}C_{13}$$

$$= \frac{52!}{(13!)^4}$$

Example 2 *In how many ways can a pack of 52 playing cards be divided in 4 groups, three of them having 17 cards each and the fourth first one card ?*

Solution If we divide 52 cards in 4 groups such that first three groups contain 17 card each and fourth first contains one card *i.e.*, $17 + 17 + 17 + 1 = 52$

Then, the number of ways of required distribution $= \dfrac{^{52}C_{17} \cdot {}^{35}C_{17} \cdot {}^{18}C_{17} \cdot {}^{1}C_{1}}{3!}$

$$= \frac{1}{3!} \cdot \frac{52!}{(17!)^3}$$

Example 3 *In how many ways 12 different books can be distributed equally among 4 persons ?*

Solution If we distribute 12 different books equally among 4 persons then each person will get 3 books. Hence, required number of distribution is

$$= {}^{12}C_3 \cdot {}^{9}C_3 \cdot {}^{6}C_3 \cdot {}^{3}C_3 = \frac{12!}{(3!)^4}$$

Example 4 *In how many ways 12 things be divided equally among four groups?*

Solution If we divide 12 different things equally among 4 groups then each groups contains 3 things. Hence required number of ways of distribution of 12 different things equally among 4 groups is given by

$$= \frac{{}^{12}C_3 \cdot {}^{9}C_3 \cdot {}^{6}C_3 \cdot {}^{3}C_3}{4!} = \frac{12!}{4!(3!)^4}$$

Example 5 *In how many ways can 10 balls be divided between two boys, one receiving two and other eight balls ?*

Solution Let B_1 and B_2 be two boys. Then the number of ways of distributing 10 balls such that one boy, say B_1 receives two balls and other boy B_2 receives eight balls or one boy B_2 receives 2 balls and other boy B_1 receives 8 balls is

$$= {}^{10}C_2 \cdot {}^{10-2}C_8 + {}^{10}C_8 \cdot {}^{10-8}C_2$$

$$= {}^{10}C_2 \cdot {}^{8}C_8 + {}^{10}C_8 \cdot {}^{2}C_2$$

$$= {}^{10}C_2 + {}^{10}C_2 = 2 \cdot {}^{10}C_2 = 90$$

Example 6 *In how many ways 10 mangoes can be distributed among 4 persons if any person can get number of mangoes ?*

Solution A person can get 0, 1, 2, 3, ... mangoes and there are four persons.

Here, $r = 4$

Hence, number of distribution of 10 mangoes among 4 persons if any person can get any number of mangoes [including zero mangoes] is

$$= \text{coefficient of } x^{10} \text{ in } (1 + x + x^2 + x^3 + \dots)^4$$

$$= \text{coefficient of } x^{10} \text{ in } \left(\frac{1}{1-x}\right)^4 = \text{coefficient of } x^{10} \text{ in } (1-x)^{-4}$$

$$[\because (1-x)^{-n} = 1 + {}^{n}C_1 x + {}^{n+1}C_2 x^2 + \dots + {}^{n+2}C_3 x^3 + \dots]$$

$$= \text{coefficient of } x^{10} \text{ in } (1 + {}^{4}C_1 x + {}^{5}C_2 x^2 + \dots)$$

$$= {}^{13}C_{10} = {}^{13}C_{13-10}\ {}^{13}C_3$$

Example 7 *In how many ways can r flags be displaced on n poles in a row, disregarding the limitation on the number of flags on a pole ?*

Solution Number of ways in which r flags be displaced on n poles (such that a pole may have 0, 1, 2, 3 and so on flags)

$$= \text{coefficient of } x^r \text{ in } (1 + x + x^2 + \dots)^n$$

$$= \text{coefficient of } x^r \text{ in } \left(\frac{1}{1-x}\right)^n$$

$$= \text{coefficient of } x^r \text{ in } (1-x)^{-n}$$

$$= 1 + {}^nC_1 x + {}^{n+1}C_2 x^2 + \ldots = {}^{n-1+r}C_r = {}^{n+r-1}C_r$$

Example 8 (*i*) *In how many ways can five different books be distributed among three student if each student is to have at least one book?*

(*ii*) *In how many ways can five different books be tied up in three bundles?*

Solution (i) We know that number of ways in which n different things can be distributed into r different parcels, no lots being blank is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-1)^n - \ldots$$

Here, $n = 5$ and $r = 3$, since 5 different books are to be distributed among 3 students. Hence, number of ways of distributing 5 books among 3 students so that each student receives at least one book is

$$= 3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5$$

$$= 3^5 - 3 \times 2^5 + 3 \cdot 1 = 150$$

(ii) Since books are to be tied up in a bundle so that books are to be kept in a group or set and hence required number of ways

$$= \frac{1}{3!}[3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5]$$

$$= \frac{1}{6}(3^5 - 3 \times 2^5 + 3 \cdot 1)$$

$$= \frac{150}{6} = 25$$

Example 9 *In how many ways can* 15 *identical Mathematics books be distributed among six students ?*

Solution We know that number of ways in which n things all alike can be distributed into r different parcels is ${}^{n+r-1}C_{r-1}$ when lots may be blank.

Hence, number of ways in which 15 identical books be distributed among six students is

$$= {}^{15+6-1}C_{6-1} = {}^{20}C_5 = \frac{20!}{5!15!}$$

$$= \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15!}{5 \times 4 \times 3 \times 2 \times 1 \times 15!}$$

$$= 15504$$

Concepts

(I) If $p_1 + p_2 + p_3$ things are such that p_1 things are alike, p_2 things are alike and p_3 things all different, then number of selections of any r things out of $p_1 + p_2 + p_3$ things

$$= \text{coefficient of } x^r \text{ in } (x^0 + x + x^2 + \ldots + x^{p_1})$$

$$\times (x^0 + x + x^2 + \ldots + x^{p_2})(x^0 + x^1)^{p_3} = \text{coefficient of } x^r \text{ in}$$

$$(1 + x + x + x^2 + \ldots + x^{p_1}) \times (1 + x + x^2 + \ldots + x^{p_2})(1 + x)^{p_3}$$

(ii) If $p_1 + p_2 + p_3$ things are such that p_1 things are alike of 1st kind, p_2 things alike of 2nd kind and p_3 things alike of 3rd kind, then we have

(a) Number of selection of any r things out of $p_1 + p_2 + p_3$ things containing at least one thing from p_1 alike things

$$= \text{coefficient of } x^r \text{ in } (x + x^2 + x^3 + \ldots + x^{p_1})$$
$$(x^0 + x + x^2 + \ldots + x^{p_2}) \times (x^0 + x + x^2 + \ldots + x^{p_3})$$
$$= \text{coefficient of } x^r \text{ in } (x + x^2 + \ldots + x^{p_1})(1 + x + x^2 + \ldots + x^{p_2})$$
$$(1 + x + x^2 + \ldots + x^{p_3})$$

(b) Number of selection of any r things containing at least one thing from p_1 alike and one things from p_2 alike things is given by

$$= \text{coefficient of } x^r \text{ in } (x + x^2 + \ldots + x^{p_1})$$
$$(x + x^2 + \ldots + x^{p_2})(x^0 + x + x^2 + \ldots + x^{p_3})$$
$$= \text{coefficient of } x^r \text{ in } (x + x^2 + x^3 + \ldots + x^{p_1})$$
$$(x + x^2 + \ldots + x^{p_2})(1 + x + x^2 + \ldots + x^{p_3})$$

(c) Number of selection of any r things containing at least one things from p_1 alike and two things from p_2 alike things is given by

$$= \text{coefficient of } x^r \text{ in } (x + x^2 + \ldots + x^{p_1})$$
$$(x^2 + x^3 + \ldots + x^{p_2})(1 + x + x^2 + \ldots + x^{p_3})$$

and so on.

Example 1 *Prove that the number of ways in which we can select four letters out of the word EXAMINATION is* 136.

Solution Letters of the word EXAMINATION are $A, A, I, I, N, N, E, X, M, T, O$. *i.e.*, 2 ($A$'s) are alike, 2 ($I$'s) are alike and 2 ($N$'s) are alike and other 5 are all different.

Now, number of selections of any four letters out of the letters of the word EXAMINATION *i.e.*, out of the letters $(A, A), (I, I), (N, N), E, X, M, T, O$.

$$= \text{coefficient of } x^4 \text{ in } (x^0 + x^1 + x^2)(x^0 + x^1 + x^2)(x^0 + x^1 + x^2)(1 + x)^5$$
$$= \text{coefficient of } x^4 \text{ in } (1 + x^1 + x^2)^3 (1 + x)^5$$
$$= \text{coefficient of } x^4 \text{ in } \left(\frac{1 - x^3}{1 - x}\right)^3 (1 + x)^5$$
$$= \text{coefficient of } x^4 \text{ in } (1 - x^3)^3 (1 + x)^5 (1 - x)^{-3}$$
$$= \text{coefficient of } x^4 \text{ in } (1 - {}^3C_1 x^3 + {}^3C_2 x^6 - x^9)(1 + {}^5C_1 x + {}^5C_2 x^2 + {}^5C_3 x^3 + {}^5C_4 x^4 + x^5)(1 - x)^{-3}$$
$$= \text{coefficient of } x^4 \text{ in } (1 - 3x^3 + 3x^6 - x^9)(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)(1 - x)^{-3}$$
$$= \text{coefficient of } x^4 \text{ in } (1 - 3x^3 + 5x + 10x^2 + 10x^3 + 5x^4 - 15x^4 + \ldots)(1 - x)^{-3}$$
$$= \text{coefficient of } x^4 (1 + 5x + 10x^2 + 7x^3 - 10x^4)(1 + {}^3C_1 x + \ldots)$$
$$= {}^6C_4 + 5\,{}^5C_3 + 10\,{}^4C_2 + 7\,{}^3C_1 - 10$$
$$= 15 + 50 + 60 + 21 - 10$$
$$= 146 - 10$$
$$= 136$$

Example 2 *A box contains two white, three black and four red balls. Find the number of ways in which we can select three balls from the box, if at least one black ball is to be included in the selection.*

Solution Considering balls of the same colour alike we have 2 (white) alike balls of one kind, 3 (black) alike balls of 2nd kind and 4 (red) alike balls of 3rd kind. Hence, number of selections of any 3 balls containing at least one black balls is

$$= \text{coefficient of } x^3 \text{ in } \underset{\text{white}}{(x^0 + x + x^2)}\underset{\text{black}}{(x + x^2 + x^3)}$$
$$(x^0 + x + x^2 + x^3 + x^4)$$
$$= \text{coefficient of } x^3 \text{ in } (1 + x + x^2)\, x\, (1 + x + x^2)$$
$$(1 + x + x^2 + x^3 + x^4)$$
$$= \text{coefficient of } x^3 \text{ in } x(1 + x + x^2)^2 \left(\frac{1 - x^5}{1 - x}\right)$$
$$= \text{coefficient of } x^2 \text{ in } \left(\frac{1 - x^3}{1 - x}\right)^2 \left(\frac{1 - x^5}{1 - x}\right)$$
$$= \text{coefficient of } x^2 \text{ in } (1 - 2x^3 + x^6)(1 - x^5)(1 - x)^{-3}$$
$$= \text{coefficient of } x^2 \text{ in } (1 - 2x^3 + x^6 - x^5 + 2x^8 - x^{11})(1 - x)^{-3}$$
$$= \text{coefficient of } x^2 \text{ in } (1 - 2x^3 - x^5 + x^6 + \ldots)(1 + {}^3C_1 x + {}^4C_2 x^2 + \ldots)$$
$$= {}^4C_2 = 6$$

Example 3 *In how many ways can an examiner assign* 30 *marks to* 8 *questions giving not less than 2 marks to any question ?*

Solution Since number of question = 8

No question has mark less than 2 and no question has marks greater than 16 *i.e.*, minimum marks = 2 and maximum marks for a question is 16.

Hence, number of ways in which 30 marks can be assigned to 8 questions

$$= \text{coefficient of } x^{30} \text{ in } (x^2 + x^3 + x^4 + \ldots + x^{16})^8$$
$$= \text{coefficient of } x^{30} \text{ in } x^{16}(1 + x + x^2 + \ldots + x^{14})^8$$
$$= \text{coefficient of } x^{14} \text{ in } \left(\frac{1 - x^{15}}{1 - x}\right)^8$$
$$= \text{coefficient of } x^{14} \text{ in } (1 - x^{15})^8 (1 - x)^{-8}$$
$$= \text{coefficient of } x^{14} \text{ in } (1 - x)^{-8}$$
$$= \text{coefficient of } x^{14} \text{ in } (1 + {}^8C_1 x + {}^9C_2 x^2 + \ldots)$$
$$= {}^{21}C_{14}$$
$$= \frac{21!}{14!\,7!}$$
$$= \frac{21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$
$$= 116280$$

Example 4 *Show that the number of ways of selecting n things out of 3n things of which n are of one kind and alike and n are of 2nd kind and rest are unike is* $(n+2)2^{n-1}$.

Solution Here, n things are alike, out of which $0, 1, 2, 3, \ldots, n$ may be selected.

Also, n other things are alike out of which $0, 1, 2, 3, \ldots\ldots, n$ may be selected and rest n things are all different.

Now, the number of selection of n things = coefficient of x^n in

$$(x^0 + x + x^2 + x^n)(x^0 + x + x^2 + x^n)\underbrace{(1+x)(1+x)\ldots(1+x)}_{n \text{ times}}$$

$$= \text{coefficient of } x^n \text{ in } (1 + x + x^2 + x^3 \ldots + x^n)^2 (1+x)^n$$

$$= \text{coefficient of } x^n \text{ in } \left(\frac{1 - x^{n+1}}{1 - x}\right)^2 (1+x)^n$$

$$= \text{coefficient of } x^n \text{ in } (1-x)^{-2}(1+x)^n$$

$$= \text{coefficient of } x^n \text{ in } (1-x)^{-2}[2-(1-x)]^n$$

$$= \text{coefficient of } x^n \text{ in } (1-x)^{-2}[2^n - n2^{n-1}(1-x) + \frac{n(n-1)}{2!}2^{n-2} + \ldots + (-1)^n(1-x)^n]$$

$$= \text{coefficient of } x^n \text{ in } 2^n(1-x)^{-2} - n\cdot 2^{n-1}(1-x)^{-1}$$

$$= 2^n \cdot {}^{n+1}C_n - n \cdot 2^{n-1} = 2^n(n+1) - n\cdot 2^{n-1}$$

$$= [2(n+1) - n]2^{n-1} = (n+2)\cdot 2^{n-1}$$

Example 5 *Suppose that there are pile of red, blue and green balls and that each pile contains at least eight balls.*

(*i*) *In how many ways can eight ball be selected?*

(*ii*) *In how many ways can eight balls be selected if at least one ball of each colour is to be selected?*

Solution There are at least 8 (red) alike balls, 8 (blue) alike balls and 8 (green) alike balls. Balls of each colour may be unlimited in numbers.

To select 8 balls in all we select

0, 1, 2, ... balls from each colours of balls. Hence, number of selection of any eight balls

$$= \text{coefficient of } x^8 \text{ in } \underbrace{(x^0 + x + x^2 + \ldots)}_{\text{red}} \underbrace{(x^0 + x + x^2 + \ldots)}_{\text{blue}} \times \underbrace{(x^0 + x + x^2 \ldots)}_{\text{green}}$$

$$= \text{coefficient of } x^8 \text{ in } (1 + x + x^2 + x^3 + \ldots)^3$$

$$= \text{coefficient of } x^8 \text{ in } \left(\frac{1}{1-x}\right)^3 = (1-x)^{-3}$$

$$= \text{coefficient of } x^8 \text{ in } (1 + {}^3C_1 x + {}^4C_2 x^2 + \ldots)$$

$$= {}^{10}C_8 = \frac{10!}{8!\,2!}$$

$$= 5 \times 9 = 45$$

Example 6 *Show that the number of different selection of 5 letters from 5 a's, 4 b's, 3c's, 2d's and one e is 71.*

Solution Here, 5 letters (a's) are alike, 4 letters (b's) are alike, 3 letters (c's) are alike and 2 letters (d's) are alike and one other letters is e.

Hence, number of different selections of 5 letters

$$= \text{coefficient of } x^5 \text{ in } (x^0 + x + x^2 + \ldots + x^5)(x^0 + x + x^2 + \ldots + x^4)(x^0 + x + x^2 + x^3)(x^0 + x + x^2)(x^0 + x)$$

$$= \text{coefficient of } x^5 \text{ in } (1 + x + x^2 + \ldots + x^5)(1 + x + x^2 + \ldots + x^4)(1 + x + x^2 + x^3)(1 + x + x^2)(1 + x)$$

$$= \text{coefficient of } x^5 \text{ in } \frac{1-x^6}{1-x}\cdot\frac{1-x^5}{1-x}\cdot\frac{1-x^4}{1-x}\cdot\frac{1-x^3}{1-x}(1+x)$$

$$= \text{coefficient of } x^5 \text{ in } (1-x^6)(1-x^5)(1-x^4)(1-x^3)(1+x)(1-x)^{-4}$$

$$= \text{coefficient of } x^5 \text{ in } (1 + x - x^3 - 2x^4 - 2x^5\ldots)(1 + {}^4C_1x + {}^5C_2x^2\ldots.)$$

$$= {}^8C_5 + {}^7C_4 - {}^5C_2 - 2\,{}^4C_1 - 2$$

$$= 56 + 35 - 10 - 8 - 2$$

$$= 91 - 20 = 71$$

Example 7 *If n objects are arranged in a row, then find the number of ways of selecting three of them objects so that no two of them are next to each other.*

Solution Let the three persons P_1, P_2, and P_3 selected $\underline{n}_1\, P_1\, \underline{n}_2\, P_2\, \underline{n}_3\, P_3\, \underline{n}_4$

Let n_1 and n_4 be the number of persons to the left of P_1 and to the right of P_3. Let n_2 and n_3 be respectively the number of persons between $P_1, P_2; P_2, P_3$

The $n_1 + n_2 + n_3 + n_4 = n - 3$ (3 are P_1, P_2, P_3) where $n_1, n_4 \geq 0$ and $n_2, n_3 \geq 1$

Hence, required number of selections

$$= \text{coefficient of } x^{n-3} \text{ in } (x^0 + x + x^2 + \ldots)(x^1 + x^2 + x^3)(x + x^2 + x^3 + \ldots)(x^0 + x + x^3 + \ldots)$$

$$= \text{coefficient of } x^{n-3} \text{ in } (1 + x + x^2 + x^3 + \ldots)^2 (x + x^2 + x^3 + \ldots)^2$$

$$= \text{coefficient of } x^{n-3} \text{ in } x^2(1 + x + x^2 + x^3 + \ldots)^2(1 + x + x^2 + \ldots)^2$$

$$= \text{coefficient of } x^{n-5} \text{ in } (1 + x + x^2 + x^3 + \ldots)^4$$

$$= \text{coefficient of } x^{n-5} \text{ in } \left(\frac{1}{1-x}\right)^4 = \text{coefficient of } x^{x-5} \text{ in } (1-x)^{-4}$$

$$= (1 + {}^4C_1x + {}^5C_2x^2 + \ldots) = {}^{n-2}C_{n-5} = {}^{n-2}C_3$$

Example 8 *A box contains 2 white balls, 3 black balls and 4 red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw?*

(i) If balls of the same colour are considered to be identical.

(ii) All balls are considered to be different.

Solution (i) In this case, there are 2 (white) alike balls, 3 (black) alike balls and 4 (red) alike balls, we have to select 3 balls including at least one black balls *i.e.*, we may select, 0 or 1or 2 balls from white balls, 1or 2 or 3 from 3 black balls and 0 or 1 or 2 or 3 or 4 from 4 red balls. Hence, number of selection of any 3 balls containing at least one black balls

$= \text{coefficient of } x^3 \text{ in } (x^0 + x + x^2)$
$(x + x^2 + x^3)(x^0 + x + x^2 + x^3 + x^4)$

$= \text{coefficient of } x^3 \text{ in } (1 + x + x^2)(1 + x + x^2)(1 + x + x^2 + x^3 + x^4)$

$= \text{coefficient of } x^2 \text{ in } (1 + x + x^2)^2 (1 + x + x^2 + x^3 + x^4)$

$= \text{coefficient of } x^2 \text{ in } \left(\frac{1-x^3}{1-x}\right)^2 \left(\frac{1-x^5}{1-x}\right)$

$= \text{coefficient of } x^2 \text{ in } \frac{(1 - 3x^2 + 3x^6 - x^9)(1 - x^5)}{(1-x)^3}$

$= \text{coefficient of } x^2 \text{ in } (1 - 3x^3 + \dots)(1 - x)^{-3}$

$= \text{coefficient of } x^2 \text{ in } (1 - x)^{-3}$

$= 1 + {}^3C_1 x + {}^4C_2 x^2 + \dots$

$= {}^4C_2$

$= 6$

(ii) In this case, all the balls are different and total number of balls = 2 (white) + 3 (black) + 4 (red) = 9 Hence, number of selections of any 3 balls from 9 = ${}^9C_3 = 84$
Also, number of selection of 3 balls containing no black balls = ${}^6C_3 = 20$
Hence, number of selection of 3 balls containing at least one black balls = 84 − 20
= 64

Example 9 *If there are three different kinds of mangoes for sale in a market. In how many ways can you purchase 25 mangoes?*

Solution Here, we have unlimited number of 3 kinds of mangoes and we can select 25 mangoes in all. Hence, we can select 0 or 1 or 2 or 3... mangoes from each kind of mangoes.

Hence, number of selection of 25 mangoes

$= \text{coefficient of } x^{25} \text{ in } (x^0 + x + x^2 + x^3 \dots)^3$

$= \text{coefficient of } x^{25} \text{ in } \left(\frac{1}{1-x}\right)^3$

$= \text{coefficient of } x^{25} \text{ in } (1 - x)^{-3}$

$= 1 + {}^3C_1 x + {}^4C_2 x^2 + \dots$

$= {}^{27}C_{25} = {}^{27}C_2 = \frac{27 \cdot 26}{2} = 351$

Example 10 *If $n (\geq 3)$ is a +ve integer, then find the number of positive integral solution of $x + y + z = n$.*

Solution The maximum value of x, y, z satisfying $x + y + z = n$ is $n - 2$.

Hence, the number of positive integral solution of $x + y + z = n$ is given by the

$= \text{coefficient of } x^n \text{ in } (x + x^2 + x^3 + \dots + x^{n-2})^3$

$= \text{coefficient of } x^n \text{ in } x^3 (1 + x + x^2 + \dots + x^{n-3})^3$

$= \text{coefficient of } x^{n-3} \text{ in } \left(\frac{1 - x^{n-2}}{1-x}\right)^3$

$$= \text{coefficient of } x^{n-3} \text{ in } (1-x^{n-2})^3 (1-x)^{-3}$$
$$= \text{coefficient of } x^{n-3} \text{ in } (1-x)^{-3}$$
$$= 1 + {}^3C_1 x + {}^4C_2 x^2 + \ldots$$
$$= {}^{n-1}C_{n-3} = {}^{n-1}C_2$$

Example 11 *Find the number of integral solution of* $x_1 + x_2 + x_3 = 0$ *with* $x_2 \geq -5$.

Solution Here, we have to choose any three integers out of $-5, -4, -3, -2, -1, 0, 1, 2\ldots$ such that sum of the three chosen integral is 0. Hence, number of integral solutions of

$$x_1 + x_2 + x_3 = 0 \qquad \ldots(\text{i})$$

is given by

$$= \text{coefficient of } x^0 \text{ in } (x^{-5} + x^{-4} + x^{-3} + \ldots)^3$$
$$= \text{coefficient of } x^0 \text{ in } \left(\frac{1}{x^5} + \frac{1}{x^4} + \frac{1}{x^3} + \ldots\right)^3$$
$$= \text{coefficient } x^0 \text{ in } \left(\frac{1 + x + x^2 + x^3 + \ldots}{x^5}\right)^3$$
$$= \text{coefficient of } x^0 \text{ in } \frac{(1 + x + x^2 + \ldots)^3}{x^{15}}$$
$$= \text{coefficient of } x^{15} \text{ in } (1 + x + x^2 + \ldots)^3$$
$$= \text{coefficient of } x^{15} \text{ in } \left(\frac{1}{1-x}\right)^3$$
$$= \text{coefficient of } x^{15} \text{ in } (1-x)^{-3} = 1 + {}^3C_1 x + {}^4C_2 x + \ldots$$
$$= {}^{17}C_{15} = \frac{17!}{2!\,15!} = \frac{17 \cdot 16 \cdot 15!}{2 \cdot 15!}$$
$$= 17 \times 8 = 136$$

Hence, number of integral solutions of $x_1 + x_2 + x_3 = 0$ with $x_2 \geq -5$ is 136.

Example 12 *Find the number of integral solutions of* $2x + y + z = 20$ *with* $x, y, z \geq 0$.

Solution Here setting $y = z = 0$, we have $2x = 20$ or $x = 10$.

Hence, maximum value of $x = 10$

i.e., x may be taken from 0, 1, 2, 3, ... to 10

such that $2x + y + z = 20$

For $x = 0$, $y + z = 20$. But integral solutions of $y + z = 20$

$$= \text{coefficient of } x^{20} \text{ in } (1 + x + x^2 + x^3 + \ldots)^2$$
$$= \text{coefficient of } x^{20} \text{ in } \left(\frac{1}{1-x}\right)^2$$
$$= \text{coefficient of } x^{20} \text{ in } (1-x)^{-2}$$
$$= \text{coefficient of } x^{20} \text{ in } (1-x)^{-2}$$
$$= \text{coefficient of } x^{20} \text{ in } (1 + {}^2C_1 x + {}^3C_2 x^2 + \ldots)$$
$$= {}^{21}C_{20} = 21$$

For $x = 1$, $y + z = 18$ and number of integral solutions of $y + z = 18$ is given by the coefficient of x^{18} in $(1 + x + x^2 + \dots)^2(1 - x)^{-2}$

$$= 1 + {}^2C_1 x + \dots = {}^{19}C_{18} = 19$$

For $x = 2$, $y + z = 16$ and number of integral solution of $y + z = 16$ is ${}^{17}C_{16} = 17$

For $\qquad x = 3$

$y + z = 14$ and number of integral solutions of this equation is ${}^{15}C_{14} = 15$ and so on.

For $\qquad x = 9$

$y + z = 2$ and number of integral solutions of this equation is ${}^3C_2 = 3$.

For $\qquad x = 10$

$y + z = 0$ and number of integral solution of this equation is $= 1$

For $\qquad y = 0, z = 0$

In this way, the number of integral solution of $2x + y + z = 20$ with $x, y, z \geq 0$ is

$$= 21 + 19 + 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 + 1$$

$$= \frac{11}{2}[2.21 + (11 - 1)(-2)]$$

$$= \frac{11}{2}(42 - 20)$$

$$= \frac{11}{2} \times 22$$

$$= 121$$

We know from the greatest coefficient in the binomial theorem that for n is even, ${}^nC_r$ is greatest, if

$$r = \frac{n}{2}$$

for n is odd, ${}^nC_r$ is greatest, if

$$r = \frac{n+1}{2} \text{ or } \frac{n-1}{2}$$

Example 13 *Out of 15 balls, of which some are white and rest are black. How many should be white so that the number of ways in which the balls can be arranged in a row may be the greatest possible? It is given that the balls of the same colour are alike.*

Solution Let number of white balls be n. Then number of black balls is $15 - n$. Now number of arrangements of these 15 balls consisting of n white (alike) and $15 - n$ black (alike) balls

$$= \frac{15!}{15! - n\, n!} = {}^{15}C_n$$

It is given that number of arrangement is greatest *i.e.*, ${}^{15}C_n$ is greatest. But ${}^{15}C_n$ is greatest, if

$$n = \frac{15-1}{2} \text{ or } \frac{15+1}{2}$$

i.e., $\qquad n = 7$ or 8

Hence, number of white balls = 7 or 8

Example 14 *A person wishes to make up as many different parties as he can out of 20 friends. Each party consists of the same number. How many should be invited at a time? In how many of these parties would the same be found?*

Solution Let the number of friends be invited be n, then number of parties = ${}^{20}C_n$.

But it is given that ${}^{20}C_n$ is maximum and hence

$$n = \frac{20}{2} = 10.$$

Let a particular friend attends P parties. Then, number of combination of 20 taken 10 at a time such that the particular friend must be included = P.

But number of combination of 20 taken 10 at a time such that the particular member is included is

$$= {}^{20-1}C_{10-1} = {}^{19}C_9$$

Hence, $P = {}^{19}C_9$

Combinatorial Identities and Binomial Coefficients

In the counting process, the counting of things in two different ways gives equal number of ways of counting and this leads to an equality called identity. for example, if we choose r objects out of a collection of n distinct objects, this can be done in $\binom{n}{r}$ ways. But to choose r objects is equivalent to reject the remaining $n - r$ objects. Hence, we have an identity $\binom{n}{r} = \binom{n}{n-r}$.

Such identities are called combinatorial identities and the process which leads to the formulation of combinatorial identities is called combinatorial argument.

The number of the type $\binom{n}{r}$ which we are using are called Binomial coefficients because they appear in the Binomial expansion $(x + y)^n$; where n is a positive integer. The proof of Binomial theorem by using the principle of inductions is already known to the students. Below we present the combinatorial proof of Binomial theorem.

Combinatorial Proof of Binomial Theorem

Consider the expansion of $(x + y)^n$, where n is a positive integer.

$$(x + y)^n = (x + y)(x + y)(x + y)\ldots(x + y)$$

This expansion consists of three steps.

Step 1. Select one term from each of n factors.

Step 2. Multiply the selections together.

Step 3. Sum the products.

For example

$$(x + y)^3 = (x + y)(x + y)(x + y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$
$$= x^3 + x^2y + x^2y + xy^2 + x^2y + x\,y^2 + y^2x + y^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

The term of the type $x^r y^{n-r}$ arises by selecting x's from r factors and y's from $n - r$ factors. But this can be done in $\binom{n}{r}$ ways. Hence, the coefficient of $x^r y^{n-r}$ is $\binom{n}{r}$.

Therefore, $$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \ldots + \binom{n}{n}x^ny^0,$$

It is the same things as

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}x^0y^n.$$

This is Binomial theorem.
If we put $x = y = 1$ in the Binomial theorem,
we get

$$(1+1)^n = \binom{n}{0}1^n\,1^0 + \binom{n}{1}1^{n-1}\,1^1 + \binom{n}{2}1^{n-2}\,1^2 + \ldots + \binom{n}{n}1^0\,1^n$$

$$\therefore \quad 2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$$

We now give combinatorial argument to prove this identity.
Consider the set S with n elements

$$S = \{x_1, x_2, \ldots, x_n\}$$

In the right hand side of the identity, the general term $\binom{n}{r}$ counts the number of r elements subset of S.

Hence, the right hand side counts the number of subsets of the set S. *Now if* A is any subset of S, then $x_1 \in A$ or $x_1 \notin A$. Similarly $x_2 \in A$ or $x_2 \notin A, \ldots$ and finally $x_n \in A$ or $x_n \notin A$. Thus the subset A is formed in 2^n different ways. This is equivalent to saying that there are 2^n different subsets of the set S.

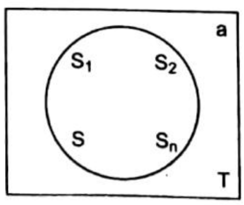
$$\therefore \quad 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

Example 1 *Give combinatorial argument to prove that* $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Solution Suppose S is a set with $n-1$ elements. Choosing an element 'a' not belonging to S, consider the set $T = S \cup \{a\}$ of n elements. Now $\binom{n}{k}$ denotes the number of k element subsets of the set T. This is left hand side of the identity.

We partition the k element subsets of T in two classes :

(i) The k element subsets of T not containing element 'a'. Here we have to choose all k elements from the set S. These subset are $\binom{n-1}{k}$ in number.



(ii) The k element subsets of T containing element 'a'. Here element 'a' is already with us. Hence to form k element subsets are $\binom{n-1}{k-1}$ in number.

Hence, the total number of k element subsets of T are $\binom{n-1}{k}+\binom{n-1}{k-1}$.

$$\therefore \qquad \binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$$

Example 2 *Give combinatorial proof of* $\binom{2n}{2}=2\binom{n}{2}+n^2$.

Solution Let $X=\{x_1, x_2, \dots, x_n\}$ and $Y=\{y_1, y_2, \dots, y_n\}$ has $2n$ elements. The two element subsets of $X \cup Y$ are $\binom{2n}{2}$ in number. This is left hand side of the identity, we partition the 2 element subsets of $X \cup Y$ into 3 classes.

1. Both the elements are from the set X. This gives 2 elements subsets of X. These are again $\binom{n}{2}$ in number.
2. Both the elements are from the set Y. This gives 2 elements subsets of Y. These are again $\binom{n}{2}$ in number.
3. One element is from X and one element is from Y. In this case the two elements can be chosen in $\binom{n}{1}\binom{n}{1}=n.n=n^2$ ways.

   So these subsets are n^2 in number. Hence, the total number of 2 elements subsets of $X \cup Y$ are

$$\binom{n}{2}+\binom{n}{2}=2\binom{n}{2}+n^2, \quad \binom{2n}{2}=2\binom{n}{2}+n^2.$$

Example 3 *Give combinatorial argument to prove that*

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\dots+\binom{n}{n}^2=\binom{2n}{n}.$$

Solution Consider two disjoint sets $X=\{x_1, x_2, \dots, x_n\}$ and $Y=\{y_1, y_2, \dots, y_n\}$ of n elements each.

Then, $X \cup Y=\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ is a set with $2n$ elements. The n element subsets of $X \cup Y$ are $\binom{2n}{n}$ in number. This is right hand side of the identity. Now any n element subset of $X \cup Y$ is formed by choosing k elements from X and $n-k$ elements form $Y: k=0,1,2,3,\dots,n$ this can be done in $\binom{n}{k}\binom{n}{n-k}$ ways.

But we know $\binom{n}{k}=\binom{n}{n-k}$

$$\therefore \qquad \binom{n}{k}\binom{n}{n-k}=\binom{n}{k}\binom{n}{k}=\binom{n}{k}^2,$$

$$k=0,1,2,\dots,n$$

Therefore, the total number of n element subsets of $X \cup Y$ in

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2$$

This is left hand side of the identity.

Example 4 *Give combinatorial argument to prove that*

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}.$$

Solution Consider a set of n person. There are $\binom{n}{k}$ different ways of forming a committee of k person $k = 1, 2, 3, \ldots, n$.

After the committee is formed, those k members decide to have a dinner party at each member's house. Thus, there will be $k\binom{n}{k}$ parties of k member's each. Putting $k = 1, 2, 3, \ldots, n$, the total number of parties will be

$$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = \sum_{k=1}^{n} k\binom{n}{k}$$

This is left hand side of the identity. Now consider the first person P_1 and the number of parties at his house. For each of remaining $n-1$ persons, there are two possibilities, being or not being included in the party at P_1' s house. So there are 2^{n-1} parties at P_1' s house. Similarly there are 2^{n-1} parties at P_2' s house, ... p_n' s house, Hence, the total number of parties is $n \cdot 2^{n-1}$. This is the right hand side of the identity

$$\therefore \quad 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = n \cdot 2^{n-1}$$

Example 5 *Give combinatorial proof of the identity*

$$\sum_{k=0}^{n} 2^k\binom{n}{k} = 3^n$$

Solution Consider three letters A, B, C. We form different strings of length n formed by these 3 letters with repetition.

Since each of n position can be filled in 3 ways, there are 3^n strings.

This is right hand side of the identity. Now, we divide the strings of length n into disjoint classes depending upon how many times the letters A appears in the string.

Suppose A appears k times. Then, letter A can choose k positions in $\binom{n}{k}$ ways. The remaining $n-k$ places are to be filled by using two letters B and C. This can be done in 2^{n-k} different ways. Thus, there are $2^{n-k}\binom{n}{k}$ strings of length n containing k A's. Put $k = 0, 1, 2, \ldots n$ and add, the total number of strings is

$$2^n\binom{n}{0}+2^{n-1}\binom{n}{1}+2^{n-2}\binom{n}{2}+\ldots+2^0\binom{n}{n}$$
$$=2^n\binom{n}{n}+2^{n-1}\binom{n}{n-1}+2^{n-2}\binom{n}{n-2}+\ldots+2^0\binom{n}{0}$$

This is the left hand side of the identity

$\therefore$ $$\sum_{n=1}^{n} 2^k\binom{n}{k}=3^n$$

Example 6 *Prove that*

$$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\ldots\binom{n}{r}=\binom{n+1}{r+1}.$$

Use this result to find the sum

$$1+2+3+\ldots+n.$$

Solution We have

$$\binom{m}{r}+\binom{m}{r+1}=\binom{m+1}{r+1}$$
$$\binom{m}{r}=\binom{m+1}{r+1}-\binom{m}{r+1}$$

Put $m=r, r+1, r+2\ldots, n$ and add

$\therefore$ $$\binom{r}{r}+\binom{r+1}{r}+\ldots+\binom{n}{r}=\binom{r+1}{r+1}-\binom{r}{r+1}+\binom{r+3}{r+1}-\binom{r+2}{r+1}$$
$$+\binom{r+2}{r+1}-\binom{r+1}{r+1}+\binom{r+4}{r+1}-\binom{r+3}{r+1}+\ldots..+\binom{n+1}{r+1}-\binom{n}{r+1}$$

On the right side, the terms cancel diagonally and also

$\therefore$ $$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\ldots+\binom{n}{r}=\binom{n+1}{r+1}$$

Deduction: Put $r=1$ in the above result

$\therefore$ $$\binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\ldots+\binom{n}{1}=\binom{n+1}{2}$$

$\therefore$ $$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

Example 7 *Give combinatorial argument and prove that*

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\ldots+\binom{n+r}{r}=\binom{n+r+1}{r}$$

Solution Consider the equation $x_1+x_2+\ldots+x_n+x_{n+1}+x_{n+2}=r$

It has $\binom{n+2+r-1}{r}=\binom{n+r+1}{r}$ solutions in non-negative integers. This gives right hand side of the identity.

Now, depending upon the value of $x_n + 2$, we divide these solution into various disjoints classes as below.

Put $x_n + 2 = 0$. Then, equation is

$$x_1 + x_2 + \dots + x_{n+1} = r$$

The number of solution is $\binom{n+1+r-1-1}{r-1} = \binom{n+r-1}{r-1}$

Put $x_{n+2} = 2$. Then, equation is

$$x_1 + x_2 + \dots + x_{n+1} = r-2$$

The number of solution is $\binom{n+1+r-2-1}{r-2} = \binom{n+r-2}{r-2}$

Put $x_{n+2} = r-1$. Then, equation is

$$x_1 + x_2 + \dots + x_{n+1} = 1$$

The number of solution is

$$\binom{n+1+1-1}{1} = \binom{n+1}{1}$$

Finally put $x_{n+2} = r$. Then, equation is $x_1 + x_2 + \dots + x_{n+1} = 0$

The number of solution is

$$\binom{n+1+0-1}{0} = \binom{n}{0}$$

Addition of these gives the total number of solutions

$$\binom{n+r}{r} + \binom{n+r-1}{r-1} + \binom{n+r-2}{r-2} + \dots + \binom{n+1}{1} + \binom{n}{0}$$

$$= \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{n+r}{r}$$

This is left hand side of the identity

$$\therefore \quad \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{n+r}{r} = \binom{n+r+1}{r}$$

The Pigeonhole Principle

Introduction The truth of the following statement is obvious :

If 4 pigeons fly into 3 pigeonholes, some pigeonhole must receive at least two pigeons.

This statement is a sample example of a very basic combinatorial principle, called the **pigeonhole principle.** The simplest form of this principle is as follows:

Pigeonhole Principle (PP1) If more than n objects are distributed into n boxes, then at least one box must receive at least two objects.

This follows by noting that, if each of the n boxes contained at most 1 object only, then the n boxes together would contain at most n objects contrary to the assumption that more than n objects were put in the boxes.

This principle was first stated formally by Dirichlet and is therefore also called **Dirichlet's box principle** or **Dirichlet's drawer principle.** Let us apply it to some simple problems.

Example 1 *A box contains three pair of socks; coloured red, blue and white. Suppose I take out the socks without looking at them. How many socks must I take out in order to be sure that they will include a matching pair ?*

Solution If I take only 2 or 3 socks, it is possible that they are all different.

For example, they may be one red and one blue or one red, one blue and one white. But if I takes 4 socks, these must include a matching pair. Here the 4 chosen socks are the "objects" and the 3 colour are the "boxes" and by PP1, it follows that at least two of the 4 chosen socks must have the same colour and hence must for a matching pair. Thus the minimum number of socks to be taken is 4.

Example 2 *Show that in a group of* 8 people, at least two will have their birthday on the same day of the week.

Solution Here the 8 people are the "objects" and the 7 days namely,

Monday, Tuesday, ..., Sunday are the "boxes". Hence, by PP1 at least two of the people must belong to the same day. Similarly, it is clear that in a group of 13 people, at least two will have their birthday in the same month.

Example 3 *How many student do you need in a school to guarantee that there are at least* 2 *students, who have the same first two initials ?*

Solution There are $26 \times 26 = 676$ different possible sets of two initials that can be obtained using the 26 letters $A, B, \ldots, Z$. So by PP1, the number of students should be greater than 676.

Example 4 *A professor tells* 3 *jokes in his ethics class each year. How large a set of jokes does the professor need in order never to repeat the exact same triple of jokes over a period of 12 years ?*

Solution Let n be the number of jokes needed. Now for a period of 12 years 12 triples are needed. Also these are to be all distinct. These are $\binom{n}{3}$ different triples that can be chosen from the n jokes. Hence n must be chosen, so that $\binom{n}{3} \geq 12$

$$\text{or} \qquad \frac{n(n-1)(n-2)}{6} \geq 12 \quad \text{or} \quad n \geq 6$$

Hence by PP1, the smallest value of n is 6.

Example 5 *In a round–robin tournament, show that there must be two players with the same number of wins, if no player loses all matches.*

Solution In a round-robin tournament of n players, each player plays every other player once and each game results in a win for one of the players. Let a_i, denote the number of games won by player i. Then, by assumption, every player wins at least one game, so that $a_i \geq 1$ for all i. Also, a player can win at most $n-1$ games.

Hence, $1 \leq a_i \leq n-1$ for $i = 1, \ldots, n$. Thus, the n 'objects' *i.e.*, the number $a_1, \ldots, a_n$ are put in the $(n-1)$ 'boxes'

$$\{1\}, \{2\}, \{3\}, \ldots, \{n-1\}$$

Hence, by PP1, two of the numbers, say a_r and a_s, must be the same box *i.e.*, they must be equal.

Example 6 *Show that a party of 20 people, there are two people who have the same number of friends.*

Solution Here it is assumed that, if A is a friend of B, then B is also a friend of A. Also, we will show that the result is true not only for 20 people but for any number n of them $(n \geq 2)$.

Let $P_1, \ldots, P_n$ denote the n people and let a_i be the number of friends of person P_i. Then since a person has either no friends or at most $n-1$ friends, we have $0 \leq a_i \leq (n-1)$, for $1 \leq i \leq n$.

First suppose that everyone at the party has at least one friend. Then, $a_i \geq 1$ for all $i = 1, \ldots, n$ and so the n number $a_1, a_2 \ldots a_n$ come from then $(n-1)$ numbers $1, \ldots, n-1$. Hence, by PP1, at least two of the a's must be equal, say $a_r = a_s$. That is the persons P_r and P_s have the same number of friends.

Secondly, suppose that one of the persons, say P_n, has no friends. Then, each of the remaining $n-1$ persons has at most $n-2$, friends so that now the n numbers $a_1, \ldots, a_n$ come from the $n-1$ numbers $0, 1, \ldots, n-2$. Hence, as before, two of the n persons must have the same number of friends.

Remark

Note that in applying the Pigeonhole Principle, the crucial step is always to decide, what are the 'objects' and what are the 'boxes'. This choice is not always easy to make. Also, PP1 does not tell us how to make this choice, but a clue is provided by the fact that the arithmetic of the numbers involved in the problem is to be exploited.

Further note that PP1 claims only the existence of at least one box containing two or more objects; it does not tell us, how to locate this box.

Second Form of the Pigeonhole Principle (PP2)

For any positive integers n, t, if $tn + 1$ or more objects are placed in n boxes, then at least one box will contain more than t objects.

Proof If each of the n boxes contained at most t objects, then the n boxes together would contain at most tn objects. This contradict the fact that, $tn + 1$ or more objects were placed in the boxes. Hence at least one box must contain the more than t objects.

As an illustration of PP2, assume that no human has more than 300000 hairs on his/her head. A city has a population over 300000. Here the residents are the 'objects' and there are $n = 300000$ 'boxes' corresponding to the different possible number of hairs. Thus, by PP1, at least 2 residents of the city must be in the same box *i.e.*, must have the same number of hair on their heads. But if the city has more than 15 million residents, then at least 50 of them must have the same number of hairs on their heads. This follows by PP2, since in this case the number of residents is more than $15000000 = 50 \times 300000$ and so some box must contain more than $t = 50$ objects.

Third Form of the Pigeonhole Principle (PP3)

If the average of n positive number is t, then at least one of the numbers is greater than or equal to t. Further, at least one of the numbers is less than or equal to t.

Proof Let $a_1, a_2, \ldots, a_n$ be the numbers. Then by data,

$$\frac{a_1 + \ldots + a_n}{n} = t$$

So that

$$a_1 + \ldots + a_n = tn \qquad \ldots(i)$$

Hence, if each of the n numbers $a_1, \ldots, a_n$ is less than t, then the sum of these numbers would be less than nt, contradicting (i).

A similar argument shows that at least one of the numbers is less than or equal to t.

Remark

If the number $a_1, a_2, \ldots, a_n$ are integers, then PP3 says that at least one of them is $\geq t_0$, where t_0 is the smallest integer not less than t; and at least one is $\leq [t]$, where $[t]$ is the integral part of t.

Fourth Form of the Pigeonhole Principle (PP4)

Let $q_1, q_2, \ldots, q_n$ be positive integers. If $(q_1 + q_2 + \ldots + q_n - n + 1)$ objects are put into n boxes, then either the first box contains at least objects or the second box contains at least q_2 objects, ... or the nth box contains at least q_n objects.

Proof Suppose we distribute

$(q_1 + q_2 + \ldots + q_n - n + 1)$ objects in n boxes and if for each $i = 1, 2, \ldots, n$, the ith box contains less than q_i object, then the total number of objects in the n boxes is

$$\leq (q_1 - 1) + (q_2 - 1) + \ldots + (q_n - 1) = q_1 + q_2 + \ldots + q_n - n$$

But this number is less than the number of object placed in the n boxes. Hence, for at least one i, ith box must contain at least q_i objects.

Recurrence Relations

A recurrence relation is a way of defining a sequence inductively.

We may define the sequence $a_0, a_1, a_2, \ldots$ (or $\{a_n\}$) inductively in 2 steps as follows.

(i) For a fixed integer $m \geq 0$, give the terms

$$a_0, \ldots, a_m \text{ explicitly and}$$

(ii) States an equation that relates a_n, where $n > m$, to certain of its predecessors $a_0, a_1, \ldots, a_{n-1}$

The values of the terms $a_0, \ldots, a_m$ in (i) are called initial conditions for $\{a_n\}$ and equation in (ii) is called recurrence relation for $\{a_n\}$.

We give 2 examples for recurrence relations.

1. The sequence of Fibonacci Numbers F_n is defined by
   (a) $F_0 = 1, F_1 = 1$ and
   (b) $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$
   This sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...
2. Derangements
   (a) $D_1 = 0, D_2 = 1$
   (b) $D_n = (n-1)(D_{n-1} + D_{n-2})$, for $n \geq 3$.
   A counting problem can sometime be modeled with recurrence relation follows :

Suppose that the number of ways of performing an action using n object is a_n. Then it may be possible to divide these an ways into mutually exclusive cases, such that these cases involve $n - 1$ or fewer objects. This allows us to relate a_n with $a_{n-1}, a_{n-2}, \ldots$ and the result is a recurrence relation satisfied by a_n. The corresponding initial conditions have to be obtained by direct calculation. Then, solving the recurrence relation we get a_n.

So, in the following we first illustrate by examples the process of obtaining a recurrence relation.

Example 1 *If there are 48 different pairs of people, who know each other at a party of 20 people, then show that some person has 4 or fewer acquaintances.*

Solution Let a_i denotes the number of acquaintances of person P_i. Now, if P_1 and P_2 know each other, then the pair (P_1, P_2) contributes 1 to the count a_1 and 1 to the count a_2. Thus, each pair of acquaintances contributes 2 to the total count of acquaintances. But there are 48 pairs of acquaintances.

Hence, the total count of acquaintances is $48 \times 2 = 96$. Hence, the average number of acquaintance per person is

$$\frac{a_1 + \ldots + a_{20}}{20} = \frac{96}{20} = 4.8$$

Therefore, by PP3, at least one person has 4 or fewer acquaintances.

Example 2 *A computer is used for 99 hours over a period of 12 days, an integral number of hours each day. Show that on some pair of consecutive days, the computer was used at least 17 hours.*

Solution Let a_i denote the number of hours the computer is used on the ith day. Consider the 6 pairs of consecutive numbers :

$$(a_1, a_2), (a_3, a_4), \ldots, (a_{11}, a_{12})$$

The average of the corresponding 6 sums is

$$\frac{(a_1 + a_2) + (a_3 + a_4) + \ldots + (a_{11} + a_{12})}{6} = \frac{99}{6} = 16.5$$

Hence, by PP3, at least one of the sum is ≥ 16.5 and hence, that sum is ≥ 17, since it is an integer.

Example 3 *Given 10 French books, 20 Spanish books, 8 German books, 15 Russian books and 25 Italian books, how many books must be chosen to guarantee that there are 12 books of the same language ?*

Solution Note first that there are less than 12 books in French and in German and so the 12 required books in the same language must come from one of the other languages. therefore, we must allow the possibility that the chosen books include all of the 10. French and 8 German books. Thus by PP4, it is enough to chose $(10 + 11 + 8 + 11 + 11) + 1 = 52$ books.

Example 4 *Find a recurrence relation for the number a_n of pairs of rabbits after n months, if (i) initially there is a new born pair (ii) a new born pair will produce their first pair of offspring (a male and a female) after 2 months and (iii) every pair older than 2 months will produce a pair every month. Assume that no deaths occur.*

Solution Let A denote a new born pair B a one month old pair C a pair 2 or more months old. Then, starting with A the pairs grow as follows.

(0) Initially A

(1) After one month B

(2) After 2 month C, A

(3) After 3 month C, B, A

(4) After 4 month C, C, B, A, A and so on

Thus, $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5$. To see how the number a_4 arises, note that at the end of the fourth month, the a_3 pairs in (3) are still there and they have become C, C, B. Further the a_2 pairs in (2) have become C, C and they produce a pair each A, A. Hence, we see that

$$a_4 = a_3 + a_2$$

Similarly for $n \geq 2$, we have the following situation

$(n - 2)$ After $n - 2$ months a_{n-2} pairs

$(n - 1)$ After $n - 1$ months a_{n-1} pairs

(n) After n months a_n pairs

The number a_n arises thus : in (n), there are a_{n-1} pairs, which were present in $(n-1)$ and each of these a_{n-2} pairs in $(n-2)$ produces a pair (since each of them has become a C pair.) Hence, we obtain

$$a_n = a_{n-1} + a_{n-2}$$

with initial conditions $a_0 = 1, a_1 = 1$

Example 5 *A man has a staircase of n stairs to climb. Each step he takes, can cover either 1 stair or 2 stairs. Find a recurrence relation for a_n, the number of different ways for the man to ascend the n stair staircase.*

Solution Let the steps covering first stair and second stairs be denoted respectively by a and b. Then, ascending an n staircase corresponds to a sequence of a's and b's .

Clearly if $n = 1$, then there is only one stair and so the only possibility is a.

Hence, $a_1 = 1$. If $n = 2$, then there are 2 stairs and so the only possibilities are aa or b. Hence, $a_2 = 2$, if $n = 3$, then there are 3 stairs and so the only possibilities are aaa or ab or ba. Hence, $a_3 = 3$.

Now, suppose that there are $n \geq 4$ stairs. Then. there are exactly 2 mutually exclusive possibilities for the first step : a or b. If the first step is a, then there remain $n-1$ stairs to be covered and this can be done in a_{n-1} ways. If the first step is b, then there remain $n-2$ stairs to be covered and this can be done in a_{n-2} ways.
Hence, by AP, we get the required recurrence relations as.

$$a_n = a_{n-1} + a_{n-2}$$

with initial conditions $a_1 = 1, a_2 = 2$.

Example 6 *Find a recurrence relation for the number a_n of binary sequences of length n that do not contain the pattern 11.*

Solution Consider the set S_n of all binary sequences of length n that do not contain the pattern 11.

Clearly $\quad S_1 = \{0, 1\}$, so that $a_1 = 2$

$\quad S_2 = \{00, 01, 10\}$, so that $a_2 = 3$

Now, let $n \geq 3$. The every sequence in the set S_n either starts with a zero or with $a1$. All such sequence starting with a zero are obtained by appending a zero at the beginning of every binary sequence of length $n-1$ and not containing 11. Hence, there are a_{n-1} such sequences, Next if a sequence x in S_n starts with $a1$, then second digit must be zero. Hence, all the sequences starting with a 1 are obtained by appending the pattern 01 at the beginning of every binary sequence of length $n-2$ and not containing 11. Hence, there are a_{n-2} such sequences. Required recurrence relation is $a_n = a_{n-1} + a_{n-2}$ with initial conditions $a_1 = 2, a_2 = 3$.

Example 7 *Find a recurrence relation for the number a_n of ternary sequences of length n that contain 2 consecutive digits that are the same. What are the initial conditions ? find a_6.*

Solution Clearly, no ternary sequence of length 1 can contain 2 consecutive identical digits and so $a_1 = 0$. Next the only ternary 2-sequences of the required type are 00, 11, 22 and so $a_2 = 3$.

Let $n \geq 3$. Every n sequence of the required form satisfies exactly one of the following conditions :

(i) Its first 2 digits are unequal.

(ii) Its first 2 digits are identical.

Let (i) hold. Then, the sequence starts with one of 01, 02, 12, 20, 21.

First suppose that it starts with 01. Now the condition that the sequence contains "2 consecutive identical digits" is symmetric w.r.t. all 3 digits 0, 1, 2. Hence, there are a_n equal number, namely, $m = \frac{1}{3} a_{n-1}$, of sequences of length $n - 1$ and starting with 0,1 or 2. So by appending 0 as first digit to each $(n - 1)$ sequence starting with 1, we get m sequences of length n which start with 01. Similarly, there are m sequences of length n starting with 02, 10,12, 20 or 21. Thus there are $6m = 2a_{n-1}$ sequences in this case, Let (ii) hold, then the sequences starts with 00 or 11 or 22 and its remaining $n - 2$ digits can form any $(n - 2)$ ternary sequence.

Hence, there are $3^{n-2}n$ sequences starting with 00; and the same holds for 11 and 22.

Thus, there are $3 \times 3^{n-2} = 3^{n-1}$ sequences in this case.

Required recurrence relation is

$$a_n = 2a_{n-1} + 3^{n-1}$$

With initial conditions $a_1 = 0, a_2 = 3$

Hence, $a_3 = 15, a_4 = 57$

$a_5 = 195, a_6 = 633$

Example 8 *Find a recurrence relation for the number a_n of ways to distribute n distinct objects into 5 boxes.*

Solution Hence, a box can hold any number of objects. Clearly the first object can be put in any one of the 5 boxes.

So $a_1 = 5$, Let $n \geq 2$

Then, again the first object can be placed in 5 ways. Then, the remaining $n - 1$ objects can be placed in 5 boxes in a_{n-1} ways. Hence, the n objects can be placed in $5a_{n-1}$ ways. This gives the recurrence relation $a_n = 5a_{n-1}$ with initial condition $a_1 = 5$.

Example 9 *A man has a staircase of n stairs to climb. Find a recurrence relation for the number a_n of different ways for the man to ascend the n staircase if each step covers either 1 or 2 or 3 stairs.*

Solution Let the steps covering 1 stair, 2 stairs and 3 stairs be denoted by respectively *a*, *b*, *c*. Then, ascending an n stair staircase corresponds to a sequence of *a′ s′*, *b′ s*, *c′ s*. Clearly $a_1 = 1$ and $a_2 = 2$. If $n = 3$. Then, the only possibilities are *aaa* or *ab* or *ba* or *c*. Hence, $a_3 = 4$.

Now suppose that there are $n \geq 4$ stairs. Then, there are exactly 3 mutually exclusive possibilities for first step : *a* or *b* or *c*.

If first step is *a*, then there are a_{n-1} ways.

If first step is *b*, then there are a_{n-2} ways.

If first step is *c*, then there are a_{n-3} ways.

Hence, by AP, we get required recurrence relation as

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

with initial conditions $a_1 = 1, a_2 = 2, a_3 = 4$

Example 10 *A child has ₹ n. Each day he buys either milk for ₹ 1 or orange juice for ₹ 2 or pineapple juice for ₹ 2. If S_n denotes the number of ways of spending all the money. Find the recurrence relation for this sequence. In how many ways can he spend ₹ 7?*

Solution If he buys milk on first day, then remaining ₹ $(n-1)$ can be spent in S_{n-1} ways. If he buys orange juice on the first day, then remaining ₹$(n-2)$ can be spent in S_{n-2} ways If he buys pineapple juice on the first day, then remaining ₹$(n-2)$ can be spent in S_{n-2} ways.

The above cases are disjoint

$$\therefore \quad S_n = S_{n-1} + S_{n-2} + S_{n-2}$$

$$\therefore \quad S_n = S_{n-1} + 2S_{n-2}$$

This is recurrence relation.

In the other part, a child can spend ₹ 7 in S_7 ways.

$$\begin{aligned} S_7 &= S_6 + 2S_5 \\ &= S_5 + 2S_4 + 2(S_4 + 2S_3) \\ &= S_5 + 4S_4 + 4S_3 \\ &= S_4 + 2S_3 + 4(S_3 + 2S_2) + 4(S_2 + 2S_1) \\ &= S_4 + 6S_3 + 12S_2 + 8S_1 \\ &= S_3 + 2S_2 + 6(S_2 + 2S_1) + (2S_2 + 8S_1) \\ &= S_2 + 2S_1 + 2S_2 + 6S_2 + 12S_1 + 12S_2 + 8S_1 \\ &= 21S_2 + 22S_1 \end{aligned}$$

But $S_1 = 1$. Also ₹ 2 can be spent in 2 ways as 1,1 or 2

$$S_7 = 21(2) + 22(1) = 64$$

Additional Solved Examples

Additional Solved Examples

Example 1. *Find the number of ordered pairs (x, y) of +ve integers such that $x + y \leq 4$.*

Solution We have to find the number of elements in the set

$$z = \{(x, y) \mid x, y + \text{ ve integers}, x + y \leq 4\}$$

$\therefore$ $$x \geq 1; y \geq 1$$

We have 3 mutually exclusive cases

$$x + y = 2$$

or $$x + y = 3 \quad \text{or} \quad x + y = 4$$

for $i = 2, 3, 4$

Let $z_i = \{(x, y) \mid x, y + \text{ ve integers}, x + y = i\}$

Then

$$z_2 = \{(1, 1)\}$$

$$z_3 = \{(1, 2), (2, 1)\}$$

$$z_4 = \{(1, 3), (2, 2), (3, 1)\}$$

These sets are pairwise disjoint sets.

$\therefore$ By Addition principle, we get

$$|z| = 1 + 2 + 3 = 6$$

Example 2. *Eight cards bearing number 1, 2, 3, 4, 5, 6, 7, 8 are well shuffled. Find in how many cases the top 2 cards will form a pair of twin prime?*

Solution Out of 8 integers 1,..., 8 the pairs of twin primes are (3, 5), (5, 3), (5, 7) and (7, 5). We consider the following 3 cases

Case I Top card bears number 3. Then second by multiplication rule, the number of arrangements of 8 cards is

$$1 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Case II Top card bears number 5. Then second card has 2 choices 3 and 7.

By Multiplication rule, the number of arrangements of 8 cards is

$$1 \times 2 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 1440$$

Case III Top card bears number 7. Then second card has only one choice *i.e.*,5

By Multiplication rule, the number of arrangements of 8 cards is

$$1 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Now, by rule of addition, the required number of arrangements is

$$720 + 1440 + 720 = 2880$$

Example 3. *How many 8 bit strings can formed by using 0 and 1? How many of them begin with 110 or have the 4th bit 1?*

Solution For each of the 8 places, there are 2 choices *viz.*, 0 and 1. Hence, the number of 8 bit strings is

$$2 \times 2 \times \ldots \times 2 = 2^8 = 256$$

For the 2nd part, we consider three cases.

Case I String starts with 110 and 4th bit is 1.

Here, first four places are fixed.

For next four places, there are 2 choices each. Thus in this case, the number of string is

$$2 \times 2 \times 2 \times 2 = 16$$

Case II String starts with 110 but 4th bit is not 1. In fact, 4th bit is zero. Here, first four places are fixed. Next four places have 2 choices each. Thus in this case, the number of strings is

$$2 \times 2 \times 2 \times 2 = 16$$

Case III String does not start with 110 and 4th bit is 1. If the group of first three bits is not 110; then it is one of the following 7 groups

$$111, 100, 010, 000, 011, 101, 001$$

The group of first three bits has 7 choices

4th bit is 1.

Remaining of strings in this case is

$$7 \times 1 \times 2 \times 2 \times 2 \times 2 = 112$$

By Addition rule, the total number of required strings is

$$16 + 16 + 112 = 144$$

Example 4. *Find, how many different + ve integers can be obtained by finding the sum of two or more from the list* 2, 5, 15, 30, 55?

Solution In the given list 2, 5, 15, 30, 55, we see that any member of this list cannot be expressed as the sum of two or more of its predecessors.

This fact suggests that all the sums of two or more will give us different + ve integers.

There are 5 elements in the given list

Sum of two elements : Out of 5 elements, we can choose 3 elements for addition in 10 ways.

Sum of three elements : Out of 5 elements, we can choose 3 elements for addition in 10 ways.

Sum of four elements : Out of 5 elements, we can choose 4 elements for addition in 5 ways.

Sum of five elements : 5 out of 5 elements can be choose for addition in only one way.

By the rule of addition, the number of different sums are

$$10 + 10 + 5 + 1 = 26$$

Example 5. *In how many ways can the letters of the word JUPITER be arranged in a row so that the vowels will appear in alphabetic order?*

Solution JUPITER has 7 letters having 3 vowels *U, I, E* and 4 consonants *J, P, T, R*.

Alphabetic order of vowels is *E, I, U*.

The condition alphabetic order to vowels implies that *E can appear in Ist, 2nd, 3rd or 5th place.*

In every arrangement of 3 vowels, the 4 consonants can be placed in the remaining 4 places in

$$4 \times 3 \times 2 \times 1 = 24 \text{ ways}$$

We consider following 5 cases.

Case I If *E* appears in Ist place, then *I* and *U* will appear in following 15 ways.

(2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 4), (3, 5), (3, 6), (3, 7),
(4, 5), (4, 6), (4, 7), (5, 6), (5, 7) and (6, 7)

Total number of arrangements is

$$15 \times 24 = 360$$

Case II If *E* appears in 2nd place, then *I* and *U* will appear in following 10 ways.

(3, 4), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7),
(5, 6), (5, 7), (6, 7)

Total number of arrangements is

$$10 \times 24 = 240$$

Case III If E appears in 3rd place, then I and U will appear in following 6 ways.

$$(4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7)$$

Total number of arrangements is

$$6 \times 24 = 144$$

Case IV If E appears in 4th place, then I and U will appear in following 3 ways.

$$(5, 6), (5, 7) \text{ and } (6, 7)$$

Total number of arrangements is

$$3 \times 24 = 72$$

Case V If E appears in 5th place, then I and U will appear in 1 way.

$$(6, 7)$$

Total number of arrangements is

$$1 \times 24 = 24$$

By Addition rule, total number of arrangements is

$$360 + 240 + 144 + 72 + 24 = 840$$

Example 6. *A binary word or a binary sequence is a sequence of length n such that each of its term is* 0 *or* 1.

(*i*) *How many binary words of length n are there?*

(*ii*) *How many binary words of length* 10 *begin with three* 0*'s ? How many ends with two* 1*'s?*

Solution (i) Each of the n terms in the word can be chosen in 2 ways 0 or 1.

By Multiplication principle

There are 2^n binary words of length n.

(ii) Let P = set of binary words of length 10 which begins with three 0's.

Q = set of binary words of length 10 which end with two 1's.

A word in P is of the form 0 0 0 – – – – – – – – whether each of the 7 dashes is either 0 or 1.

Hence, $|P| = 2^7$

Likewise, a word in Q is of the form.

– – – – – – – – 1 1

where each of the 8 dashes is either 0 or 1.

Hence, $|Q| = 2^8$

Example 7. *Show that the number of ways of making a non-empty collection by choosing some or all of* $n_1 + \ldots + n_k$ *objects, where* n_1 *are alike of one kind,* n_2 *alike of second kind,* $\ldots$, n_k *alike of kth kind is*

$$(n_1 + 1)(n_2 + 1) \ldots (n_k + 1) - 1.$$

Solution To make a collection, we have to select a certain number of objects of each kind.

Now, for each value of r, $1 \le r \le k$, from n_r like objects we can choose 0, 1,....., or n_r objects *i.e.*, there are $n_r + 1$ choices.

Hence, by Multiplication principle, there are in all $(n_1 + 1)(n_2 + 1) \ldots (n_k + 1)$ collections.

So, the number of non-empty collections is $(n_1 + 1)(n_2 + 1) \ldots (n_k + 1) - 1$.

Example 8. *Show that the total number of subsets of a set Z with n elements is* 2^n.

Solution Let $$Z = \{a_1, a_2, \ldots, a_n\}$$

Note that we form a subset y of Z in n stages as follows.

We have two choices for a_1, either a_1 is included in y or a_1 is not included in y.

Similarly, we have two choices for a_2, either a_2 is included in y or a_2 is not included in y etc., finally, we have 2 choices for a_n either a_n is included in y or a_n, is not included in y.

For example

If $n = 4$, then subset $\{a_2, a_4\}$ corresponds to the sequence of choices no, yes, no, yes.

Hence, by Multiplication principle the total number of subsets is

$$2 \times 2 \times \ldots \times 2 \text{ (}n\text{ factors) } i.e., 2^n$$

Example 9. *How many times is digit 0 written when listing all numbers from 1 to 3333?*

Solution We have to consider integers t such that

$$1 \le t \le 3333$$

There largest number t having 0 in the units place is 3330.

There are 333 numbers t having 0 in the units place they are 10, 20,..., 3330.

We can describe these numbers as $t = x\,0$

where x is anyone of 1, 2,..., 333.

Similarly, number $t = x0y$ *i.e.*, number having 0 in the ten's place are in all 33×10 because x can be anyone of 1, 2,..., 33

$$y \text{ can be anyone of } 0, 1, 2, \ldots, 9$$

There are $33 \times 10 = 330$ number like $x0yz$

In the same way there are $3 \times 10^2 = 300$ number with 0 in the hundreds place *i.e.*,

$$x0yz \text{ where } 1 \le x \le 3$$
$$0 \le y,\ z \le 10$$

Hence, the total number of times 0 is written is $333 + 330 + 300 = 963$

Example 10. *Find the number of ways in which 20 passengers can be put in 3 rooms, such that no room is vacant. Assume that there is no restriction on the capacity of accommodation in any room.*

Solution The total number of ways of accommodating 20 passengers in 3 rooms is 3^{20}.

Let A_i be the event that the ith room is vacant.

In this case, the 20 passengers are accommodated in 2 rooms in 2^{20} ways.

$$\therefore \quad |A_1| = 2^{20}$$
$$|A_2| = 2^{20}$$
$$|A_3| = 2^{20}$$

If 2 rooms i and j are vacant, then 20 passengers are accommodated in the remaining one room in only one way.

$$\therefore \quad |A_1 \cap A_2| = 1$$
$$|A_1 \cap A_3| = 1$$
$$|A_2 \cap A_3| = 1$$

It is impossible that all the rooms remain vacant

$$\therefore \quad |A_1 \cap A_2 \cup A_3| = 0$$

Now, $$|A_1 \cup A_2 \cup A_3| = \sum_{i=1}^{3} |A_i| \sum_{1 \le i \le j \le 3} |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|$$

$$\therefore \quad |A_1 \cap A_2 \cap A_3| = 3 \cdot 2^{20} - 3 + 0$$

If no room is vacant, this happens in

$$3^{20} - 3 \cdot 2^{20} + 3 \text{ ways}$$

Example 11. *Find the number of +ve integers less than* 30 *which are relatively prime with* 30.

Solution Prime divisor of 30 are 2, 3, 5.

A multiple of any of these 3 numbers is not relatively prime with 30.

We have

$$A = \{\text{multiples of } 2\} = \{2, 4, 6, \dots, 30\}$$
$$\therefore \quad |A| = 15$$
$$B = \{\text{multiples of } 3\} = \{3, 6, 9, \dots, 30\}$$
$$\therefore \quad |B| = 10$$
$$C = \{\text{multiples of } 5\} = \{5, 10, \dots, 30\}$$
$$\therefore \quad |C| = 6$$

$A \cap B$ is the set of multiples of 2 and 3.

Hence, $$A \cap B = \{\text{multiples of } 6\}$$
$$\therefore \quad A \cap B = \{6, 12, \dots, 30\}$$
$$\therefore \quad |A \cap B| = 5$$

$A \cap C$ is set of multiples of 2 and 5

$$\therefore \quad A \cap C = \{\text{multiples of } 10\}$$
$$\therefore \quad A \cap C = \{10, 20, 30\}$$
$$\therefore \quad |A \cap C| = 3$$

$B \cap C$ is the set of multiples of 3 and 5.

$$\therefore \quad B \cap C = \{\text{multiples of } 15\}$$
$$\therefore \quad B \cap C = \{15, 30\}$$
$$\therefore \quad |B \cap C| = 2$$

$A \cap B \cap C$ is the set of multiples of 2, 3 and 5.

$$\therefore \quad A \cap B \cap C = \{\text{multiples of } 30\}$$
$$\therefore \quad A \cap B \cap C = \{30\}$$
$$\therefore \quad |A \cap B \cap C| = 1$$

Now, $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 15 + 10 + 6 - 5 - 3 - 2 + 1 = 22$$

$\therefore$ There are $30 - 22 = 8$ integers ≤ 30 which are relatively prime with 30.

Example 12. 60 *persons form a syndicate and purchase* 1 *lottery ticket each. The syndicate includes* 4 *member from the family of Mr. X. If* 3 *winning tickets are drawn without replacement, find in how many cases the family of Mr. X will be happy?*

Solution Let A_i be the event that the number i of the family of Mr. X wins the lottery;

$$i = 1, 2, 3, 4$$

The family will be happy in

$$|A_1 \cup A_2 \cup A_3 \cup A_4| \text{ cases.}$$

If member i of family X wins, then remaining 2 tickets are drawn from 59 tickets.

This happens in $\binom{59}{2} = 1711$ ways.

Here, $$i = 1, 2, 3, 4$$

If 2 members i and j of family X win, then remaining 1 ticket is drawn in $\binom{58}{1}$ ways *i.e.*, 58 ways.

The 2 members out of 4 are chosen in $\binom{4}{2} = 6$ ways.

If 3 members i, j, k of family X win, then this happen in only one way and 3 members can be chosen in $\binom{4}{3} = 4$ ways.

It cannot happen that all 4 members win as only 3 tickets are drawn.

Now, $|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{i=1}^{4} |A_i| - \sum_{1 \le i \le j \le 4} |A_i \cap A_j|$

$$+ \sum_{1 \le i \le j \le k \le 4} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$\therefore$

$$|A_1 \cup A_2 \cup A_3 \cup A_4|$$
$$= 4\,(1711) - 6\,(58) + 4\,(1) - 0$$
$$= 6844 - 348 + 4$$
$$= 6848 - 348 = 6500$$

The family X is happy in 6500 cases.

Example 13. *How many +ve integers n are there such that n is a divisor of one of the numbers* 10^{40}, 20^{30} ?

Solution We first note that the number of +ve divisors of a +ve integer n is

$$(a_1 + 1)(a_2 + 1) \ldots (a_k + 1)$$

If $$n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$$

where $p_1, \ldots, p_k$ are integers.

Now, $$a = 10^{40} = 2^{40}\, 5^{40}$$

$$b = 20^{30} = 2^{60}\, 5^{30}$$

GCD of a, b is $c = 2^{40}\, 5^{30}$

Let A, B denote the sets of divisor of a, b respectively.

Then $A \cap B$ is set of divisors of c.

$$|A| = 41^2$$
$$|B| = 61 \times 31$$
$$|A \cap B| = 41 \times 31$$

Hence, $$|A \cup B| = 1681 + 1891 - 1271 = 2301$$

Example 14. *Suppose in a poll from* 150 *people following information obtained.* 70 *of them read 'The Hindustan Times',* 80 *read 'The Times of India',* 50 *read 'Indian Express',* 30 *read both 'The Hindustan Times' and the 'Times of India'.* 20 *read both 'The Hindustan Times' and the 'Indian Express'.* 25 *read both 'The Times of India' and the 'Indian Express'. Find at most, how many of them read all the three?*

Solution Let H, I, D respectively the set of those who read 'The Hindustan Times', The Times of India and the 'Indian Express'.

Then

$$|H \cup I \cup D| \le 150$$
$$|H| = 70$$
$$|I| = 80$$
$$|D| = 50$$

$$|H \cap I| = 30$$
$$|H \cap D| = 20$$
$$|I \cap D| = 25$$

We need to find the maximum possible value of $|H \cap I \cap D|$

$$150 \geq |H \cup I \cup D| = |H| + |I| + |D| - |H \cap I| - |I \cap D| - |H \cap D| + |H \cap I \cap D|$$

$\Rightarrow$ $150 - 70 - 80 - 50 + 30 + 20 + 25 \geq |H \cap I \cap D|$

$\therefore$ $|H \cap I \cap D| \leq 25$

At most 25 of them read all the three.

Example 15. *Let Z be the set of pensioners. E is the set of those that lost an eye. H those that lost an ear. A those that lost an arm. L those that lost a leg. Given that $n(E) = 70\%$, $n(H) = 75\%$, $n(A) = 80\%$, $n(L) = 85\%$, find, what percentage at least must have lost all the four?*

Solution Let $n(Z)$ be 100,

$$n(Z) \geq n(E \cup H) = n(E) + n(H) - n(E \cap H)$$
$$100 \geq 70 + 75 - n(E \cap H)$$
$$n(E \cap H) \geq 45$$

Similarly,

$$100 \geq n(L \cup A) = n(L) + n(A) - n(L \cap A) = 80 + 85 - n(L \cap A)$$
$$n(L \cap A) \geq 65$$

Now, $n(Z) = 100 \geq n[(E \cap H) \cup (L \cap A)] = n[(E \cap H) + n(L \cap A) - n(E \cap H \cap L \cap A)]$

$\Rightarrow$ $100 \geq 45 + 65 - n(E \cap H \cap L \cap A)$

$\Rightarrow$ $n(E \cap H \cap L \cap A) \geq 110 - 100 = 10$

At least 10% of the people must have lost all the four.

Example 16. *a, b, c, d be integers ≥ 0, $d \leq a$, $d \leq b$ and $a + b = c + d$. Prove that there exists sets A and B satisfying $n(A) = a$ and $n(B) = b$. $n(A \cup B) = c$, $n(A \cap B) = d$*

Solution $(A \cap B) \subseteq A$

$\Rightarrow$ $n(A \cap B) \leq n(A)$ or $d \leq a$

$(A \cap B) \subseteq B$

$\Rightarrow$ $n(A \cap B) \leq n(B)$

$d \leq a$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow$ $n(A \cup B) + n(A \cap B) = n(A) + n(B)$

$c + d = a + b$

Example 17. *Find the number of numbers $\leq 10^8$ which are neither perfect squares, nor perfect cubes, nor perfect fifth powers.*

Solution Let $[z]$ denotes greatest integer $\leq z$

Then number of integers from 1 to 10^8

which are perfect squares $= \left[\sqrt{10^8}\right] = 10^4$

which are perfect cube $= \left[\sqrt[3]{10^8}\right] = 464$

$$\text{which are perfect fifth power} = \left[\sqrt[5]{10^8}\right] = 39$$

$$\text{which are both squares and cubes} = \left[\sqrt[6]{10^8}\right] = 21$$

$$\text{which are both cubes and fifth power} = \left[\sqrt[15]{10^8}\right] = 3$$

$$\text{which are both squares and fifth power} = \left[\sqrt[10]{10^8}\right] = 6$$

$$\text{which are squares, cubes and fifth powers} = \left[\sqrt[30]{10^8}\right] = 1$$

By principle of exclusion and inclusion

$$10^8 - [10^4 + 464 + 39 - 21 - 3 - 6 + 1]$$

is the required number.

Example 18. *Consider the following picture*

Z_1	Z_2	Z_3	Z_4

How many ways are there to colour each of the squares Z_1, Z_2, Z_3, Z_4 with n different colours provided that no two adjacent squares can receive the same colour?

Solution Let U = set of all possible ways to colour the squares.

A = set of ways in which Z_1, Z_2 receive the same colour.

C = set of ways in which Z_3, Z_4 receive the same colour.

Now, we can colour the squares using any of the n colours with repetition allowed.

So, $$|U| = n^4$$

$$|A| = n^3 = |B| = |C|$$

$$|A \cap B| = n^2 = |B \cap C| = |C \cap A|$$

$$|A \cap B \cap C| = n$$

Hence, $$|A' \cap B' \cap C'| = n^4 - 3 \cdot n^3 + 3 \cdot n^2 - n$$

Example 19. *Find the number of ways to choose an ordered pair (a, b) of numbers from the set $\{1, 2, \ldots, 10\}$, such that $|a - b| \le 5$.*

Solution Let $$A_1 = [(a, b) | a, b \in \{1, 2, 3, \ldots, 10\}$$

$$|a - b| = \{i\}, i = 0, 1, 2, 3, 4, 5$$

$$A_0 = \{(i, i) | i = 1, 2, \ldots, 10\} \text{ and } |A_0| = 10$$

$$A_1 = \{(i, i+1) | i = 1, 2, 3, \ldots, 9\} \cup \{(i, i-1) | i = 2, 3, \ldots, 10\}$$

and $$|A_1| = 9 + 9 = 18$$

$$A_2 = \{(i, i+2) | i = 1, 2, 3, \ldots, 8\} \cup \{(i, i-2) | i = 3, 4, \ldots, 10\}$$

and $$|A_2| = 8 + 8 = 16$$

$$A_3 = \{(i, i+3) | i = 1, 2, \ldots, 7\} \cup \{(i, i-3) | i = 4, 5, \ldots, 10\}$$

and $$|A_3| = 7 + 7 = 14$$

$$A_4 = \{(i, i+4) | i = 1, 2, \ldots, 6\} \cup \{(i, i-4) | i = 5, 6, \ldots, 10\}$$

and $$|A_4| = 6 + 6 = 12$$

$$A_5 = \{(i, i+5) | i = 1, 2, \ldots, 5\} \cup \{(i, i-5) | i = 6, 7, \ldots, 10\}$$

and $$|A_5| = 5 + 5 = 10$$

$\therefore$ The required set of pairs $(a, b) = \bigcup_{i=0}^{5} A_i$

The number of such pairs, (which are disjoint)

$$= \left| \bigcup_{i=0}^{5} A_i \right| = \sum_{i=0}^{5} |A_i| = 10 + 18 + 16 + 14 + 12 + 10 = 80$$

Example 20. *How many numbers from* 1 *to* 1000 *are not divisible by* 2, 3 *and* 5?

Solution Let a_1 being the property of being divisible by 2, a_2 the property of being divisible by 3, a_3 the property of being divisible by 5.

Then

$a_1\ a_2$ denote the property of being divisible by 6
$a_1\ a_3$ denote the property of being divisible by 10
$a_2\ a_3$ denote the property of being divisible by 15
$a_1\ a_2\ a_3$ denote the property of being divisible by 30

We have to find $n(a_1'\ a_2'\ a_3') = n - n(a_1) - n(a_2) - n(a_3) + n(a_1\ a_2) + n(a_1\ a_3) + n(a_2\ a_3) - n(a_1 a_2\ a_3)$ ...(i)

We are given that $n = 1000$

$$n(a_1) = \left[\frac{1000}{2}\right] = 500$$

$$n(a_2) = \left[\frac{1000}{3}\right] = 333\ ; n(a_3) = \left[\frac{1000}{5}\right] = 200$$

$$n(a_1 a_2) = \left[\frac{1000}{2 \cdot 3}\right] = 166\ ; n(a_1 a_3) = \left[\frac{1000}{2 \cdot 5}\right] = 100$$

$$n(a_2 a_3) = \left[\frac{1000}{3 \cdot 5}\right] = 66\ ; n(a_1 a_2 a_3) = \left[\frac{1000}{30}\right] = 33$$

Substitute these values in Eq. (i), we get

$$n(a_1'\ a_2'\ a_3')$$
$$= 1000 - 500 - 333 - 200 + 166 + 100 + 66 - 33$$
$$= 266$$

Example 21. *A man has* 6 *friends. At dinner in a certain restaurant, he has met each of them* 12 *times, every* 2 *of them* 6 *times, every* 3 *of them* 4 *times, every* 4 *of them* 3 *times, every* 5 *twice and all* 6 *only once. He has dined out* 8 *times without meeting any of them. How many times has he dined out altogether?*

Solution Let U be the set of all days on which the man has dined out.

Let $a_1, a_2, \ldots, a_6$ be the 6 friends.

Let us say that a day in U has property p_i, if the man meets a_i on that day

Let N be the number of days with at least one of the properties $p_1, \ldots, p_6$

Then we are given that

$$N(p_1) = N(p_2) = \ldots = 12$$
$$N(p_1 p_2) = N(p_1 p_3) = \ldots = 6$$
$$N(p_1 p_2 p_3) = N(p_1 p_2 p_4) = \ldots = 4$$
$$N(p_1 \ldots p_4) = 3$$
$$N(p_1 \ldots p_5) = 2$$
$$N(p_1 \ldots p_6) = 1$$
$$|U| - N = 8$$

$\therefore$
$$N = \binom{6}{1}12 - \binom{6}{2}6 + \binom{6}{3}4 - \binom{6}{4}3 + \binom{6}{5}2 - \binom{6}{6}1$$
$$N = 28$$
So,
$$|U| - N = 8$$
$$|U| = N + 8 = 36$$

Example 22. *Let A, B be finite sets with $|A| = r$ and $|B| = n$. Find the number of functions from A onto B.*

Solution Let
$$A = \{a_1, \ldots, a_r\}$$
$$B = \{b_1, \ldots, b_n\}$$
Let A_1 be the set of functions from A into $B - \{b_1\}$
A_2 be the set of functions from A into $B - \{b_2\}$, ... A_n be the set of functions from A into $B - \{b_n\}$.
Let U be the set of functions from A into B.
Let
$$f \in U \text{ i.e., let } f : A \to B$$
Then f is onto B if and only if the range of f is the whole of B. *i. e.*, if and only if f is not in any of the sets $A_1, \ldots, A_n$.
For example, if $f \in A_1$ then b_1 is not in the range of f.
Hence, the number of functions from A onto B is
$$|A_1' \cap \ldots \cap A_n'|$$
We have
$$|U| = n^r$$
Similarly,
$$|A_i| = (n-1)^r \text{ for } 1 \le i \le n$$
$\therefore$
$$|B - \{b_i\}| = n - 1$$
Hence,
$$S_1 = \Sigma |A_i| = n(n-1)^r$$
$f \in A_1 \cap A_2$ if and only if f is from A into $B - \{b_1, b_2\}$.
Hence,
$$|A_1 \cap A_2| = (n-2)^r$$
Similarly,
$$|A_i \cap A_j| = (n-2)^r$$
for each of the $\binom{n}{2}$ pairs A_i, A_j with $1 \le i \le j \le n$

Hence,
$$S_2 = \Sigma |A_i \cap A_j| = \binom{n}{2}(n-2)^r$$
$\therefore$
$$S_k = \binom{n}{k}(n-k)^r, \text{ for } 3 \le k \le n-1$$
So, by principle of inclusion-exclusion, we have
$$|A'_i \cap A'_2 \cap \ldots \cap A'_m| = |U| - S_1 + S_2 - \ldots + (-1)^{n-1} S_{n-1}$$
$$= n^r - n(n-1)^r + \binom{n}{2}(n-2)^r - \ldots + (-1)^{n-1} n$$

Example 23. *Let $\phi(n)$ denote Euler's totient function prove that if $p_1^{e_1}\, p_2^{e_2} \ldots\ldots p_k^{e_k}$ is the factorization of the +ve integers n into distinct primes p_i and e_i are +ve integers, then $\phi(n)$ is given by*
$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right).$$

Solution $\phi(n)$ is the number of integers $< n$ and relatively prime to n.

Let
$$U = \{1, 2, \ldots, n\}$$
Let A_i be the set of those integers in U which are divisible by the prime p_i.

Then
$$|A_i| = \frac{n}{p_i}$$

$|A_i \cap A_j|$ = numbers in U divisible by p_i and p_j
$$= \frac{n}{p_i p_j}$$
$$|A_i \cap A_j \cap A_k| = \frac{n}{p_i p_j p_k} \text{ and so on.}$$

Hence, principle of inclusion-exclusion gives the number of numbers in U not divisible by any of the primes p_i i.e., $\phi(n)$

$$\phi(n) = n - \Sigma \frac{n}{p_i} + \Sigma \frac{n}{p_i p_j} - \Sigma \frac{n}{p_i p_j p_k} + \ldots + (-1)^k \frac{n}{p_1 p_2 \cdots p_k}$$
$$= n\left(1 - \Sigma \frac{1}{p_i} + \Sigma \frac{1}{p_i p_j} - \Sigma \frac{1}{p_i p_j p_k} + \ldots + (-1)^k \frac{1}{p_1 p_2 \cdots p_k}\right)$$
$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right)$$

Example 24. *A, B and C are the sets of all +ve divisors of* $10^{60}, 20^{50}$ *and* 30^{40} *respectively. Find* $n(A \cup B \cup C)$.

Solution Let $n(A)$ = *number of positive divisors of* 10^{60}
$$= 2^{60} \times 5^{60} \text{ is } 61^2$$

$n(B)$ = number of +ve divisors of 20^{50}
$$= 2^{100} \times 5^{50} \text{ is } 101 \times 51$$

$n(C)$ = number of +ve divisors of 30^{40}
$$= 2^{40} \times 3^{40} \times 5^{40} = 41^3$$

The set of common factors of A and B will be of the form $2^m\ 5^n$, where $0 \le m \le 60$, $0 \le n \le 50$

So,
$$n(A \cap B) = 61 \times 51$$
Similarly, the common factors of B and C, A and C are also of the form $2^m \times 5^n$

and in the former case $0 \le m \le 40$, $0 \le n \le 40$

and in the latter case $0 \le m \le 40$, $0 \le n \le 40$

$\therefore$
$$n(B \cap C) = 41^2 \text{ also, } n(A \cap C) = 41^2$$
and
$$n(A \cap B \cap C) = 41^2$$
$\therefore$
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 61^2 + 101 \times 51 + 41^3 - 61 \times 51 - 41^2 - 41^2 + 41^2$$
$$= 61(61 - 51) + 41^2(41 - 1) + 101 \times 51$$
$$= 610 + 1681 \times 40 + 101 \times 51$$
$$= 73001$$

Example 25. *I have 6 friends and during a certain vacation, I met them during several dinners. I found that I dined with all the 6 exactly on 1 day, with every 5 of them on 2 days, with every 4 of them on 3 days, with every 3 of them on 4 days and with every 2 of them on 5 days. Further every friends was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?*

Solution For $i = 1, 2, 3, \ldots, 6$. Let A_i be the set of days on which *ith* friend is present at dinner then given $n(A_i)$ or $|A_i| = 7$ and $|A'_i| = 7$

So,
$$|A_i \cap A_j| = 5, |A_i \cap A_j \cap A_k| = 4$$
$$|A_i \cap A_j \cap A_k \cap A_l| = 3,$$
$$|A_i \cap A_j \cap A_k \cap A_l \cap A_m| = 2$$
and
$$|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| = 1$$
where i, j, k, l, m vary between 1 to 6 and are distinct.

$$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_6|$$
$$= \sum_{i=1}^{6} |A_i| - \Sigma |A_i \cap A_j| + \Sigma |A_i \cap A_j \cap A_k| - \Sigma |A_i \cap A_j \cap A_k \cap A_l|$$
$$+ \Sigma |A_i \cap A_j \cap A_k \cap A_l \cap A_m| - |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6|$$
$$= \binom{6}{1}(7) - \binom{6}{2}(5) + \binom{6}{4}(4) - \binom{6}{4}(3) + \binom{6}{5}(2) - \binom{6}{6}(1)$$
$$= 42 - 75 + 80 - 45 + 12 - 1 = 13$$

Total number of dinners
$$= |A_i| + |A'_i| = 7 + 7 = 14$$

The number of dinners in which at least one friend was present.
$$= |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = 13$$

Number of dinner *i* dine alone = 14 - 13 = 1

Example 26. *A student on vacation for d days observed that*

(i) *it rained 7 times morning or afternoon*

(ii) *when it rained in the afternoon it was clear in the morning*

(iii) *there were 5 clear afternoons, and*

(iv) *there were 6 clear mornings.*

Find d.

Solution Let the set of days it rained in the morning be $M2$.

Let A_r be the set of days it rained in afternoon.

M' be the set of days, when there were clear morning.

A'_r be the set of days when there were clear afternoon.

By condition (ii), we get $\quad M_r \cap A_r = \phi$

By (iv), we get $\quad M'_r = 6$

By (iii), we get $\quad A'_r = 5$

By (i), we get $\quad M_r \cup A_r = 7$

M_r and A_r are disjoint sets
$$n(M_r) = d - 6$$
$$n(A_r) = d - 5$$

Applying the principle of inclusion-exclusion
$$n(M_r \cup A_r) = n(M_r) + n(A_r) - n(M_r \cap A_r)$$

$$\Rightarrow \quad 7 = (d - 6) + (d - 5) - 0$$
$$\Rightarrow \quad 2d = 18$$
$$\Rightarrow \quad d = 9$$

Example 27. *Find, how many number of* 4 *different digits out of the digits* 1, 2, 3, 4, 5, 6, 7 *can be made? Also find, how many of them are greater than* 3400?

Solution Number of digits, $n = 7$ for the numbers of 4 digit number of places to be filled up $r = 4$

Hence, number of numbers of 4 different digits out of the digits 1, 2, 3, 4, 5, 6, $7 = {}^7P_4$

$$= \frac{7!}{(7-4)!} = 840$$

Now, numbers in which thousands place is filled with 3 and hundred with 4 and remaining 2 places with any must be greater than 3400.

Now, for the number of numbers of 4 digits in which thousands *i.e.,* (Ist from left) places is filled with 3 and hundred (2nd from left) with 4 then for the remaining 2 places there are (1, 2, 5, 6, 7,) 5 digits *i.e.,* $n = 5$ and $r = 2$

Number of 4 digits such that 3 is at thousands place and 4 at hundreds place ${}^5P_2 = \frac{5!}{3!} = 20$

Similarly, number of numbers of 4 digits such that 3 comes at thousands and 5 at hundred place is

$${}^5P_2 = 20$$

Number of numbers of 4 digits such that 3 comes at thousand and 6 at hundreds place is

$${}^5P_2 = 20$$

Number of numbers of 4 digits such that 3 comes at thousand and 7 at hundred places

$$= {}^5P_2 = 20$$

Number of numbers of 4 digits beginning with 3 (and in which 4, 5, 6 or 7 come at 100 place) and greater than 3400

$$= 20 + 20 + 20 + 20 = 80$$

Also, number of 4 digits beginning with 4, 5, 6 or 7 are all greater than 3400.

Number of numbers of 4 digits beginning with 4

Number of places remain to be filled up is $r = 3$

Number of digits remain to be utilised is $n = 6$

Number of numbers of 4 digits beginning with 4 is

$$= {}^6P_3 = \frac{6!}{3!} = 120$$

Also, number of numbers of 4 digits beginning with $5 = {}^6P_3 = 120$

Number of numbers of 4 digits beginning with

$$6 = 120$$

Number of numbers of 4 digits beginning with

$$7 = 120$$

Number of numbers of 4 digits and greater than $3400 = 80 + 480 = 560$

Example 28. *How many +ve numbers can be formed by using any number of the digit* 0, 1, 2, 3, *and* 4 *no digit being repeated?*

Solution Number of digits, $n = 5$

Number of numbers of 1 digit = 4 ...(i)

To find the number of numbers of 2 digits number of places to be filled up $r = 2$
Number of digits, $n = 5$
Number of numbers of 2 digits

$$= {}^5P_2 = \frac{5!}{3!} = 20$$

But out of these 20 numbers of 2 digits, some numbers begins with 0 which are not of our purpose.
To find the number of such numbers of digits beginning with 0, number of places and number of digits remains to be utilised = $n = 4$
Hence, number of numbers of 2 digits which begin with 0 = ${}^4P_1 = 4$
Number of numbers of 2 digits

$$= 20 - 4 = 16 \qquad \text{...(ii)}$$

For the number of numbers of 3 digits, number of places to be filled up = $r = 3$
Number of objects, $n = 5$
Number of numbers of 3 digits

$$= {}^5P_3 = \frac{5!}{2!} = 60$$

But of these 60 numbers, the number which begin with 0 cannot be regarded as the number of 3 digits.
Now, to find the number of such number of 3 digits which begin with 0.
Number of places = $r = 2$
Number of things remains to be utilised $n = 4$
Number of numbers of 3 digits beginning with 0 is ${}^4P_2$ *i.e.*,

$$\frac{4!}{2!} = 12$$

Number of numbers of 3 digits = 60 – 12 = 48
Also, for the number of numbers of 4 digits number of places to be filled up = $r = 4$. Number of digits to be utilised, $n = 5$
Number of numbers of 4 digits = ${}^5P_4 = \frac{5!}{1!} = 120$
But out of these 120 numbers, there are some numbers which begin with zero.
For the numbers of 4 digits beginning with 0.
Number of places remain to be filled up, $r = 3$
Number of digits remain to be utilised $n = 4$
Number of numbers of 4 digits beginning with 0

$$= {}^4P_3 = \frac{4!}{1!} = 24$$

Number of numbers of 4 digits = 120 – 24 = 96
For the number of numbers of 5 digits, number of places to be filled up, $r = 5$
Number of digits, $n = 5$
Number of numbers of 5 digits

$$= {}^5P_5 = \frac{5!}{0!} = 120$$

But out of these 120 numbers, there are some numbers which begin with zero.
Number of places to be filled up $r = 4$
Number of digits remain to be utilised, $n = 4$
Number of numbers of 5 digits which begin with 0 = ${}^4P_4 = \frac{4!}{0!} = 24$

Number of numbers of 5 digits = 120 − 24 = 96

Number of +ve numbers formed 0, 1, 2, 3, 4, is

$$4 + 16 + 48 + 96 + 96 = 260$$

Example 29. *How many numbers can be formed with the digits* 1, 3, 7, 9, *when taken all at a time? Find their sum.*

Solution Number of given digits $n = 5$

Number of places to be filled $= r = 5$

Numbers are to be formed taken all at a time

Number of different numbers of 5 digits formed with the digits 1, 3, 5, 7, 9 is ${}^5P_5 = \dfrac{5!}{0!} = 120$

Also, keeping 1 at unit place, number of numbers of 5 digits in which 1 comes at unit place

$${}^4P_4 = \frac{4!}{0!} = 24$$

Number of places remain to be filled up $= r = 4$

Number of digits remain to be utilised $= n = 4$

i.e, in the number of numbers of 5 digits 1 may come at unit place in 24 number of times.

Similarly, 3, 5, 7, 9, each may come at (unit, tens, hundreds, thousands and ten thousands) each place in 24 number of times.

Sum of the digits at unit place

$$= (1 + 3 + 5 + 7 + 9) \times 24 \times 1$$

Sum of the digits at tens place

$$= (1 + 3 + 5 + 7 + 9) \times 24 \times 10$$

Sum of the digits at 100 place

$$= (1 + 3 + 5 + 7 + 9) \times 24 \times 100$$

Sum of the digits at 1000 place

$$= (1 + 3 + 5 + 7 + 9) \times 24 \times 1000$$

Sum of the digits at 10000 place

$$= (1 + 3 + 5 + 7 + 9) \times 24 \times 10000$$

Sum of all the number of 5 digits

$$= (1 + 3 + 5 + 7 + 9) \times 24$$

$$(1 + 10 + 100 + 1000 + 10000)$$

$$= 25 \times 24 \times 11111 = 600 \times 11111 = 6666600$$

Example 30. *Find the sum of all the four digits number that formed with the digits* 3, 2, 3, 4.

Solution We first keep one '3' at unit place, then number of places to be filled up, $r = 3$

Number of digits remain to be utilised, $n = 3$ (3, 2, 4)

Number of numbers of 4 digits formed with (3, 2, 3, 4) in which 3 comes at unit place

$$= {}^3P_3 = \frac{3!}{0!} = 6$$

Hence, 3 may come at unit place in 6 ways.

Similarly, 3 may come at (tens, hundreds and thousands) each place in 6 number of times.

But keeping 2 at unit place, number of places be filled up, $r = 3$

Number of digits remain to be utilised $= n = 3$ (3, 3, 4)

Number of numbers of 4 digits in which 2 occurs at unit place $= \frac{^3P_3}{2!} = \frac{6!}{2}$

Similarly, 2 may come at (10's, 100's, 1000's) each place 3 number of times.

Similarly, 4 may come at (unit, 10's, 100's, 1000's) each place 3 number of times.

Sum of digits at unit place

$$= [(2+4)\times 3 + 3\times 6]\times 1 \quad \text{...(i)}$$

Sum of digits at 10's place

$$= [(2+4)\times 3 + 3\times 6]\times 10 \quad \text{...(ii)}$$

Sum of digits at 100's place

$$= [(2+4)\times 3 + 3\times 6]\times 100 \quad \text{...(iii)}$$

Sum of digits at 1000's place

$$= [(2+4)\times 3 + 3\times 6]\times 1000 \quad \text{...(iv)}$$

Adding (1), (2), (3), (4), altogether we get 39996

Sum of all four digits number formed with the digits 3, 2, 3, 4 is 39996.

Example 31. *Find the sum of all the* 4 *digit numbers, that can be formed with the digits* 1, 2, 2 and 3.

Solution Keeping 1 or 3 at unit place

Number of places remain to be filled $= r = 3$

Number, of digits remain to be utilised $(2, 2, 3) = n = 3$

Hence, number of numbers of 4 digits in which 1 or 3 comes at unit place $= \frac{^3P_3}{2!} = \frac{6!}{2} = 3$

Similarly, 1 or 3 may come at (10's, 100's, 1000's) each place in 3 number of times.

i.e., 1 or 3 may come at (unit or 10's or 100's or 1000's) place 3 number of times.

Also, keeping 2 at unit place

Number of places remain, to be filled up $= r = 3$

Number of digits remain to be utilised $= n = 3$ (1, 2, 3)

Number of numbers of 4 digits such that 2 comes at unit place $= {}^3P_3 = \frac{3!}{0!} = 6$

i.e., 2 may come at unit (or 10's or 100's or 1000's) place 6 number of times.

Sum of digits at unit place $= [(1+3)\times 3 + 2\times 6]\times 1$

Sum of digits at 10's place $= [(1+3)\times 3 + 2\times 6]\times 10$

Sum of digits at 100's place $= [(1+3)\times 3 + 2\times 6]\times 100$

Sum of at 1000's place $= [(1+3)\times 3 + 2\times 6]\times 1000$

Sum of all the 4 digit numbers $= [(1+3)\times 3 + 2\times 6][1 + 10 + 100 + 1000]$

$= 24 \times 1111 = 26664$

Example 32. *How many odd numbers greater than* 600000 *can be formed from the digits* 5, 6, 7, 8, 9, 0 *if repetition of digits is allowed?*

Solution Numbers greater than 600000 and formed with the digits 5, 6, 7, 8, 9, 0 are of 6 digits but begin with 6, 7, 8 or 9.

Also, the numbers which end with 5, 7, 9 are odd.

Hence, first place can be filled by 4 ways (out of 6, 7, 8 or 9). Last place can be filled by 3 ways.

Hence, first and last places can be filled by 4×3 ways.

Also, 2nd place can be filled by 6 ways.

3rd place can be filled by 6 ways.

4th place can be filled by 6 ways.
5th place can be filled by 6 ways.
Hence, all the 6 places can be filled by

$$4 \times 3 \times 6 \times 6 \times 6 \times 6 = 15552 \text{ ways.}$$

Example 33. *In a car plate number containing only 3 or 4 digits not containing the digit 0. What is the maximum number of cars that can be numbered?*

Solution Here, repetition of digits is allowed.

Also, numbers are formed with the digits 1, ..., 9.

Case I When car plate numbers contain 3 digits number of places to be filled up $r = 3$

Out of the 9 digits first place can be filled by 9 ways.
Similarly, 2nd and 3rd place can be filled in 9 ways respectively.
So, when car plate number contains 3 digit, maximum number of cars $= 9^3$

Case II When car plate number contains 4 digit in this case number of cars to be filled up $= r = 4$

Ist place can be filled by 9 ways,
2nd place can be filled by 9 ways and so on.
Maximum number of cars that can be numbered $= 9^3 + 9^4 = 7290$

Example 34. *m men and n women are to be seated in a row so that no two women sit together, if $m > n$. Show that the number of ways in which they can be seated is*

$$\frac{m!\,(m+1)!}{(m-n+1)!}.$$

Solution Number of arrangements of m men in a line

$(n =$ number of thing (men)^m)

$(r =$ number of place)

is
$$^mP_m = \frac{m!}{0!} = m!$$

Since, no 2 women are to be seated together so that number of women may be $m + 1$.
Hence, number of arrangements of n women in the $(m + 1)$ place is $^{m+1}p_n$.

Required number of arrangements

$$= m! \times {}^{m+1}P_n = \frac{m!(m+1)!}{(m+1-n)!}$$

$$= \frac{m!(m+1)!}{(m-n+1)!}$$

Example 35. *Show that number of ways in which p +ve and n -ve signs may be placed in a row so that no two -ve signs shall be together is $^{p+1}C_n$.*

Solution Number of arrangements of all the p +ve signs in a row.

$$= \frac{^pP_p}{p!} \text{ (since } n = p, r = p \text{)}$$

$$= \frac{p!}{p!}$$

Since, two -ve signs come together so that number of places for the –ve signs is $p + 1$.
Number of arrangements of the n -ve signs according to the condition

is $\frac{^{p+1}P_n}{n!}$ (since $n = p + 1$; $r = p + 1$) (all the n are identical)

$$= \frac{(p+1)!}{n!\,(p+1-n)!}$$

Finally, required number of arrangements of p +ve and n -ve signs in a row so that no two –ve signs are together

$$= \frac{p!}{p!} \times \frac{(p+1)!}{(p+1-n)!\,n!}$$

$$= \frac{(p+1)!}{n!\,(p+1-n)!} = {}^{p+1}C_n \qquad \left[\because {}^nC_r = \frac{n!}{(n-r)!\,r!}\right]$$

Example 36. *If the letters of the word 'LATE' to be permutated and the words so formed be arranged as in a dictionary what with be the rank of the word LATE.*

Solution Arrange alphabetically LATE, we have AELT. Now, for the words beginning with A, number of places to be filled up $= r = 3$.

Number of letters to be utilised, $n = 3$.

Number of words beginning with $A = {}^3P_3 = 3!$ number of words beginning with $E = 3!$.

Also, for the words beginning with LA, number of places to be filled up $= r = 2$.

Number of letters remain to be utilised, $n = 2$. Hence, number of word LATE.

Rank of the word LATE in the dictionary $= 3! + 3! + 2! = 14$

Example 37. *On a new year day, every students of a class sends a card to every other student. The postman delivers* 600 *cards. How many students are there in the class?*

Solution Total number of students $= n$

Number of pair of students $= {}^nC_2$

Two students out of n can be selected in ${}^nC_2$ ways. Here, for each pair of students, number of cards sent is 2.

If P sends card to Q, then Q also sends a card to P. Number of cards sent $= 2 \cdot {}^nC_2$.

According to the problem $2 \cdot {}^nC_2 = 600$

or $$2 \cdot \frac{n(n-1)}{2!} = 600$$

or $$n^2 - n - 600 = 0$$

$$(n-25)(n+24) = 0$$

$$n = 25 \qquad [\because n \neq -24]$$

Example 38. *You are given $n \geq 3$ circles. Find the number of radical axis and radical centres of these circles. Find the value of n to which number of radical axis is equal to number of radical centres.*

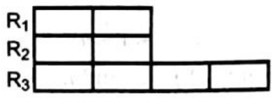
Solution Radical axis is the locus of the intersection of the two tangents equal in length to the 2 circles. Hence, radical axis associates with 2 circles. Number of radical axis of n circles is ${}^nC_2$. Also, number of radical centres of n circles is ${}^nC_3$.

We are given $${}^nC_2 = {}^nC_3 \Rightarrow {}^nC_3 = {}^nC_{n-3}$$

or $$n - 3 = 2$$

or $$n = 5$$

Example 39. *In how many ways the letters of the word PERSON can be placed in the squares of the given figure shown so that no row remain empty?*



Solution There are 6 different letters in the word PERSON.

There are 3 rows, we have to select 6 squares taking at least one from each row.

Selection of 6 squars 1 from Ist row, 1 from 2nd row and 4 from 3rd row can be made in

$$^2C_1 \cdot {}^2C_1 \cdot {}^4C_4 = 4 \text{ ways.}$$

Selection of 6 squares, 1 from 1st row, 2 from 2nd row and 2 from the 3rd row can be made in $^2C_1 \cdot {}^2C_2 \cdot {}^4C_3 = 8$ ways.

Selection of 6 squares 2 from 1st row, 1 from 2nd row and 3 from 3rd row can be made in $^2C_2 \cdot {}^2C_1 \cdot {}^4C_3 = 8$ ways.

Selection of 6 squares, 2 from 1st row, 2 from 2nd row and 2 from the 3rd row can be made in $^2C_2 \cdot {}^2C_2 \cdot {}^4C_2 = 6$ ways.

Total number of selection of 6 squares

$$= 4 + 8 + 8 + 6$$
$$= 26$$

Now, for each selection of 6 squares, the number of arrangements of 6 letters

$$= 6! = 720$$

Required number of ways $= 26 \times 720$

$$= 18720$$

Example 40. (*i*) *In how many ways can* 16 *constables be divided into 8 batches of 2 each?*

(*ii*) *In how many ways* 16 *constables can be assigned to petrol* 8 *villages* 2 *for each?*

Solution (i) In case of dividing 16 constables into 8 batches of 2 each, order of formation of groups does not matter. Hence, 16 constables can be divided into 8 batches of 2 each in

$$= \frac{^{16}C_2\ {}^{14}C_2\ {}^{12}C_2\ {}^{10}C_2\ {}^{8}C_2\ {}^{6}C_2\ {}^{4}C_2\ {}^{2}C_2}{8!}$$

$$= \frac{16!}{8!\,(2!)^8}$$

(ii) When 16 constables are to be assigned to petrolling 8 villages 2 for each. Then we first divide 16 constables into 8 batches as in (i) in $\frac{16!}{8!\,(2!)^8}$ ways.

But in this case, a question that which batch petrols the first village and which the 2nd and so on, arises automatically *i.e.*, order of formation of batches must be taken into account. Hence, number of ways of assigning 8 batches to petrolling 8 villages

$$= \frac{16}{8!\,(2!)^8}\,8!$$

$$= \frac{16}{(2!)^8}$$

Example 41. *Find the number of ways of dividing 2n people into n couples.*

Solution Here, we have to make a group of 2 people out of $2n$ people. The number of ways in which we can divide $2n$ people into n group of 2.

$$^{2n}C_2\ ^{2n-2}C_2\ ^{2n-4}C_2 \ldots \frac{^2C_2}{n^2!}$$
$$= \frac{2n!}{2!2n!-2} \cdot \frac{2n!-2}{2!2n!-4} \cdot \frac{2n!-4}{2!2n!-6} \cdots \frac{4!}{2!2!} \cdot \frac{2!}{2!0!}$$
$$= \frac{2n!}{2^n\,(2n!)^2}$$

Example 42. *Find the number of ways in which we can distribute mn students equally among m sections.*

Solution When we distribute mn students equally among m sections, then each section has n students.
Hence, number of ways of distribution

$$= (^{mn}C_n\ ^{mn-n}C_n\ ^{mn-2n}C_n \ldots ^{2n}C_n\ ^nC_n)\frac{1}{m!}m!$$
$$= \frac{mn!}{n!\,mn!-n} \cdot \frac{mn!-n}{n!\,mn!-2n} \cdots\cdots \frac{2n!}{n!\,n!} \cdot \frac{n!}{n!\,0!}$$
$$= \frac{mn!}{(n!)^m}$$

Example 43. *From a given number of 4n books, there are three sets of n identical books each on Physics, Chemistry and Mathematics. The remaining n books are distincts books of other subjects. Find the number of ways of choosing n books out of the 4n books.*

Solution There are n identical books of one kind, n identical books of 2nd, n identical books of 3rd kind and rest n books are all different.

Hence, number of selections of any n books out of these $4n$ books

$$= \text{coefficient of } x^n \text{ in } (x^0 + x + x^2 + x^3 + \ldots + x^n)$$
$$(x^0 + x + x^2 + \ldots + x^n)(x^0 + x + x^2 + x^3 + \ldots + x^n)(1+x)^n$$
$$= \text{coefficient of } x^n \text{ in } (1 + x + x^2 + x^3 + \ldots + x^n)^3 (1+x)^n$$
$$= \text{coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^3 \cdot (1+x)^n$$
$$= \text{coefficient of } x^n \text{ in } (1-x^{n+1})^3 (1-x)^{-3} (1+x)^n$$
$$= \text{coefficient of } x^n \text{ in } (1-x)^{-3} [2-(1-x)]^n$$
$$= \text{coefficient of } x^n \text{ in } (1-x)^{-3} [2^n - {}^nC_1 2^{n-1}(1-x)$$
$$+ {}^nC_2 n^{n-2} (1-x)^2 - {}^nC_3\, 2^{n-3}(1-x)^3 + \ldots + (-1)^n (1-x)^n]$$
$$= \text{coefficient of } x^n \text{ in } 2^n (1-x)^{-3} - {}^nC_1\, 2^{n-1}(1-x)^{-2} + {}^nC_2\, 2^{n-2}(1-x)$$
$$= \text{coefficient of } x^n \text{ in } [2^n (1 + {}^3C_1 x + {}^4C_2 x^2 + \ldots) - n \cdot 2^{n-1}(1 + {}^2C_1 x + {}^3C_2 x^2 + \ldots)$$
$$+ \frac{n(n-1)}{2} 2^{n-2} (1 + {}^1C_1 x + {}^2C_2 x^2 + \ldots)]$$
$$= 2^n\ {}^{n+2}C_n - n \cdot 2^{n-1}\ {}^{n+1}C_n + \frac{n(n-1)}{2} \cdot 2^{n-2}$$
$$= 2^n\ {}^{n+2}C_2 - n \cdot 2^{n-1}\ {}^{n+1}C_1 + 2^{n-3}(n^2 - n)$$

$$= 2^n \frac{(n+2)(n+1)}{2} - n \cdot 2^{n-1}(n+1) + 2^{n-3}(n^2 - n)$$
$$= 2^{n-1}(n^2 + 3n + 2) - 2^{n-1}(n^2 + n) + 2^{n-3}(n^2 - n)$$
$$= 2^{n-3}[2^2(n^2 + 3n + 2) - 2^2(n^2 + n) + n^2 - n] = 2^{n-3}(n^2 + 7n + 8)$$

Example 44. *In an examination of the maximum marks for each of the three papers are* 50 *each and maximum marks for the fourth paper is* 100. *Find the number of ways in which a candidate secure* 60% *marks in aggregate.*

Solution Here, aggregate of marks

$$= 50 + 50 + 50 + 100 = 250$$

and 60% of aggregate 250

$$= \frac{60}{100} \times 250 = 150$$

Now, number of ways in which we can get 150 marks in the examination of 4 papers such that first three have maximum marks 50 and 4th has maximum marks 100 is

$$= \text{coefficient of } x^{150} \text{ in } (x^0 + x + x^2 + \ldots + x^{50})^3 (x^0 + x + x^2 + \ldots + x^{100})$$
$$= \text{coefficient of } x^{150} \text{ in } (1 + x + x^2 + \ldots + x^{50})^3 (1 + x + x^2 + \ldots + x^{100})$$
$$= \text{coefficient of } x^{150} \text{ in } \left(\frac{1 - x^{51}}{1 - x}\right)^3 \left(\frac{1 - x^{101}}{1 - x}\right)$$
$$= \text{coefficient of } x^{150} \text{ in } \frac{(1 - 3x^{51} + 3x^{102} - x^{153})(1 - x^{101})}{(1 - x)^4}$$
$$= \text{coefficient of } x^{150} \text{ in } (1 - 3x^{51} - x^{101} + 3x^{102} + \ldots)(1 - x)^{-4}$$
$$= \text{coefficient of } x^{150} \text{ in } (1 - 3x^{51} - x^{101} + 3x^{102} + \ldots)$$
$$(1 + {}^4C_1 x + {}^5C_2 x^2 + \ldots)$$
$$= {}^{153}C_{150} - 3\ {}^{102}C_{99} - {}^{52}C_{49} + 3\ {}^{51}C_{48}$$
$$= \frac{153 \cdot 152 \cdot 151}{6} - 3 \cdot \frac{102 \cdot 101 \cdot 100}{6} - \frac{52 \cdot 51 \cdot 50}{6} + 3 \cdot \frac{51 \cdot 50 \cdot 49}{6}$$
$$= 76 \cdot 51 \cdot 151 - 100 \cdot 101 \cdot 51 - 50 \cdot 26 \cdot 17 + 51 \cdot 49 \cdot 25$$
$$= 51(76 \times 151 - 100 \times 101) + 17(75 \times 49 - 50 \times 26)$$
$$= 51(11476 - 10100) + 17(3675 - 1300)$$
$$= 51 \times 1376 + 17 \times 2375$$
$$= 70176 + 40375$$
$$= 110551$$

Example 45. *In how many ways can* 16 *apples be distributed among* 4 *persons, each receiving not less than* 3 *apples?*

Solution The apples are considered to be alike.

Since, a person will not receive less than 3 apples so that one person can get 3, 4, ... apples.

Hence, number of ways of distributing 16 apples among 4 persons.

$$= \text{coefficient of } x^{16} \text{ in } (x^3 + x^4 + x^5 + \ldots)^4$$
$$= \text{coefficient of } x^{16} \text{ in } x^{12}(1 + x + x^2 + \ldots)^4$$
$$= \text{coefficient of } x^4 \text{ in } (1 + x + x^2 + \ldots)^4$$

$= \text{coefficient of } x^4 \text{ in } \left(\frac{1}{1-x}\right)^4$

$= \text{coefficient of } x^4 \text{ in } (1-x)^{-4}$

$= \text{coefficient of } x^4 \text{ in } (1 + {}^4C_1\, x + {}^5C_2\, x^2 + \ldots) = {}^7C_4$

$= \frac{7!}{4!3!} = \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2} = 35$

Example 46. *There are* 12 *intermediate stations between two places A and B. In how many ways can a train be made to stop at* 4 *of these* 12 *intermediate stations, no two of which are consecutive?*

Solution Let S_1, S_2, S_3, S_4 be the 4 stopping stations.

Let $\ldots S_1 \ldots S_2 \ldots S_3 \ldots S_4 \ldots$

$\quad n_1 \quad n_2 \quad n_3 \quad n_4 \quad n_5$

Let n_1 and n_5 be the number of stations respectively before S_1 and after S_4 and let n_2, n_3 and n_4 be the number of stations between S_1 and S_2, S_2 and S_3, S_3 and S_4 respectively. Then

$$n_1 + n_2 + n_3 + n_4 + n_5 = 12 - 4 = 8$$

where $n_1 n_5 \geq 0; n_3, n_4 \geq 1$

Now, the number of ways in which a train stops at 4 of these 12 stations such that no two of four are consecutive is

$= \text{coefficient of } x^8 \text{ in } (x^0 + x + x^2 + \ldots)(x + x^2 + x^3 + \ldots)$
$(x + x^2 + x^3 + \ldots)(x + x^2 + x^3 + \ldots)(x^0 + x + x^2 + \ldots)$

$= \text{coefficient of } x^8 \text{ in } (1 + x + x^2 + \ldots)^2 (x + x^2 + x^3 + \ldots)^3$

$= \text{coefficient of } x^8 \text{ in } x^3 (1 + x + x^2 + \ldots)^2 (1 + x + x^2 + \ldots)^3$

$= \text{coefficient of } x^5 \text{ in } (1 + x + x^2 + \ldots)^5$

$= \text{coefficient of } x^5 \text{ in } \left(\frac{1}{1-x}\right)^5$

$= \text{coefficient of } x^5 \text{ in } (1-x)^{-5}$

$= \text{coefficient of } x^5 \text{ in } (1 + {}^5C_1 x + {}^6C_2 x^2 + \ldots)$

$= {}^9C_5$

$= \frac{9!}{5!\,4!}$

$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{4 \cdot 3 \cdot 2 \cdot 5!}$

$= 9 \times 2 \times 7$

$= 126$

Example 47. *How many integral Solutions are there to* $x + y + z + t = 29$, *when* $x > 0, y > 1, z > 2$ *and* $t \geq 0$?

Solution We have to select four integral for x, t, z and t. For x we may choose (> 0) 1, 2, 3, 4,...

For y we may choose (> 1) 2, 3, 4, ...

For z we may choose (> 2) 3, 4, 5, ...

For t we may choose (≥ 0) 0, 1, 2, ...

Such that sum of the four chosen integers for x, y, z and t is $= 29$ *i.e.*, $x + y + z + t = 29$

Hence, number of ways of choosing four integers for x, y, z and t = number of integral solutions of $x + y + z + t = 29$ for $x > 0, y > 1, z > 2, t \geq 0$ is given by the coefficient of x^{29} in $(x + x^2 + x^3 + ...)$ $(x^2 + x^3 + x^4 + ...)(x^3 + x^4 + x^5 + ...)(x^0 + x + x^2 + ...)$

$$= \text{coefficient of } x^{29} \text{ in } x(1 + x + x^2 + ...)\, x^2 (1 + x + x^2 + ...)$$
$$x^3 (1 + x + x^2 + x^3 + ...)(1 + x + x^2 + ...)$$
$$= \text{coefficient of } x^{29} \text{ in } x^6 (1 + x + x^2 + ...)^4$$
$$= \text{coefficient of } x^{23} \text{ in } \left(\frac{1}{1-x}\right)^4$$
$$= \text{coefficient of } x^{23} \text{ in } (1-x)^{-4}$$
$$= \text{coefficient of } x^{23} \text{ in } (1 + {}^4C_1 x + {}^5C_2 x^2 + {}^6C_3 x^3 + ...)$$
$$= {}^{26}C_{23} = \frac{26!}{23!3!}$$
$$= \frac{26 \cdot 25 \cdot 24}{3 \cdot 2 \cdot 1}$$
$$= 26 \times 100$$
$$= 2600$$

Example 48. *How many integral solutions are there to the system of equations*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

and $x_1 + x_2 + x_3 = 5$, *when* $x_2 \geq 0$?

Solution Here,
$$x_1 + x_2 + x_3 + x_4 + x_5 = 20 \quad ...(i)$$
$$x_1 + x_2 + x_3 = 5 \quad ...(ii)$$
so that
$$x_3 + x_4 = 15 \quad ...(iii)$$
with $x_2 \geq 0$

Now, obviously Eqs. (i) and (ii) hold if Eqs. (ii) and (iii) hold. Hence, we find the integral solutions of
$$x_1 + x_2 + x_3 = 5$$
and
$$x_3 + x_4 = 15 \quad ...(iv)$$
with $x_2 \geq 0$

Now, number of integral solutions of Eq. (i) is given by the coefficient of x^5 in $(x^0 + x + x^2 + ...)^3$

$$= \text{coefficient of } x^5 \text{ in } (1 + x + x^2 + ...)^3$$
$$= \text{coefficient of } x^5 \text{ in } \left(\frac{1}{1-x}\right)^3$$
$$= \text{coefficient of } x^5 \text{ in } (1-x)^{-3}$$
$$= 1 + {}^3C_1 x + {}^4C_2 x^2 + ...$$
$$= {}^7C_5 = \frac{7 \cdot 6}{2 \cdot 1} = 21$$

Also, number of integral solutions, if Eq. (ii) is given by the coefficient of x^{15} in $(x^0 + x + x^2 + ...)^2$

$$= \text{coefficient of } x^{15} \text{ in } (1 + x + x^2 + ...)^2$$
$$= \text{coefficient of } x^{15} \text{ in } \left(\frac{1}{1-x}\right)^2$$

$$= \text{coefficient of } x^{15} \text{ in } (1-x)^{-2}$$
$$= 1 + {}^2C_1 + {}^3C_2 + \ldots = {}^{16}C_{15}$$

Hence, number of solutions $= {}^3C_2 \times 16 = 336$

Example 49. *Find the number of non-negative integral solutions of*
$$x_1 + x_2 + x_3 + 4x_4 = 20$$

Solution We have to find the number of non-negative integral solutions of
$$x_1 + x_2 + x_3 + 4x_4 = 20 \quad \ldots\text{(i)}$$

Hence, we have to take x_1, x_2, x_3, x_4 from 0, 1, 2, 3, ..., so that Eq. (i) holds

Now, putting $x_1 = x_2 = x_3 = 0$, we have $4x_4 = 20$

$\therefore \quad x_4 = 5$

Hence, maximum value of $x_4 = 5$

i.e., we may take $x_4 = 0$ or 1 or 3 or 4 or 5

For $x_4 = 0$, $x_1 + x_2 + x_3 = 20$ has number of integral solutions

$$= \text{coefficient of } x^{20} \text{ in } (1 + x + x^2 + x^3 + \ldots)^3$$
$$= \text{coefficient of } x^{20} \text{ in } \left(\frac{1}{1-x}\right)^3$$
$$= \text{coefficient of } x^{20} \text{ in } (1-x)^{-3}$$
$$= \text{coefficient of } x^{20}\ (1 + {}^3C_1\, x + {}^4C_2\, x^2 + \ldots)$$
$$= {}^{22}C_{20} = \frac{22!}{20!\,2!} = \frac{22 \times 21}{2}$$
$$= 231$$

For $x_4 = 1$, number of integral solutions of $x_1 + x_2 + x_3 = 16$ is given by the coefficient of x^{16} in $(1 + x + x^3 + \ldots)^3$

$$= \left(\frac{1}{1-x}\right)^3 = (1-x)^{-3}$$
$$= 1 + {}^3C_1 x + {}^4C_2 x^2 + \ldots = {}^{18}C_{16}$$

For $x_4 = 2$, number of integral solutions of $x_1 + x_2 + x_3 = 12$ is ${}^{14}C_{12} = {}^{14}C_{12}$

For $x_4 = 3$, number of integral solutions of $x_1 + x_2 + x_3 = 8$ is ${}^{10}C_8$.

For $x_4 = 4$, number of integral solutions of $x_1 + x_2 + x_3 = 4$ is ${}^{16}C_4$.

For $x_4 = 5$, number of integral solutions of
$$x_1 + x_2 + x_3 = 0 \text{ is } 1 \text{ (for } x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0)$$

Hence, total number of integral solutions of
$$x_1 + x_2 + x_3 + 4x_4 = 20 \text{ with } x_i \geq 0$$

is
$$ {}^{22}C_{20} + {}^{18}C_{16} + {}^{14}C_{12} + {}^{10}C_8 + {}^6C_4 + 1$$
$$= \frac{22 \cdot 21}{2} + \frac{18 \cdot 17}{2} + \frac{14 \cdot 13}{2} + \frac{10 \cdot 9}{2} + \frac{6 \cdot 5}{2} + 1$$
$$= 231 + 153 + 91 + 45 + 15 + 1$$
$$= 536$$

Example 50. *How many integers between* 1 *and* 1000000 *have the sum of their digits equal to* 18?

Solution Integers between 1 and 1000000 may be of 1 digits or of 2 digits or of 3 digits or of 4 digits or of 5 digits or of 6 digits and

Let the digits be $x_1, x_2, x_3, x_4, x_5, x_6$

Then $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18 \quad \text{...(i)}$$

with $0 \le x_2 \le 9$ for $i = 1, 2, 3, 4, 5, 6$

Hence, we have to find the number of integral solutions of Eq. (i) with the digits 0, 1, 2, ..., 9.

$$= \text{coefficient of } x^{18} \text{ in } (1 + x + x^2 + \ldots + x^9)^6$$

$$= \text{coefficient of } x^{18} \text{ in } \left(\frac{1 - x^{10}}{1 - x}\right)^6$$

$$= \text{coefficient of } x^{18} \text{ in } (1 - x^{10})^6 (1 - x)^{-6}$$

$$= \text{coefficient of } x^{18} \text{ in } (1 - {}^6C_1 x^{10} + {}^6C_2 x^{20} + \ldots)(1 + {}^6C_1 x + {}^7C_2 x^2 + \ldots)$$

$$= {}^{23}C_{18} - {}^6C_1 \times {}^{13}C_8 = \frac{23!}{18!\,5!} - 6 \cdot \frac{13!}{8!\,5!}$$

$$= \frac{23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 18!} - 6 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 8!}$$

$$= 23 \times 11 \times 7 \times 19 - 6 \times 13 \times 11 \times 9$$

$$= 33649 - 6 \times 1287$$

$$= 33649 - 7722$$

$$= 25927$$

Example 51. *There are m seats in the first row of a theature, of which n are to be occupied. Find the number of ways of arranging n persons so that*

(i) *no two person sit side by side* (ii) *each person has exactly one neighbour.*

Solution (i) Let $P_1, P_2, \ldots, P_n$ be the n persons and S_1, S_2, S_m be the m seats.

Let the n persons $P_1, P_2, \ldots, P_n$ be seated as

$$\underset{n_1}{\ldots} P_1 \underset{n_2}{\ldots} P_2 \underset{n_3}{\ldots} P_3 \underset{n_n}{\ldots} P_n \underset{n_{n+1}}{\ldots}$$

Let n_1 be the number of empty seats to the left of P_1 and n_{n+1} be the number of seats to the right of P_n, let n_2 be the number of seats between P_1 and P_2, n_3 between $P_2, P_3 \ldots n_n$ between P_n and P_{n+1}, then

$$n_1, n_{n+1} \ge 0 \text{ and } n_2, n_3, \ldots, n_n \ge 0$$

Then $$n_1 + n_2 + \ldots + n_n = m - n$$

Now, solution if (i), given by coefficient of

$$x^{m-n} (1 + x + x^2 + \ldots)(x + x^2 + x^3 + \ldots)^{n-1} (1 + x + x^2 + \ldots)$$

$$= \text{coefficient of } x^{m-n} \text{ in } (1 + x + x^2 + \ldots)^2 (x + x^2 + x^3 + \ldots)^{n-1}$$

$$= \text{coefficient of } x^{m-n} \text{ in } x^{n-1}(1 + x + x^2 + \ldots)^2 (1 + x + x^2 + \ldots)^{n-1}$$

$$= \text{coefficient of } x^{m-2n+1} \text{ in } (1 + x + x^2 + \ldots)^{n+1}$$

$$= \text{coefficient of } x^{m-2n+1} \text{ in } (1 + x + x^2 + \ldots)^{n+1}$$

$$= \text{coefficient of } x^{m-2n+1} \text{ in } \left(\frac{1}{1 - x}\right)^{n+1}$$

$$= \text{coefficient of } x^{m-2n+1} \text{ in } (1 - x)^{-(n+1)}$$

$$= \text{coefficient of } x^{m-2n+1} \text{ in } (1 + {}^{n+1}C_1x + {}^{n+2}C_2x^2 + \ldots)$$

$$= {}^{m-n+1}C_{m-2n+1} = {}^{m-n+1}C_n$$

Now, since n persons can be permuted in ${}^nP_n = n!$ ways, Hence, required number of ways in the case

$$= {}^{m-n+1}C_n \; {}^nP_n = \frac{m! - n + 1}{n!\,m! - 2n + 1} n!$$

$$= {}^{m-n+1}P_n$$

(ii) It is important to notice that no such arrangement exist when n is odd. Hence, we suppose that n is even. Let $n = 2k$, k being a +ve integer. Let x_0 denote the number of empty seats to the left of first pair and x_i $(1 \le i \le k-1)$ the number of empty seats to the right of *mth pair*. Here, $x_0 > 0, x_1 \ge 0$; $x_i \ge 0$, $x_2 \ge 1$ $(1 \le i \le k-1)$ and $n_0 + n_1 + n_2 + \ldots + n_k = m - 2k \ldots$

Now, as in part (i) number of integers solution of (ii) is $m - 2k + 1$.

Also, n person can be permuted in ${}^nP_n = n!$ ways. Hence, in the case required number of ways are

$${}^{m-2k+1}C_m \cdot {}^{2k}P_{2k} = \frac{m! - 2k + 1}{k!\,(m! - 3k + 1)} \cdot 2k!$$

$$= \frac{2k!\,(m! - 2k + 1)}{k!\,(m! - 3k + 1)}$$

$$= {}^{2k}P_k \; {}^{m-2k+1}P_k = {}^nP_{n/2} \; {}^{m-n+1}P_{n/2}$$

Example 52. *'A' is a set consisting of n elements A subset 'P' of 'A' is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is chosen. Find the number of ways of choosing P and Q solution so that $P \cap Q$ contains exactly one element.*

Solution We choose one element out of n elements of A is ${}^nC_1 = n$ ways. Then $(n-1)$ element remains and these $(n-1)$ elements should not be in $P \cap Q$ in 3 ways.

(i) $a_r \in P$, $a_r \notin Q$ (ii) $a_r \notin P$, $a_r \in Q$ (iii) $a_r \notin P$, $a_r \notin Q$

Hence, remaining $(n-1)$ elements are not in $P \cap Q$ in 3^{n-1} ways.

Hence, number of ways of choosing the subset P and Q.

So that $P \cap Q$ contains exactly one element is $n \cdot 3^{n-1}$ ways.

Example 53. *A is a set containing n elements. A subset P of A is chosen. Then the set A is reconstructed by replacing the elements of P_1. Then a subset P_2 of A is chosen and again the set A is reconstructed by the elements of P_2. In this way m (>1) subsets P_1, P_2, P_m of A are chosen. Find the number of ways of choosing $P_1, P_2, \ldots, P_m$ so that*

(i) $P_1 \cap P_2 \cap \ldots \cap P_m = \phi$ (ii) $P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_n = A$.

Solution Let $A \supset \{a_1, a_2, a_3, \ldots, a_n\}$. In course of choosing $P_1, P_2, \ldots, P_n$ for any element a_r of A', we have two cases either $a_r \in P_i$ or $a_r \notin P_i$ $(1 \le i \le m)$

Hence, total number of ways in which a_r may be in P_i' are 2^m. Also, out these 2^m ways there is only one ways in which $a_r \notin P_i$ for $i = 1, 2, \ldots, m$ which does not just the condition $P_1 \cap P_2 \cap P_m = \phi$

Thus, for any element $a_r \in A$ $a_r \notin P_1 \cap P_2 \cap \ldots \cap P_m$ in $2^n - 1$ ways. Hence, total number of ways is $(2^m - 1)^n$.

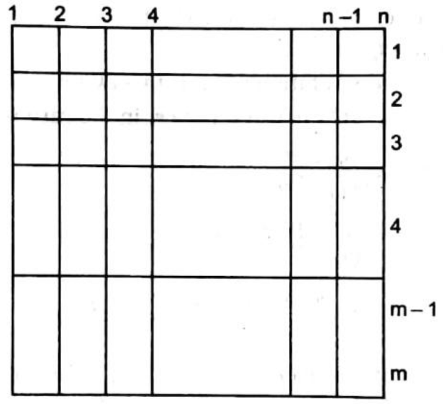
Also, there is only one way *i.e.*, $a_r \notin P_i$ which does not suit $P_1 \cup P_2 \cup \ldots \cup P_m = A$. Hence, a_r $P_1 \cup P_2 \cup \ldots \cup P_m$, in $2^n - 1$ ways. Also, since these n elements in the set A and hence the number of ways in which

$$P_1 \cup P_2 \cup \ldots \cup P_m = A \text{ is } (2^m - 1)^n.$$

Example 54. *If a city has m parallel roads running East-West and n parallel roads running North-South. How many rectangles are formed with sides along these roads? If the distance between every consecutive pair of parallel roads is the same, how many shortest possible routes are there to go from one corner of the city to its diagonally opposite corner?*

Solution A rectangle is formed by two lines from $E - W$ and two lines from $N - S$ but two lines out of m lines running $E - W$ can be selected in ${}^{m}C_2$ ways and 2 lines out of n running $N - W$ can be selected in ${}^{n}C_2$ ways. Hence, number of rectangles thus formed $= {}^{m}C_2 \cdot {}^{n}C_2$

$$= \frac{mn(m-1)(n-1)}{4}$$



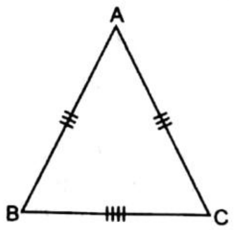
Now, if distance between two consecutive parallel roads is (= 1 unit). Now, total distance a man has to travel to go from one corner to a diagonally opposite corner is $(m + n - 2)$ units. Of these $(m + n - 2)$ units, he has travel $(m - 1)$ units in one direction and $(n - 1)$ units in the other direction.

Hence, the number of ways of arranging the steps

$$= {}^{m+n-2}C_{m-1}\ {}^{n-1}C_{n-1} = {}^{m+n-2}C_{m-1}$$

Example 55. *The sides AB, BC, CA of a $\Delta\ ABC$ have 3, 4, 5 interior points respectively on them. Find the number of triangle formed by any three of the 12 points as vertices.*

Solution Here, total number of points $= 3 + 4 + 5 = 12$. If no any three out of 12 points are collinear thus, total number of triangles formed by taking any 3 points out of 12 points vertices $= {}^{12}C_3$. Thus, ${}^{12}C_3$ include the case of selection of 3 points from the 3 points on a side say AB.



But no triangle is formed by taking 3 points lying on the side AB.

Hence, we subtract ${}^{3}C_3$ from ${}^{12}C_3$. Similarly, no triangle is formed with the 4 points lying on the other side BC and no triangle is formed with 5 points lying on the side AC. Hence, we also subtract ${}^{4}C_3$ and ${}^{12}C_3$.

Finally required number of triangles

$$= {}^{12}C_3 - ({}^3C_3 + {}^4C_3 + {}^5C_3) = {}^{12}C_3 - (1 + 4 + 10)$$

$$= \frac{12!}{3!\,9!} - 15 = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2} - 15 = 195$$

Example 56. *If n points in a plane be joined in all possible ways by indefinite straight lines and if no two of straight lines be coincident or parallel and no three pass through the same point with the exception of the n points. Find the number of distinct point of intersection.*

Solution Out of n points, number of different straight lines formed = x (say)

Now. since no three straight lines pass through the same point and no two straight lines are parallel. We see that each pair of lines from the ${}^nC_2$ lines intersect at one point.

Hence, number of points of intersection = ${}^xC_2$. Now, let $P_1, P_2, \ldots, P_n$ be the n given points. It is here important to notice that through the point P_1 $(n-1)$ lines by joining P_1 to other $(n-1)$ points, pass. But any two of these $(n-1)$ lines give P_1 as point of intersection.

Hence, we see the point P_1 occurs ${}^{n-1}C_2$ times. Similarly, each point $P_2, P_3, \ldots, P_n$ occurs $n \cdot {}^{n-1}C_2$ times *i.e.*, n given points occur $n \cdot {}^{n-1}C_2$ times as point of intersection.

Now, number of points of intersection except n given points

$$= {}^xC_2 - n \cdot {}^{n-1}C_2$$

$$= \frac{1}{2} x \cdot (x-1) - n \cdot \frac{(n-1)(n-2)}{2}$$

$$= \frac{1}{2} \cdot \frac{n(n-1)}{2} \left[\frac{n \cdot (n-1)}{2} - 1\right] - \frac{n \cdot (n-1)(n-1)}{2}$$

$$= \frac{1}{8} n \cdot (n-1)(n^2 - n - 2) - \frac{n \cdot (n-1)(n-2)}{2}$$

$$= \frac{n \cdot (n-1)}{8} [n^2 - n - 2 - 4(n-2)]$$

$$= \frac{n \cdot (n-1)}{8} (n^2 - 5n + 6)$$

Hence, total number of points of intersection including the n given points

$$= n + \frac{n \cdot (n-1)(n^2 - 5n + 6)}{8}$$

$$= \frac{1}{8} n \cdot (n^3 - 6n^2 + 11n + 2)$$

Example 57. *Find the number of rectangles that you can have on a chess-board.*

Solution We observe that there are 9 horizontal and 9 vertical lines on a chess-board such that each horizontal line intersect each vertical line. A rectangle is formed by two horizontal and two vertical lines.

But if we select 2 horizontal lines out of 9 horizontal lines in ${}^9C_2$ ways and we can select 2 vertical lines out of 9 vertical lines in ${}^9C_2$ ways.

Hence, total number of rectangles on a chess-board = ${}^9C_2 \times {}^9C_2 = 36 \times 36 = 1296$.

Example 58. *The straight lines l_1, l_2 and l_3 are parallel and lie on the same plane. A total number m points are taken on l_1; n points on l_2, k points on l_3. Find the maximum number of triangles formed with the vertices at these points.*

Solution Total number of points $= m+n+k$

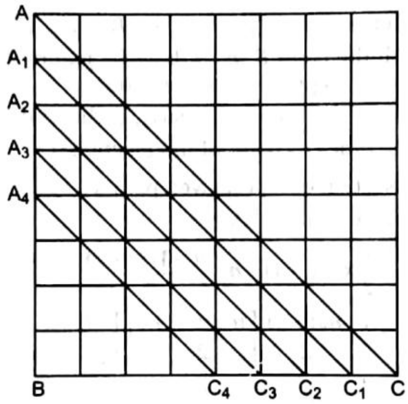
Now, number of triangles formed by the points $m+n+k = {}^{m+n+k}C_3$

But as m points be on the same straight line ${}^mC_3$ give no triangle, n points be on the same straight line so that ${}^nC_3$ give no triangle and k points lie on the same straight line so that ${}^kC_3$ also give no triangle.

Hence, required number of triangles $= {}^{m+n+k}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$.

Example 59. *In how many ways 4 square are can be chosen on a chess-board, such that all the squares lie in a diagonal line.*

Solution Let us consider the ΔABC. Number of ways in which 4 selected squares are along the lines A_4C_4, A_3C_3, A_2C_2, A_1C_1 and AC are



${}^4C_4$, ${}^5C_4$, ${}^6C_4$, ${}^7C_4$ and ${}^8C_4$ respectively.

Similarly, in ΔACB, number of ways in which 4 selected squares are along the diagonal line parallel to AC are ${}^4C_4, {}^5C_4, {}^6C_4, {}^7C_4$ and ${}^8C_4$.

But ${}^8C_4$ triangles occur only once.

Hence, the total number of ways in which the 4 selected squares are in a diagonal line parallel to AC are

$$2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4$$

Also, same is the case of selecting 4 squares in a diagonal line parallel to BD.

Hence, the number of ways of selecting 4 squares on a chess-board. Such that the 4 squares are in a diagonal line

$$= 2[2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4]$$

Example 60. *Find the number of positive divisor of $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_r^{k_r}$, where $p_1, p_2, \ldots, p_r$ are distinct prime numbers and $k_1, k_2, \ldots, k_r$ are positive integers.*

Solution A divisor d of n is of the form $d = p_1^{l_1} \cdot p_2^{l_2} \ldots p_r^{l_r}$

where $\qquad 0 \le l_i \le k_i$ for $i = 1, 2, \ldots, r$.

Associate each divisor of n with an r tuple $(l_1, l_2, \ldots, l_r)$, such that $0 \le l_i \le k_i$. Therefore, the numbers of divisors is the same as the number of r tuples $(l_1, l_2, \ldots, l_r)$, $0 \le l_i \le k_i$, $i = 1, 2, \ldots, r$.

Since, l_1 can have $k_1 + 1$ possible values $0, 1, 2, \ldots, k_1$ and l_2 can have $k_2 + 1$ values $0, 1, 2, \ldots, k_2$ and l_r can have $k_r + 1$ values. The number of r triples $(l_1, l_2, \ldots, l_r)$ is

$$(k_1+1) \times (k_2+1) \times (k_3+1) \times \ldots \times (k_r+1)$$

or $$\prod_{i=1}^{r} (k_i + 1)$$

That is total number of divisors of

$$n = p_1^{k_1}, p_2^{k_2}, \ldots p_r^{k_r}$$

$$= (k_1 + 1)(k_2 + 1)\ldots(k_r + 1) \text{ or } \prod_{i=1}^{r} (k_i + 1)$$

Example 61. *Prove that, there are $2(2^{n-1} - 1)$ ways of dealing n cards to two persons. (The players may receive unequal number of cards).*

Solution Let us number of the cards, for the moment. Let us accept the case, where all the cards, go to one of the two players, also.

With just two cards we have the possibilities.

$$AA, AB, BA, BB \quad \ldots\text{(i)}$$

Here, AA means A gets 1 card and also card 2,
AB means A gets card 1 and B gets card 2,
BA means B gets card 1 and A gets card 2,
BB means B gets card 1 and also card 2.

Thus, for two cards we have 4 possibilities. Here, for three cards

$$AAA, ABA, BAA, AAB, ABB, BAB, BBB \quad \ldots\text{(ii)}$$

i.e., for three cards there are $2^3 = 8$ possibilities. Here, if the third card goes to A, then in Eq. (i) annex A at the end, thus getting AAA, ABA, BAA, BBA,. If it goes to B then in Eq. (i) annex A at the end, which gives AAB, ABB, BAB, BBB.

Thus, the possibilities doubled, when a new card (third card) is included.

In fact just with one card it may either go to A or B, giving AB.

By annexing the second card, it may give

AA, BA, giving Eq. (i)
AB, BB

Thus, every new card doubles the existing number of possibilities of distributing the cards.

Hence, the number of possibilities with n cards is 2^n. But this include the distribution where one of them gets all the cards and the other none.

So, total number of possibilities is $2^n - 2 = 2(2^{n-1} - 1)$

Example 62. *In a plane, a set of* 8 *parallel lines intersects a set of n other parallel lines, giving rise to* 420 *parallelograms (many of them over lap with one another). find the value of n.*

Solution If two lines which are parallel to one another (in one direction) intersect another two lines, which are parallel, we get one parallelograms. Thus, we can choose $C(8, 2)$ pair of parallel line in one direction and the number of parallel lines intersecting there will be $C(n, 2)$ pair.

So, the number of parallelograms thus obtained is $C(n, 2) \times C(8, 2) = 420$

[With three sets of parallel lines, we have 9 parallelograms (verify) *i.e.*, ${}^3C_2 \times {}^3C_2 = 9$]

$$\frac{n(n-1)}{2 \times 1} \times \frac{8 \times 7}{1 \times 2} = 420$$

$$n(n-1) = 30$$

or $$n = 6$$

or $n = -5$ (which is not admissible).

Thus, $n = 6$ is the solution.

Example 63. *Let A be an 2n element set, where $n \geq 1$. Find the number of pairing of A.*

Solution A pair is set of 2 elements. These are thus finally n pairs.

Now, the first pair can be selected in $\binom{2n}{2}$ ways

i.e., $$\frac{(2n)!}{(2n-2)!\,21}\text{ ways.}$$

The second pair can be selected from the remaining $(2n-2)$ elements in $\binom{2n-2}{2}$ ways, i.e.,

$$\frac{(2n-2)!}{(2n-4)!\times 2!}\text{ and so on up to }\binom{2}{2}=\frac{2!}{0!\times 2!}$$

Number of pairing of A is, therefore,

$$=\frac{\frac{2n\times(2n-1)}{2}\times\frac{(2n-2)(2n-3)}{2}\times\frac{(2n-4)(2n-5)}{2}\times..\times\frac{2\times 1}{5}}{n!}$$

$$=\frac{2n\times 2(n-1)\times 2(n-2)\times\ldots\times(2n-1)(2n-3)\times(2n-5)\times\ldots\times 1}{2^n\times n!}$$

$$=\frac{2^n\times n(n-1)\times(n-2)\times\ldots\times 1\times(2n-1)(2n-3)(2n-5)\times\ldots\times 1}{2^n\times n!}$$

$$=\frac{2^n\times n!\times(2n-1)(2n-3)(2n-5)\times\ldots\times 1}{2^n\times n!}$$

$$=(2n-1)(2n-3)(2n-5)\times\ldots\times 1\text{ ways}$$

Example 64. *Find the number of 6 digits natural numbers, where each digit appears at least twice.*

Solution We consider number like 222222 or 233200 but not 212222, since the digit 1 occurs only once. The set of all such 6 digits can be divided into the following classes

S_1 = the set of all 6 digit numbers where a single digit is repeated 6 times.

$n(S_1) = 9$, since '0' cannot be significant number when all its digits are zero

Let S_2 be the set of all 6 numbers, made up of three distinct digits.

Here, we should have two cases $S_2(a)$ one with the exclusion of zero as a digit and other $S_2(b)$ with the inclusion of zero as a digit.

S_2 (a) the number of ways, three digits could be chosen from $1, 2, \ldots, 9$ is ${}^9C_3$. Each of these three digits occur twice. So, the number of 6 digit numbers in the case is

$$={}^9C_3\times\frac{6!}{2!\times 2!\times 2!}=\frac{9\times 8\times 7}{1\times 2\times 3}\times\frac{720}{8}$$

$$=9\times 8\times 7\times 15=7560$$

S_2 (b), the three digits used include one zero, implying, we have to choose the other two digits from the non-zero digits.

This could be done in ${}^9C_2=\frac{9\times 8}{1\times 2}=36$. Since, zero cannot be in the leading digit. So, let us fix one of the digit (non-zero) in the extreme left. Then the other five digits are made up of 2 zeroes, 2 fixed non-zero number and the another non-zero number, one of which is put in the extreme left. In this case the number of six digit numbers that could be formed is $\frac{5!}{2!\times 2!\times 1!}\times 2$ (Since, from either of the pairs of fixed non-zero numbers, one can occupy the extreme digits $= 60$

So, the total numbers in this case $= 36 \times 60 = 2160$

$\therefore \quad n(S_2) = n(S_2\, a) + n(S_2\, b) = 7560 + 2160 = 9720.$

Now, let S_3 be the set of 6 digit numbers, whose digits are made up of two distinct digit, each of which occurs thrice. Here again, there are two cases : S_3 (*a*) excluding the digit S_3 (*b*) including the digit zero.

S_3 (*a*) is the set of six digit numbers whose digits are made up of two non-zero digits, each occurring thrice.

$$\therefore \quad n[S_3(a)] = {}^9C_2 \times \frac{6!}{3!3!} = 36 \times 20 = 720$$

S_3 (*b*) consist of 6 digits numbers whose digits are made up of three zeroes and one of non-zero digit, occurring thrice. If you fix one of the nine non-zero digit, use that digit in the extreme left. This digit should be used thrice. So, in the remaining 5 digits, this fixed non-zero digit is use twice and the digit zero occurs thrice.

So, the number of 6 digit numbers formed in this case is $9 \times \dfrac{5!}{3! \times 2!} = 90$

$$\therefore \quad n(S_3) = n(S_3(a)) + n(S_3(b)) = 720 + 90 = 810$$

Now, let us take S_4, the case where the six digit number consist of exactly two digits, one of which occurs twice and the other four times.

Here again, there are two cases : S_4 (*a*) excluding zero and S_4 (*b*) including zero.

If *a* and *b* are the two non-zero numbers '*a*' used twice and '*b*' four times, then we get $\dfrac{6!}{2! \times 4!}$ and when '*a*' used four times; *b* twice, we again get $\dfrac{6!}{2! \times 4!}$

So, when 2 of the nine non-zero digits are used to form the six digit number in this case, the total number got is

$${}^9C_2 \times 2 \times \frac{6!}{4! \times 2!} = 36 \times 5 \times 6 = 1080$$

Thus, $\quad n[S_4(a)] = 1080$

for counting the numbers in S_4 (*b*)

In this case we use 4-zeros and a non-zero number twice 2-zeros and a non-zero number four times. In the former case, assuming the one of the fixed non-zero digit occupying extremely left, we get the other five digits consisting of 4-zeroes and one non-zero number.

This result in $\dfrac{9 \times 5!}{4! \times 1!} = 45$ six digit number.

When we use the fixed non-zero digit 4 times and use zero twice, then we get $\dfrac{9 \times 5!}{3! \times 2!} = 90$ six digit numbers, as the fixed number occupies the extreme left and for the remaining 3 times it occupies 3 of the remaining digits, other digits being occupied by the two zeroes.

So, $\quad n(S_4) = n[S_4(a)] + n[S_4(b)] = 1080 + 45 + 90 = 1215$

Hence, the total number of six digit number satisfying the given condition

$$= n(S_1) + n(S_2) + n(S_3) + n(S_4) = 9 + 720 + 810 + 1215 = 2754.$$

Example 65. *Let $S = \{1, 2, 3, ...(n+1)\}$, where $n \le 2$ and let $T = \{(x, y, z) : x, y, z \in S, x < z, y < z\}$. By counting the members of T in two different ways, prove that* $\sum_{K=1}^{n} K^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$.

Solution T can be written as $T = T \cup T_2, T_1 = \{(x, x, z) : x, z \in S, x < z\}$ and $T_2 = \{(x, y, z) : x, y, z \in S, x + y < z\}$.

The number of elements in T_1 is the same as choosing two elements from the sets where $n(S) = (n+1)$,

i.e., $n(T_1) = \binom{n+1}{2}$ (as every subset of two elements, the large element will be x and the smaller will be x and y).

In T_2, we have $2\binom{n+1}{3}$ elements, after choosing 3 elements from the set S, two of the smaller elements will be x and y and they may be either taken as (x, y, z) or as (y, x, z) or in other words, every three elements subset of S_1 say $\{a, b, c\}$, the greatest is z, and the other two can be placed in two different ways in the first two positions.

$$\therefore \qquad n(T) \text{ or } (|T|) = \binom{n+1}{2} + 2\binom{n+1}{2}$$

T can also be considered as $\bigcup_{i=2}^{n+1} S_i$, where $S_i = \{(x, y, i) : x, y < i,\ x, y \in S\}$. All these sets are pairwise disjoint as for different i, we get different ordered triplets (x, y, i).

Now, in S_i, the first two components of (x, y, i), namely (x, y) can be any element from the set $1, 2, 3, \ldots, i-1$.

x and y can be any member from $1, 2, 3, \ldots, (i-1)$ equal or distinct.

$\therefore$ The number of ways of selecting $(x, y), x, y \in \{1, 2, 3, \ldots, (i-1)\}$ is $(i-1)^2$.

Thus, $n(S_i)$ for each i is $(i-1)^2, i \geq 2$. For example $n(S_2) = 1, n(S_3) = 2^2 = 4$ and so on.

Now,
$$n(T) = n\left(\bigcup_{i=2}^{n+1} S_i\right) = \sum_{i=2}^{n+1} n(S_i)$$

(because all S_i's are pairwise disjoint)

$$= \sum_{i=2}^{n+1} (i-1)^2 = \sum_{i=1}^{n} i^2$$

and hence,
$$\binom{n+1}{2} + 2\binom{n+1}{3} = \sum_{k=1}^{n} k^2$$

Example 66. *Among the integers 1, 2,...., 200 if any 101 integers are chosen, then show that there are 2 among the chosen integers, such that one is divisible by other.*

Solution Let each of 101 integers chosen from 1, 2, ..., 200 be factorized in the form $m_i = 2^{k(i)} \cdot a_i$ for $I = 1, 2, \ldots, 101$, where a_i is an odd number and $k(i)$ is a non-negative integers. Each a_i is one of the 100 odd numbers integers only. Hence by PP1, among the chosen 101 numbers m_i at least 2 have equal odd parts.

Thus, let $m_t = 2^{k(t)} \cdot a_t$ and $m_s = 2^{k(s)} \cdot a_s$ with $a_t = a_s$. Then if $k(t) < k(s)$, then m_t divides m_s.

Example 67. *Show that every sequence $a_1, a_2, \ldots, a_{mn+1}$ of $mn + 1$ distinct real numbers contains either an increasing subsequence of length $m + 1$ or decreasing subsequence of length $n + 1$.*

Solution Let x_i denote the length of the longest increasing subsequence beginning at a_i and let y_i be the length of longest decreasing subsequence beginning at a_i. Consider the $mn + 1$ ordered pairs (x_i, y_i), $1 \leq i \leq mn + 1$. We will prove that all these $mn + 1$ ordered pairs are distinct.

For any distinct integers i, j such that $1 \leq i, j \leq mn + 1$ we are given that $a_i \neq a_j$, since the terms of the sequence are distinct. So first let $a_i < a_j$. Then it follows that $x_i > x_j$ since we can append a_i at the beginning of every increasing subsequence beginning with a_j and obtain a longer subsequence of the same type. Similarly, if $a_i > a_j$ then $y_i > y_j$. Hence, if $i \neq j$, then (x_i, y_i) and (x_j, y_j) are distinct ordered pairs.

Now, suppose that stated result is not true. Then for each i, we must have $1 \le x_i \le m$ and $1 \le y_i \le n$, thus, there are only m possible different values of x_i and only n possible different value of y_i. Hence, by the multiplication principle only mn of the $mn + 1$ ordered pair (x_i, y_i) can be distinct. Hence, by PP1 at least 2 of the $mn + 1$ ordered pairs must be equal, a contradiction. Hence, result is true.

Example 68. *Suppose the numbers 1 to 10 are randomly positioned around a circle. Show that m_2 of some set of 3 consecutive numbers be at least 17.*

Solution Let the number be $a_1, \ldots, a_{10}$. Note that these are the numbers1, 2, ..., 10 in some order. Now, there are 10 triples consisting of 3 consecutive numbers namely.

$$(a_1, a_2, a_3), (a_2, a_3, a_4), \ldots, (a_{10}, a_1, a_2)$$

Note that each a_i occurs in exactly 3 triples. Hence, average sum per triple is

$$\frac{3(a_1 + \ldots + a_{10})}{10} = \frac{3(1 + \ldots + 10)}{10}$$

$$= \frac{3(10)(11)/2}{10} = 16.5$$

Hence, by PP3, the sum of the numbers in at least one triple must be ≥ 16.5. Hence, that sum must be ≥ 17 since it is an integer.

Example 69. *13 persons have first names, Bapu, Chandru, Damu and last names Kale, Late, Mate and Natu. Show that at least 2 persons have the same first and last names.*

Solution By the multiplication principle, these are $4 \times 3 = 12$ possible names. Now, regard the 13 persons as 13 objects and 12 names as 12 boxes. Then by PP1, it follows that at least 2 objects are in the same box *i.e.*, at least 2 persons have the same name.

Example 70. *18 persons have first name Ekta, Ganesh and Hari and last names Patil and Rathi. Show that at least 3 persons have the same first and last names.*

Solution By multiplication principle, there are $3 \times 2 = 6$ names, such as Ekta, Ganesh etc. Now, since there are 18 persons and $18 = 3 \times 6$, it follows by PP2 that at least 3 persons have the same first and last names.

Example 71. *The members of a class of 27 pupils each go swimming on some of the days from Mon-Fri in a certain week. If each pupil goes at least twice, show that there must be 2 pupils, who go swimming on exactly the same days.*

Solution The set {Mon, ..., Fri} of 5 days has

$$\binom{5}{5} + \binom{5}{4} + \binom{5}{3} + \binom{5}{2} = 1 + 5 + 10 + 10 = 26$$

subsets each containing 2 or more days. Regard the 27 pupils as objects and these 26 subsets as 'boxes'. Then by PP1, there must be at least one box containing at least 2 pupils *i.e.*, at least 2 pupils must go swimming on the same days.

Example 72. *Let A be any set of 20 distinct integers chosen from AP 1, 4, 7, ..., 100. Prove that there must be 2 distinct integers in A, whose sum is 104.*

Solution The given AP 1, 4, ..., 100 has nth term $3n - 2$ and hence contains 34 terms. Arrange these terms in 18 boxes as follows.

$$\boxed{1} \quad \boxed{52} \quad \boxed{4 \ 100} \quad \boxed{7 \ 97} \cdots\cdots \boxed{49 \ 55}$$

Note that the sum of the numbers in each of the last 16 boxes is 104. Then by PP1, if 20 numbers are taken from these 18 boxes, then at least 2 of the numbers must come from the same box (and that box must be one of the last 16 boxes since the first 2 contain only 1 number each) and this sum is 104 as required.

Example 73. *Let* $S = \{3, 7, 11, \ldots, 103\}$. *How many elements must we select from* S *to ensure that there will be at least* 2 *distinct integers among them whose sum is* 110.

Solution The set S contains 26 elements which form an AP. These numbers can be put in following 14 $\boxed{3}$ $\boxed{55}$ $\boxed{7\ 103}$ $\boxed{11\ 99}$ ········ $\boxed{51\ 59}$ boxes where the sum of the numbers in each of the last 12 boxes is 110. Hence, by PP1 is enough to select 15 elements from S.

Example 74. *If* 11 *integers are selected from* $\{1, 2, \ldots, 100\}$, *prove that there are at least* 2 *say* x, y *such that* $0 < |\sqrt{x} - \sqrt{y}| < 1$.

Solution Consider the integral part $[t]$ of the real number t. Then $t - [t] = f$ is the fractional part of t and $0 \le f < 1$. Now, since $1 \le x \le 100$, we have $1 \le \sqrt{x} \le 10$, and so $1 \le [\sqrt{x}] \le 10$, thus for elements 1, 2,...,10. Hence, if 11 numbers are taken from S, then by PP1 at least 2 of them say x, y must be such that $\sqrt{x}$ and $\sqrt{y}$ have the same integral part say i.

So, if
$$\sqrt{x} = i + f_1 \text{ and } \sqrt{y} = i + f_2,$$
$$0 \le f_1, f_2 < 1, \text{ then } 0 < |\sqrt{x} - \sqrt{y}|$$
$$= |f_1 - f_2| < 1$$

Example 75. *Let* n *be an odd +ve integer. If* $i_1, i_2, \ldots, i_n$ *is a permutation of* 1, 2, ... , n. *Prove that* $(1 - i_1)(2 - i_2) \ldots (n - i_n)$ *is an even integer.*

Solution Since, n is odd, let $n = 2m + 1$, where m is a non-negative integer. Then set $S = \{1, 2, \ldots, n\}$ contains $m + 1$ odd numbers namely $1, 3, \ldots, 2m + 1$ but only m even numbers, namely $2, 4, \ldots, 2m$. The same is true of the permutation $i_1, i_2, \ldots, i_n$ of S. Consider the $m + 1$ numbers $1 - i_1, 3 - i_3, \ldots, n - i_n$ which are of the form $r - i_r$, where r is odd. Since, i_s is even for only m values of S, by PP1, one of the $m + 1$ numbers $i_1, i_2, \ldots, i_n$ say i_t is odd, where t is also odd. Hence, $t - i_t$ is even and so the product
$$(1 - i_1) \cdot (2 - i_2) \ldots (n - i_n) \text{ is even.}$$

Example 76. *Let* $n \ge 3$ *be an odd number. Show that there is a number in the set* $\{2^1 - 1, 2^2 - 1, \ldots, 2^{n-1} - 1\}$, *which is divisible by* n.

Solution Consider the n numbers $2^0, 2^1, \ldots, 2^{n-1}$ since n is odd, none of these numbers is divisible by n and so modulo n, they can leave only $n - 1$ different remainders $1, 2, \ldots, n - 1$. Hence, by PP1, 2 of them say 2^r and 2^s, $0 \le s < r \le n - 1$ must leave the remainder modulo n, so that n divider $2^r - 2^s = 2^s(2^{r-s} - 1)$. Hence, n divides $2^{r-s} - 1$, since n is odd and so n and 2^s are coprime. So, the result follows since $1 \le r - s \le n - 1$.

Example 77. *Given* m *integers* $a_1, a_2, \ldots, a_m$, *show that there exist integers* k, s *with* $0 \le k < s \le m$ *such that* $a_{k+1} + a_{k+2} + \ldots + a_s$ *is divisible by* m.

Solution Consider the sequence $a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_m$. If anyone of these m sums is divisible by m, then we are through. Otherwise suppose that none of them is divisible by m so each leaves a non-zero remainder $1, 2, \ldots, (m - 1)$. Since, there are m sums and $(m - 1)$ possible values of the remainders by the pigeonhole principle 2 of the sums leave the same remainder after division by m. So, let
$$a_1 + a_2 + \ldots + a_k = b_m + r$$
and
$$a_1 + a_2 + \ldots + a_s = c_m + r$$

This gives (if $k < s$), $a_{k+1} + a_{k+2} + ... + a_s = m\,(c - b)$

Thus, m divides $a_{k+1} + a_{k+2} + ... + a_s$

Example 78. *Suppose numbers 1 to 20 are placed in any order around a circle. Show that the sum of some 3 consecutive numbers must be at least 32.*

Solution Let $a_1, ..., a_{20}$ be the numbers placed around the circle. Since, the mean of the 20 sums of 3 consecutive numbers namely

$$a_1 + a_2 + a_3, a_2 + a_3 + a_4, ..., a_{19} + a_{20} + a_1,$$
$$a_{20} + a_1 + a_2$$

is
$$\frac{1}{20}[3(a_1 + a_2 + ... + a_{20})]$$
$$= \frac{3(20)(21)}{2(20)} = 31.5$$

We see by the alternate form of PP that at least one of the sums must be ≥ 32.

Example 79. *A storekeeper's list consist of 115 items each marked "available" or "unavailable". There are 60 available items. Show that there are at least 2 available items in the list exactly 4 items apart.*

Solution Let the position of the available items be $a_1, a_2, ..., a_{60}$. Since $a_{60} \leq 115$, we see that 120 numbers $a_1 < a_2 < ... < a_{60}$ and $a_1 + 4 < a_2 + 4 < ... < a_{60} + 4$ lie between 1 and 119. Hence, by PP1, 2 of these numbers must be equal. But the numbers in the first are all distinct. Similarly, the numbers in the second row are all distinct. Hence, the some number in the first row must be equal to a number in the second row *i.e.*, for some i, j we must have $a_i = a_j + 4$, so that $a_i - a_j = 4$ as required.

Example 80. *Prove that, if in a group of 6 persons, each pair is either of mutual friends or mutual enemies, then there are either three mutual friends or 3 mutual enemies. Also, show that the result is not true in case of a group of 5 persons.*

Solution Consider a fixed person A of the other five. By PP, there are either 3 who are friends of A or 3 who are not. In the first case, the 3 friends of A are either mutual enemies or 2 of them are friends and form a triplet of friends with A. The other case is similar.

Now, consider a group of 5 persons and suppose that exactly the following are pairs of friends AB, BC, CD, DE, EA. Then it is easy to see that no three are mutual friends or mutual enemies.

Example 81. *Let $(x_i, y_i), 1 \leq i \leq 5$ be set of 5 distinct points with integer coordinates in x-y plane. Show that the mid-point of the line joining at least one pair of these points has integer coordinates.*

Solution Since an integer must be either even or odd. Every point (a, b) with integer coordinates must be put in one of the 4 pigeonholes (even, even), (even, odd), (odd, even) and (odd, odd). Hence, 2 of the given 5 points [say $A\,(x_1, y_1)$ and $B(x_2, y_2)$] must lie in the same pigeon hole, so that their x coordinate must have same parity (*i.e.*, they are either both even or both odd) and their y coordinates must have same parity. Hence, $x_1 + x_2$ and $y_1 + y_2$ are even. Thus, $(x_1 + x_2)/2$ and $(y_1 + y_2)/2$ are both integers coordinates.

Example 82. *Prove that when a rational number a/b in lowest term is expressed as a decimal, the decimal must either terminates or recur.*

Solution Divide $10a$ by b to get, $10a = x_1 b + r_1$, $0 \leq r_1 < b$. Next divide $10r$ by b to get, $10r_1 = x_2 b + r_2$, $0 \leq r_2 < b$. Divide $10r_2$ by b to get $10r_2 = x_3 b + r_3$, $0 \leq r_3 < b$ and so on. Then we get $a/b = 0.\ x_1 x_2 x_3 \ldots$.

Now, if the decimal does not terminate, then we must obtain non-zero remainder at each stage. Since, there are only $b-1$ possible different non-zero remainders. By PP1 some remainder must be repeated after at most b steps. Hence, the expansion will recur from this point onwards.

Example 83. *Prove that, if given a set of any 7 distinct integers, there must exist 2 integers in this set, whose sum or difference is a multiple of 10.*

Solution Consider the 6 boxes labelled as (0, 0), (1, 9), (2, 8), (3, 7), (4, 6), (5, 5). Let A be the set of 7 distinct integers.

We know that every integer is congruent to 0 or 1 or 2 or...or 9 module 10. If integer from A is congruent to k or $10-k$ module 10, then we shall put that integers in the box lablled $(k, 10-k)$. Since, there are 7 integers and only 6 boxes. By the Pigeonhole principle, at least one box say $(k, 10-k)$ contains 2 integers say x and y. If x and y both are congruent to k modulo 10, then $x-y \equiv 0 \pmod{10}$.

$\therefore$ $$10 \text{ divides } x-y$$

If x and y both are congruent to $10-k$ mod 10 then again $x-y=0 \pmod{10}$

$\therefore$ $$10 \text{ divides } x-y$$

If one of x and y is congruent to k and the other is congruent to $10-k$ mod 10, then

$$x+y \equiv 10 \pmod{10}$$

i.e., $$x+y \equiv 0 \pmod{10}$$

$\therefore$ $$10 \text{ divides } x+y$$

Thus, in any case either $x-y$ or $x+y$ is divisible by 10.

Example 84. *Show that any subset of 8 distinct integers between 1 and 14 contains a pair of integers k and l, such that either k divides l or l divides k.*

Solution 8 integers are chosen from 1 to 14. By taking the factor 2 as many times as possible from each of these 8 integers, we can write them in the form $2^{\alpha}a$; 'a' is an odd integer. Possible values of 'a' are 1, 3, 5, 7, 9, 11, 13 which are 7 in numbers.

Since, there are 8 integers and only 7 values of 'a' by Pigeonhole principle, there must be 2 integers k and l having same value 'a'.

Suppose $k=2^{\alpha}a$ and $l=2^{\beta}a$, if $\alpha<\beta$, then k divides l. If $\beta<\alpha$, then l divides k.

Example 85. *Given m consecutive integers. Prove that there is one which is divisible by m.*

Solution Consider m consecutive integers $k+1, k+2, k+3, \ldots, k+m$. Assume that none of these is divisible by m. Therefore, after dividing each of them by m, we shall get non-zero remainders $1, 2, 3, \ldots, m-1$ (not necessary in this order).

Since, there are m integers and only $m-1$ remainders. By Pigeonhole principle, at least 2 integers will leave the same remainder.

Suppose, $k+i$ and $k+j$ leave the same remainder r. when divided by m, where $i<j$

$\therefore$ $$k+i=xm+r \quad \text{and} \quad k+j=ym+r$$

subtraction gives $j-i=(y-x)m$

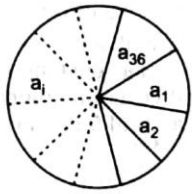
This shows that m divides +ve integer $j-i$ which is itself less than m.

But this is impossible. Therefore, our assumption must be wrong. Hence, one of $k+1, k+2, \ldots, k+m$ must be divide by m.

Example 86. *The circumference of a wheel is divided into 36 sectors and the numbers 1, 2,... 36 are assigned to them in arbitrary manner. Show that there are 3 consecutive sectors such that sum of their assigned number is at least 56.*

Solution Let $a_1, a_2, \ldots, a_{36}$ be arbitrary assignment of numbers 1, 2, 3, ..., 36 to the 36 sectors.

We group them into the collection of 3 consecutive and find 36 sums as below.



$$S_1 = a_1 + a_2 + a_3,\ S_2 = a_2 + a_3 + a_4,$$
$$S_3 = a_3 + a_4 + a_5, \ldots, S_{36} = a_{36} + a_1 + a_2$$

The sum of these 36 numbers is 3 times the sum $1 + 2 + 3 + 4 + \ldots + 36$ because each a_i is counted 3 times.

$\therefore$ $$S_1 + S_2 + \ldots + S_{36} = 3(1 + 2 + \ldots + 36) = 3(666) = 1998$$

Sum of 36 +ve integers is 1998 and $\frac{1998}{36} = 55.5$ This implies that at least one $S_i \geq 56$. Hence, there are 3 consecutive sectors, the sum of whose assigned numbers is at least 56.

Example 87. *8 composite integers are chosen from 1 to 360. Prove that the selection includes 2 integers which are not relatively prime.*

Solution Consider the first 7 prime numbers 2, 3, 5, 7, 11, 13, 17. The next prime number is 19 $19 \times 19 = 361 > 360$ from this, we conclude that any composite number from 1- 360 must have one of the above 7 prime number as a factor.

We write 8 composite numbers in the form xp, where p is smallest prime factor of that composite number.

Now, p can take 7 values and there are 8 composite numbers. By Pigeonhole principle, there exist 2 composite numbers in the selection such that they have p as smallest prime factor.

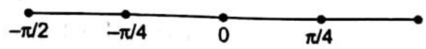
GCD of these 2 integers is $\geq p \geq 2$.

So, they are not relatively prime.

Example 88. *If 11 distinct integers are chosen from among 1, 2, 3, ..., 20. Show that selection includes at least one pair of integers which are relatively prime.*

Solution The selection of 11 distinct integers from 1 - 20 must contain at least one odd integer. If the selection includes 1, 3, 5, 7, 11, 13, 17, 19 then clearly the conclusion holds because these are prime numbers.

So, we assume that above primes are absent in the selection. Hence, the selection must contain either 9 or 15 or both 9 and 15. If the selection includes 9 but not 15, then remaining 10 integers in the selection are all even and 4 is one of them. Now, 4 and 9 are relatively prime and the conclusion holds. Suppose the selection includes 15 but not 9. Then as before remaining 10 integers in the selection are all even and 4 is one of them. Now, 4 and 15 are relatively prime and conclusion holds. Finally, suppose selection includes 9 and 15 both. The remaining 8 integers in selection are all even. The set {2, 4, 6, 8, 10, 12, 14, 16, 18, 20} contains 5 integers which are multiples of 3 or 5 (factors of 9 and 15).



They are 6, 12, 18, 10, 20. Hence, 8 even integers required for our selection must contain at least 3 integers from 2, 4, 8, 14, 16. Each of them is relatively prime with 9 and 15 and the conclusion holds.

In any case, we see that there is a pair of integers which are relatively prime.

Example 89. *Given any* 5 *distinct real numbers. Prove that there are two of them say* x *and* y *such that* $0<(x-y)/(1+xy)$.

Solution Here, we are using the property of tangent functions of trigonometry.

Given a real number a, we can find a unique real number A lying between $-\pi/2$ and $\pi/2$ *i.e.*, lying in the real interval $(-\pi/2, \pi/2)$ such that $\tan A = a$ as the tangent function in the open interval $(-\pi/2, \pi/2)$ is continuous and strictly increasing and covers R completely. Therefore, corresponding to the five given real numbers $a_i (i = 1, 2, \dots, 5)$, we can find 5 distinct real numbers $a_i (i = 1, 2, .., 5)$, lying between $-\pi/2$ and $\pi/2$ such that $\tan A_i = a_i$.

Divide the open interval $(-\pi/2, \pi/2)$ into 4 equal intervals, each of length $\pi/4$. Now, by Pigeonhole principle, at least 2 of the A_i's must lie in one of the 4 intervals. Suppose A_k and A_l with $A_k > A_l$ lie in the same interval, then

$$0 < A_k - A_l < \pi/4$$

so $$\tan 0 < \tan (A_k - A_l) < \tan \pi/4$$

[It is so because tan functions increases in the interval $(-\pi/2, \pi/2)$]

i.e., $$0 < \frac{\tan A_k - \tan A_l}{1 + \tan A_k \tan A_l} < 1$$

$$0 < \frac{a_k - a_l}{1 + a_k a_l} < 1$$

There are 2 real numbers $x = a_k, y = a_l$ such that

$$0 < \frac{x - y}{1 + xy}$$

Example 90. *Show that given* 12 *integers, there exists two of them whose difference is divisible by* 11.

Solution Possible remainders, when any integer is divided by 11 are 0, 1, 2, ..., 10. Treat these remainders as 'holes' and the 12 integers as pigeons. By PHP, two of them should lie in the same hole, *i.e.*, should leave the same remainder, when divided by 11. Hence, their difference is divisible by 11.

Example 91. *Five points are marked at random in a square plate of length* 2 *units. Show that a pair of them are apart by not more than* $\sqrt{2}$ *units.*

Solution By taking the mid-points of the edges and joining the points on opposite edge, we get a grid of 4 units squares. By PHP two of the points lie in one of these grids and the maximum distance between these points is $\sqrt{2}$ units (the length of the diagonal of the unit square).

Example 92. *Prove that any set of* 55 *integers* $\{x_1, \dots, x_{55}\}$ *such that* $1 \le x_1 \le x_2 < \dots < x_{55} = 100$, *there will be some two that differ by* 9, *some two that differ by* 10, *a pair that differ by* 12 *and a pair that differ by* 13.

Solution Consider the sets

$$A_1 = \{1, 10\}, A_2 = \{2, 11\}, \dots, A_9 = \{9, 18\}$$
$$A_{10} = \{19, 28\}, A_{11} = \{20, 29\}, \dots, A_{18} = \{27, 36\},$$
$$A_{19} = \{37, 46\}, A_{20} = \{38, 47\}, \dots,$$
$$A_{27} = \{45, 54\}\ A_{28} = \{55, 64\}, A_{29} = \{56, 65\}, \dots, A_{36} = \{63, 72\},$$
$$A_{37} = \{73, 82\}, A_{38} = \{74, 83\}, \dots, A_{45} = \{81, 90\},$$
$$A_{46} = \{91, 100\}$$
$$A_{47} = \{92\}, A_{48} = \{93\}, A_{49} = \{94\}$$
$$A_{50} = \{95\}, A_{51} = \{96\}, A_{52} = \{97\}$$
$$A_{53} = \{98\}, A_{54} = \{99\}$$

When 55 numbers are chosen from $\{1, 2, \ldots, 100\}$ some two of them come from the same set. Two such numbers differ by 9. Thus, any set of 55 numbers will have a pair differing by 9. Proof is similar for 10, 12 and 13.

Example 93. *A chess player plays at least one game of chess a day, but in order to avoid overstrain he plays no more than 12 games a week. Prove that, in period of 77 days, there must be a period of several consecutive days during which he plays exactly 20 games.*

Solution Let us suppose that the chess player plays a_1, games on Monday, a_2 game during Monday and Tuesday, a_3 games during the first three days etc...., and finally a_{77} games during 77 days.

Consider $a_1, a_2 \ldots a_{77}, a_1 + 20, a_2 + 20, \ldots, a_{77} + 20$. This is a sequence of 154 numbers, each of which does not exceed $11 \times 12 + 20 = 152$ (the 77 days, period has 11 weeks and in each week at most 12 games are played). Consequently, at least two of these numbers are equal to each other. Since $a_i \neq a_j$ for i, j, we must have k, l such that $a_k = a_l + 20$. Thus, $a_k - a_l = 20$ and it follows that during $k - l$ days from $(l + 1)$th day to the kth day inclusive, the player plays exactly 20 games.

Example 94. *Show that there exist two powers of 1999 whose difference is divisible by 1998.*

Solution Any number, when divided by 1998 leaves one of 0, 1, ... ,1997 as remainder. If we consider $1999^1, \ldots, 1999^{1999}$ the first 1999 powers of 1999, they all cannot leave distinct remainders when divided by 1998. Two of them must thus leave the same remainder and their difference is a multiple of 1998.

Example 95. *Let 'a' be any irrational number. Show that there exist infinitely many rational numbers $r = p / q$ much that $|a - r| < \frac{1}{q^2}$.*

Solution We can assume that $a > 0$ let Q be a positive integer and consider fractional parts of $0, a, 2a, \ldots Qa$ of the first $(Q + 1)$ multiples of a. By PHP, two of these must fall into one of Q interval $\left[0, \frac{1}{Q}\right], \left[\frac{1}{Q}, \frac{2}{Q}\right], \ldots, \left[\frac{Q-1}{Q}, 1\right]$, whereas usual $[A, B] = \{x : A \leq x < B\}$. In other words, there exist integers q_1 and q_2 such that (where $\{x\}$ denotes the fractional part of x)

$$\{q_1 a\}, \{q_2 a\} \in \left[\frac{S}{Q}, \frac{S+1}{Q}\right], q_1 \neq q_2$$

since $$\{q_1 a\} = q_1 a - [q_1 a],\ \{q_2 a\} = q_2 a - [q_2 a],$$

we get, $$|qa - b| = |\{q_1, a\} - \{q_2, a\}| < \frac{1}{Q}$$

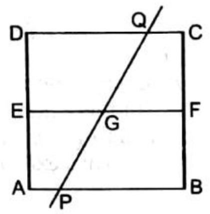
where $$q = q_1 - q_2, p = [q, a] - [q_2 - a]$$

since $0 < |q| \leq Q$, this gives $\left|q - \frac{p}{q}\right| < \frac{1}{Q|q|} \leq \frac{1}{q^2}$

We have to show that the number of such pair (p, q) is infinite. Assume on the contrary that only for a finite number of $r_i = \frac{p_i}{q_i}, i = 1, 2, \ldots, N$ we have $|q - r_i| < \frac{1}{q_i}$. Since, none of the differences $a - r_i$ exactly 0, there exist an integer Q such that $|a - r_i| > \frac{1}{Q}$ for all $i = 1, 2, \ldots, N$ apply our starting arguments to this Q and produce $r = p / q$, such that $|a - r| < \frac{1}{qQ} \leq \frac{1}{Q}$. Hence, r cannot be one of r_i, $i = 1, 2, \ldots, N$ on other hand $|a - r| < \frac{1}{q^2}$ contradicting this assumption that the fractions r_i, $i = 1, 2, \ldots, N$ where all the fraction with this property.

Example 96. *Each of the given 9 lines cuts a given square into two quadrilaterals, whose areas are in the ratio 2 : 3. Prove that at least there of these lines pass through the same point.*

Solution If a line cuts a square AB (1) into two parts, it must intersect two of its sides (internally); the two parts are quadrilaterals only if it intersect a pair of opposite sides. At least five of the nine lines must meet the pair AB, CD or else, the pair AD, BC. Suppose that five meet AB, CD, and let PQ be one of them, let E, F be the mid-points of AD, BC and PQ meets EF at G. The quadrilaterals $APQD$, $PBCQ$ formed by PQ are trapeziums. So,



$$\frac{\frac{1}{2}AD(AP+DQ)}{\frac{1}{2}BC(PB+QC)}=\frac{2}{3} \text{ or } \frac{3}{3}$$

but $$\frac{1}{2}(AP+DQ)=EG \text{ and } \frac{1}{2}(PB+QC)=GF$$

Hence, $$\frac{EG}{GF}=\frac{2}{3} \text{ or } \frac{3}{2}$$

If G_1 and G_2 are the points dividing EF in the ratio 2 : 3 and 3 : 2, then PQ contain G_1 or G_2 of the five lines meeting AB, CD, at least there must pass through the same point G_1 or G_2.

Example 97 *Given a set of 25 points in the plane such that among any there of them there exist 'a' pair at the distance less than 1. Prove that there exist a circle of radius 1 that contain at least 13 of the given points.*

Solution Let $A_1, A_2, \dots, A_{25}$ be the 25 points. If the circle with centre A_1 and radius 1 contain 12 more points other than A_1, we are done. So, suppose that there are 13 points $A_2, A_3, \dots, A_{14}$ outside the circle. Then $A_1A_2 > 1$ and $A_1A_i > 1$ for $3 \le i \le 14$. So, the given condition, applied to the three points $A_1, A_2, \dots, A_i$ implies that $A_2A_i \le 1$ for $3 \le i \le 14$. Hence, the circle with centre A_2 and the radius 1 contain 13 points $A_i, 2 \le i \le 14$.

Example 98. *Given 6 points inside a circle of radius 1, some two of the 6 units are with in 1 point of each other.*

Solution Let O be the centre of the circle, P and Q be two of the six points inside the circle. If P or Q is the same as O, then $PQ < 1$. So let O, P, Q be distinct, we can choose P, Q such that $\angle POQ \le 60°$, let $\angle POQ = \theta$, then by cosine formula,

$$PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ\cos\theta$$
$$\le OP^2 + OQ^2 - 2OP \cdot OQ(1/2)$$

If $OQ \le OP$, then we get

$$PQ^2 \le OP^2 + OQ^2 - OP \cdot OQ \le OP^2 < 1$$

Similarly, we can prove $OP \le OQ$.

Example 99. *Given* 14 *or more integer from* $\{1, 2, \ldots, 28\}$ *there exist four of the given integers which can be split into two groups of two numbers each with the same sum.*

Solution Let $a_1, a_2, \ldots, a_{14}$ be 14 numbers in $\{1, 2, \ldots, 28\}$, clearly, $1 + 2 \le a_i + a_j \le 27 + 28$ for $i \ne j$. There are $\binom{14}{2} = 91$ pairs (a_i, a_j) and there are only 53 possible values for the sum $a_i + a_j$. So, there exist two pairs (a_i, a_j) and (a_r, a_s) such that $a_i + a_j = a_r + a_s$. If $a_r = a_i$, then $a_s = a_j$ and the pairs are equal. So, $a_r \ne a_i$ similarly, $a_r \ne a_j, a_s \ne a_i$ and $a_s \ne a_j$. The four numbers a_i, a_j, a_r, a_s have the desired property.

Example 100. *A person takes at least one aspirin a day for* 30 *days. If he takes* 45 *aspirin all together. Show that in some sequence of consecutive days he takes exactly* 14 *aspirins.*

Solution Let a_i be the number of aspirin taken in the ith day, and let $S_n = a_1 + \ldots + a_n$, where $S_1 = a_1$. Then, $1 \le S_n \le 45$ for $n = 1, 2, \ldots, 30$. Consider the numbers $S_n, S_n + 14$ for $1 \le n < 30$. These 60 numbers must belong to $\{1, 2, \ldots, 59\}$. So some two of them are equal. Since $a_i \ge 1$ for all i, we see that $S_n \ne S_m$ for $n \ne m$. It follows that $S_n = S_m + 14$ for some $n > m$. Hence,

$$a_{m+1} + \ldots + a_n = S_n - S_m = 14.$$

Example 101. *Prove that we can choose a subset of a set of ten given integers, such that, their sum is divisible by* 10.

Solution Let $a_1, a_2, \ldots, a_{10}$ be 10 integers. Let $S_n = a_1 + \ldots + a_n$. The 10 numbers $S_1 \ldots S_{10}$ are either in congruent modulo 10, or else, two of them congruent modulo 10. In the first case, some S_n is congruent to 0, and hence $a_1 + \ldots + a_n$ is divisible by 10. In the other case, let $S_i \equiv S_j$ (mod 10), when $i < j$. Then, $a_{i+1} + \ldots + a_j = S_j - S_i$ is divisible by 10.

Example 102. *In a group of 7 people, the sum of the age of the members is* 332 *years. Prove that these members can be chosen, so that the sum of their ages is not less than* 142 *years.*

Solution Let $a_1, a_2, \ldots, a_7$ be the ages. Then

$$a_1 + \ldots + a_7 = 332$$

Suppose that $a_i + a_j + a_k < 142$ for distinct i, j, k there are $\binom{7}{2}$ such inequalities. Add them. Each a_i is repeated $\binom{6}{2}$ times. Thus, we get

$$\binom{6}{2}(a_1 + a_2 + \ldots + a_7) < \binom{7}{3} 142,$$

that is, $15.332 < 35.142$. Since, this is wrong, we must have $a_i + a_j + a_k \ge 142$ for some distinct i, j, k.

Example 103. *Suppose a coin is flipped until* 2 *heads appear (2 heads need not be consecutive) and then the experiment stops. Find a recurrence relation for the number* a_n *of experiments that end on the nth flip or sooner.*

Solution Clearly, $a_1 = 0, a_2 = 1$. Also $a_3 = 3$. since the possibilities are *HH, THH, HTH.*

Let $n \ge 2$. Each of the a_n experiments starts with T or H. Those which starts with T are obtained by adding T at the beginning of the a_{n-1} experiments with at most $n - 1$ flips.

Those which start with H are $n - 1$ in number as these are of the form *HTT....TH*, where there are r, T_3 and $0 \le r \le n - 2$.

Hence, $$a_n = a_{n-1} + n - 1$$

Example 104. *Find a recurrence relation for the number a_n of n digit quaternary sequences (i.e., sequence with terms 0, 1, 2, or 3) with at least one 1, and the first 1 occurring before the first 0 (possibly no 0's).*

Solution $a_1 = 1$ since 1 is only possibility.

Also for $n = 2$, the possibilities are 10, 11, 12, 13, 21, 31 so that $a_2 = 6$.

Let $n \geq 2$

Let $x = x_1x_2 \ldots x_n$ be anyone of the a_n sequences of required type, then x cannot start with 0 since the form 0...1... is not allowed while 1 must appear in x.

So, $x_1 = 1, 2$ or 3. If $x_1 = 2$ or 3, then sequences $x_2 \ldots x_n$ can be anyone of the a_{n-1} required sequences. Thus, $2a_{n-1}n$ sequences start with 2 or 3. If $x_1 = 1$, then each of $x_2, x_3 \ldots$ can be 0, 1, 2, 3,. Hence, there are $4^{n-1}n$ sequences starting with 1. Thus, $a_n = 2a_{n-1} + 4^{n-1}$

Example 105. *Find a recurrence relation for the number $a_n : r$ of ways of selecting r integers from the ordered set $x = \{1, 2, \ldots, n\}$, so that consecutive integers are not selected.*

Solution For $n = r = 1$, the only possible selection is {1}.

for $n = 2, r = 1$ the only possible selections are {1} and {2}.

Hence,
$$a_{1,1} = 1; a_{2,1} = 2$$

Let $s = (i_1, i_2, \ldots, i_r)$ be a selection from x of the required form. Assume that
$$1 \leq i_1 \leq i_2 < \ldots < i_r \leq n$$

Then, we have 2 mutually exclusive cases.

(i) $i_r \neq n$ (ii) $i_r = n$

In case (i), s is a selection of required form and contains r numbers from $x - \{n\}$.

So, there are a_{n-1}, r selections of this type. In case (ii) the element i_r in s has already been chosen to be n.

So, to obtain s, we only have to choose the selection $t = (i_1, \ldots, i_{r-1})$ of required type from $x - \{n-1, n\}$.

Hence, there are a_{n-2}, r_{n-1} selections of this type. Therefore, we get.
$$a_{n,r} = a_{n-1}, r + a_{n-2}, r-1$$

Example 106. *For every real number $x_1^{\bullet}$, construct the sequence $x_1, x_2, \ldots$ by setting $x_{n+1} = x_n\left(x_n + \frac{1}{n}\right)$ for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n.*

Solution Let $P_1(x) = x$
$$P_{n+1}(n) = P_n(n)\left[P_n(x) + \frac{1}{2}\right] \text{for } n = 1, 2 \qquad \ldots\text{(i)}$$

from this recursive definition, we see inductively that

(i) P_n is a polynomial of degree 2^{n-1}

(ii) P_n has positive coefficients is therefore an increasing convex function for $x \geq 0$.

(iii) $P_n(0) = 0, P_n(1) \geq 1$

(iv) $P_n(x_1) = x_n$

Since the condition $x_{n+1} > x_n$ is equivalent to $x_n > 1 - \frac{1}{n}$, we can reformulate the problem as follows show that there is unique positive real number t such that.
$$1 - \frac{1}{n} < P_n(t) < 1 \quad \text{for every } n.$$

Since P_n is continuous and increases from 0 to a value of ≥ 1 for $0 \leq x \leq 1$, there is unique values a_n and b_n such that

$$a_n < b_n,\ P_n(a_n) = 1 - \frac{1}{n},\ P_n(b_n) = 1 \qquad \text{...(ii)}$$

By definition (i),
$$P_{n+1}(a_n) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n} + \frac{1}{n}\right) = 1 - \frac{1}{n}$$

$$P_{n+1}(a_{n-1}) = 1 - \frac{1}{n+1}$$

We see that
$$a_n < a_{n+1} \qquad \text{...(iii)}$$

Also since
$$P_{n+1}(b_n) = 1 + \frac{1}{n} \text{ and } P_{n+1}(b_{n+1}) = 1$$

$$b_n > b_{n+1} \qquad \text{...(iv)}$$

Since P_n is convex, the graph of $P_n(x)$ lies below the chord $y = \frac{1}{b_n} x$ for $0 \leq x \leq b_n$

In particular
$$P_n(a_n) = 1 - \frac{1}{n} \leq \frac{a_n}{b_n}$$

from this and the fact that
$$b_n \leq 1,$$

we find that
$$b_n - \frac{b_n}{n} \leq a_n$$

$$b_n - a_n \leq \frac{b_n}{n} \leq \frac{1}{n} \text{ for all } n.$$

Thus, we have 2 infinite bounded sequences $\{a_n\}$, $\{b_n\}$ the first is increasing the second decreasing $a_n < b_n$ and the difference between their nth members approaches 0 as n increases. We conclude that there is a unique common value t that they approach

$$a_n < t < b_n \ \forall n$$

Number uniquely satisfies

$$1 - \frac{1}{n} < P_n(t) < 1 \ \forall n$$

Example 107. *Let $P_1, P_2, \ldots, P_n$ be distinct 2 element of the set of elements $\{a_1, a_2, \ldots, a_n\}$ such that if $P_i \cap P_j \neq \phi$, then $\{a_i, a_j\}$ is one of the P's. Prove that each of a's appear in exactly two of the P's.*

Solution Denote by m_j the number of P's containing a_j, $j = 1, 2, \ldots, n$, since each pair consists of two distinct elements.

$$m_1 + m_2 + \ldots + m_n = 2n \qquad \text{...(i)}$$

The number of pairs $\{P_i, P_j\}$ both containing a_k is $\binom{m_k}{2}$. Now consider the one to one mapping $(P_i, P_j) \leftrightarrow (a_i, a_j)$. By hypothesis (a_i, a_j) is a pair p_k, if and only if $P_i \cap P_j \neq \phi$. Since there are only n pairs $P_1, \ldots, P_n$, it follows that

$$\Sigma \binom{m_k}{2} = \frac{1}{2}(\Sigma m_k^2 - \Sigma m_k) \leq n \qquad \text{...(ii)}$$

By the Power mean or Cauch inequality.

$$(\Sigma m_k)^2 \leq n \Sigma m_k^2$$

Using Eqs. (i) and (ii) we conclude that

$$n \leq \frac{1}{2}(\Sigma m_k^2 - \Sigma m_k) \leq n \qquad \text{...(iii)}$$

Thus, equality must hold in the power mean inequality. But this occurs if and only if all the m_k are equal.
By Eq. (i),
$$m_1 = m_2 = \ldots = m_n = 2$$

Example 108. *Three distinct vertices are chosen at random from the vertices of a given regular polygon of $(2n+1)$ sides. If all such choices are equally likely. What is the probability that the center of the given polygon lies in the interior of the Δ determined by 3 chosen random points?*

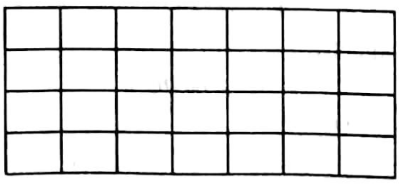
Solution Let the vertices in order be $V_0, V_1, \ldots, V_{2n}$. We can assume that the first vertex chosen is fixed at V_0. Then, the number of ways of picking 2 more vertices is $\binom{2n}{2}$. Now, if one of the remaining 2 random vertices is V_k, $1 \le k \le n$, there will k Δs possible that contain the centre. If say $k = 3$, then the only possible triangles with vertices V_0, V_3 which. Contain the centre are $V_0V_3V_{n+1}$, $V_0V_3V_{n+2}$ and $V_0V_3V_{n+3}$. Thus, the number of favourable cases is
$$\sum_{k=1}^{n} k = n(n+1)/2$$

Finally the desired probability is
$$P = \frac{n(n+1)}{2\binom{2n}{2}} = \frac{n+1}{2(2n-1)}$$

Example 109. *There are n people at a party. Prove that, there are 2 people such that of the remaining $n-2$ people there are at least $(n/2)-1$ of them, each of whom knows both or else knows neither of two. Assume that knowing is a symmetrical relation $[x]$ denotes the greatest integer less than or equal x.*

Solution Given 2 people at the party, we describe a third person as "mixed" w.r.t. that pair, if that person knows exactly one of the 2. Thus a person, who know exactly k people at the party is mixed with respect to $k(n-1-k)$ pairs. By AM-GM inequality, each person is mixed w.r.t. at most $(n-1)^2/4$ pairs. Thus, there are at most
$$\frac{n(n-1)^2}{4} = \frac{n-1}{2}\binom{n}{2}$$
person mixed pair combinations and so for at least one of the $\binom{n}{2}$ pairs at most $[(n-1)/2]$ of remaining people are mixed. For this pair there are at least
$$n-2-\left[\frac{n-1}{2}\right] = \frac{n}{2} - 1$$
other who knows either both or neither of two.

Example 110. (i) *Suppose that each square of a 4×7 chessboard is coloured either black or white. Prove that, with any such colouring, the, board must contain a rectangle, whose 4 distinct unit corner squares are all of same colour.*
(ii) *Exhibit a black white colouring of a 4×6 board in which 4 corner squares of every rectangle are not all of same colour.*



Solution (i) The stated result is even valid for a 3×7 board. We call any column (consisting of 3 unit squares) black, if it has more black unit squares than white ones otherwise we call it white. Since there are 7 colours, at least 4 of theam are of the same kind, say black. We now show that there is a rectangle, whose vertices are these 4 black columns and whose corner squares are all black. We can even assume that each of these 4 black column has one white unit square, since the white square can be in one of only 3 positions it follows that 2 of these 4 columns are identically coloured giving desired result.

(ii) For the 4×6 chessboard there are $\binom{4}{2} = 6$ different arrangements of two *B's* and two *W's* in a column. One such coluring is given in the figure below, it contains no rectangle with identically colourd corner squares

B	B	B	W	W	W		
B	W	W	W	B	B		
W	B	W	B	W	B		
W	W	B	B	B	W		P

Example 111. 9 *mathematicians meet at an international conference and discover that among any 3 of them at least two speak a common language. If each of the mathematician can speak at most* 3 *language, prove that there are at least* 3 *of the mathematician who can speak the same language.*

Solution We assume that at most 2 mathematician speak a common language. Each mathematician can speak to at most 3 others, one for each language he or she knows. Suppose mathematician M_1 can only speak with M_2, M_3 and M_4. Now M_5 can speak with at most 3 of M_2, M_3 and M_4 or at most 3 of M_6, M_7, M_8. This leaves one of the last 4 who cannot speak with M_1 or M_5 giving desired contradiction.

Example 112. *In a party with* 1982 *persons, among any group of* 4, *there is at last one person, who knows each of other* 3. *What is the minimum number of people in the party, who know everyone else?*

Solution

Case I We assume 'Knowing' is not a symmetrical relation *i.e., A* may know *B* but *B* does not know *A*. For this case no one need know everyone else. Just consider all the people arranged in a circle such that each person knows everyone else except the person next to him clockwise.

Case II We assume 'Knowing' is a symmetrical relation.

Let $\{P_1, P_2\}$ be a pair who do not know each other. Then if $\{P_3, P_4\}$ is a pair disjoint from the first pair, P_3 and P_4 must know each other, since one of P_1, P_2, P_3, P_4 knows the other 3 by hypothesis. So if there is a third person P_3 who does not know everyone, it must be P_1 or P_2 he (she) does not know. If there were a fourth person P_4 who did not know everyone, it would again be P_1 or P_2 he (she) did not know, but then $\{P_1, P_2, P_3, P_4\}$ would violate hypothesis. Thus all except at most 3 people must know everyone else.

Example 113. *Find the number of permutations* $(p_1, \ldots, p_6)$ *of* 1, 2, 3, 4, 5, 6 *such that for any* k, $1 \le k \le 5$, $(p_1, p_2, \ldots, p_k)$ *is not a permutation of* $(1, 2, \ldots, k)$ *i.e.,* $p_1 \ne 1$. (p_1, p_2) *is not a permutation of* $(1, 2)$, (p_1, p_2, p_3) *is not a permutation of* $(1, 2, 3)$ *etc.*

Solution For each positive integer k, $1 \le k \le 5$. Let N_k denote the number of permutations $(p_1, \ldots p_6)$ such that $p_1 \ne 1$, (p_1, p_2) is not a permutation $(1, 2)$, ..., $(p_1, \ldots, p_5)$ is not a permutation of $(1, 2, \ldots, 5)$. We are required to find N_5.

Since out of the 6! permutations of $(1, 2, \ldots, 6)$ there are 5! permutations having 1 in the first place. We have to find the number of permutations left after removing from the set of all permutations of $(1, 2, \ldots, 6)$ the one that begins with 1.

Now $N_1 = 6! - 5! = 600$. To get N_2 we now remove those permutations in which (p_1, p_2) is a permutation of (1, 2). Since all permutations of the type $(1, 2, p_3, \dots, p_6)$ have already been removed. We have to further remove permutations of type $(2, 1, p_3, \dots, p_6)$. The number of such permutation being 4!.

We get $$N_2 = 600 - 4! = 576.$$

To get N_3, we have to subtract from N_2 the no. of those permutation $(p_1, \dots, p_6)$ in which $(p_1 p_2 p_3)$ is a permutation of (1, 2, 3) and which have not already been removed. Since all permutations of the form $(1, p_2, \dots, p_6)$ and $(2, 1, p_3, \dots, p_6)$ have already been removed, we have to count the number of permutations of the form $(2, 3, 1, p_4, \dots, p_6)$, $(3, 1, 2, p_4, p_5, p_6)$ and $(3, 2, 1, p_4, p_5, p_6)$. The number of all such permutation is $3 \times (3!)$ *i.e.*, 18

$$\therefore \qquad N_3 = 576 - 18 = 558$$

To get N_4, we have to remove those permutations $(p_1, p_2, \dots, p_6)$ in which $(p_1, \dots, p_4)$ is a permutation of (1, 2, 3, 4) and which have not already been removed *i.e.*, permutations of the type $(1, p_2, \dots, p_6)$, $(2, 1, p_3, \dots, p_6)$, $(2, 3, 1, p_4, \dots, p_6)$ $(3, 1, 2, p_4, \dots, p_6)$ and $(3, 2, 1, p_4, \dots, p_6)$. This means that we have to remove all permutations of type $(4, p_2, \dots, p_6)$, $(2, 4, p_3, \dots, p_6)$ $(3, 4, p_3, \dots, p_6)$ $(2, 3, 4, p_4, \dots, p_6)$, $(3, 2, 4, p_4, \dots, p_6)$ in which the first 4 elements are a permutation of (1, ..., 4). There are $(3! + 2! + 2! + 1 + 1) \times 2 = 24$ permutations in all to be removed. $N_4 = 558 - 24 = 534$. To get N_5, we have to further remove permutations $(p_1, \dots, p_6)$ in which $(p_1, \dots, p_5)$ is a permutation of (1,...,5) and which have not been removed. There are 24(4!) such permutations for which $p_1 = 5$, 18 with $p_2 = 5$, 16 with $p_3 = 5$ and 13 with $p_4 = 5$ so that in all there are 71 such permutations.

Thus, $N_5 = 534 - 71 = 463$, which is desired number of permutations.

Example 114. *In a list of* 200 *number, everyone (except the end ones) is equal to the sum of the two adjacent number in the list. The sum of all the numbers is equal to the sum of the first* 100 *of them. Find that sum if the* 35*th number in the list in* 6.

Solution $\because$

$$a_2 = a_1 + a_3 \quad \therefore \; a_3 = a_2 - a_1$$
$$a_3 = a_2 + a_4 \quad \therefore \; a_4 = a_3 - a_2 = -a_1$$
$$a_4 = a_3 + a_5 \quad \therefore \; a_5 = a_4 - a_3 = -a_2$$
$$a_5 = a_3 + a_6 \quad \therefore \; a_6 = a_5 - a_4 = -a_3 = a_1 - a_2$$

Also, $$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0$$

Again, since $$a_n = a_{n-1} + a_{n+1}$$
$$a_{n+1} = a_n - a_{n-1} = -a_{n-2} \;\forall n > 2.$$

Also, $$a_{n+1} = -a_{n-2} = (-1)^2 a_{n-5} = a_{n-5} \;\forall n > 5.$$

This shows that $a_7 = a_1, a_8 = a_2, a_9 = a_3, a_{10} = a_4$

$a_{11} = a_5, a_{12} = a_6$ *i.e.*, in the list the first 6 elements are repeated so that the list is

$$a_1, a_2, a_2 - a_1, -a_1, -a_2, a_1 - a_2, a_1, a_2 \dots$$

Clearly $a_{35} = a_5 = -a_2$, since $a_{35} = -6$.

Therefore it follows that $a_2 = 6$.

Also, the sum of the first hundred elements (s_{100}, say) is equal to the sum of all the elements in the list.

$$\therefore \qquad a_{101} + a_{102} + \dots + a_{200} = 0.$$

Since, the sum of the first 6 elements is zero.

$$\therefore \qquad a_{k+1} + a_{k+2} + \dots + a_{k+6} = 0 \quad k = 0, 1, 2, \dots$$

This shows that $$a_{103} + a_{104} + \dots + a_{198} = 0.$$

Also, since $$a_{101} = a_5, a_{102} = a_6, a_{199} = a, a_{200} = a_2$$

$$\therefore \qquad a_1 + a_2 + a_5 + a_6 = 0$$

So that $$a_1 + a_2 + (-a_2) + (a_1 - a_2) = 0$$

i.e., $$2a_1 = 6$$
which gives $a_1 = 3$.

$$\begin{aligned} S_{200} &= S_{100} = a_1 + \ldots + a_{100} \\ &= a_{97} + a_{98} + a_{99} + a_{100} \\ &= a_1 + a_2 + a_3 + a_4 \\ &= 2a_2 - a_1 = 9 \end{aligned}$$

Sum of all the numbers in the list is 9.

Example 115. *A $2 \times 2 \times 12$ hole in a wall is to be filled with twenty four $1 \times 2 \times 2$ bricks. In how many different ways can this be done, if the bricks are in distinguishable?*

Solution Let T_n be the number of ways of filling up a $2 \times 2 \times n$ holes with $1 \times 1 \times 2$ bricks. We obtain a recursion relation for T_n by expressing in terms of the numbers of ways of filling smaller holes. Assume that, the long axis of hole is vertical. The other two edge directions will be called to the right and forward.

First, we count the number of ways of filling up the hole, if the bottom layer consist of two bricks lying on their sides. They can lie in two ways, with their long axes in the left - right or backward - forward position the number of these packings is $2T_{n-1}$.

Next consider the packing, in which some of the bricks of the bottom layer stick out, but the bricks in layer 1 and 2 fill this sub-box with nothing sticking out. There are 5 such packings; one in which all four bricks are vertical, one in which the two bricks in front are vertical and the two in the back are horizontal and the three more packings of this sort in which the two standing bricks are on the left in the back, and on the right, this gives $5T_{n-2}$ more packings.

Next let K be an integer, $2 < K \le n$. We count the number of ways to fill one hole in such a manner that the lowest bottom sub - hole which is completely filled with nothing sticking out has K layers. The bottom layer of such a packing must consist of two vertical and one horizontal bricks. There is no further choice until we get above the Kth-layer: the two empty spaces in each layer must be filled with the two, vertical bricks until we reach layer K, when the two spaces must be filled with a horizontal brick thus for all, the values of K in the given range, there are 4 ways of filling the K lowest layers. Since for every packing there is one and only one value of K which satisfies the above conditions we get

$$T_n = 2T_{n-1} + 5T_{n-2} + 4T_{n-2} + \ldots + 4T_0 \qquad \ldots\text{(i)}$$

Here, we set

$$T_0 = 1 \text{ (Also)}$$

$$T_{n-1} = 2T_{n-2} + 5T_{n-3} + 4T_{n-4} + \ldots + 4T_0 \qquad \ldots\text{(ii)}$$

Combining Eqs. (i) and (ii), we get

$$T_n = 3T_{n-1} + 3T_{n-2} - T_{n-3} \qquad \ldots\text{(iii)}$$

The characteristic equation of the latter recurrence relation is $x^3 - 3x^2 - 3x + 1 = 0$

its roots are $-1, 2 + \sqrt{3}, 2 - \sqrt{3}$. Thus, T_n has the form

$$T_n = C_1(-1)^n + C_2(2 + \sqrt{3})^n + C_3(2 - \sqrt{3})^n.$$

Since the initial values are $T_0 = 1$, $T_1 = 2$ and

$T_2 = 9$

$$T_n = \frac{(-1)^n}{3} + \frac{(2 + \sqrt{3})^{n+1}}{6} + \frac{(2 - \sqrt{3})^{n+1}}{6}$$

In particular,

$$T_{12} = 1/3 + (2 + \sqrt{3})^{13} + (2 - \sqrt{3})^{13}$$

the nearest integer to $(2 + \sqrt{3})^{13} = 4,541,161$

Example 116. *29th Feb of year 2000 will fall on a Tuesday show that after this date 29th Feb will fall on Tuesdays thrice in the whole next century. What are the 3 years when this will happen?*

Solution $365 \equiv 1 \pmod 7$.

Therefore, 28th Feb (the last day Feb) of 2001 will be a Wednesday (the day next to Tuesday). Let us agree to express this by saying that there is an excess of one day in any ordinary year. With this terminology. There will be an excess of two days in a leap year the next leap year after year 2000 will be year 2004, there will be $1+1+1+2$ *i.e.*, 5 excess days upto 29th Feb 2004.

The day of the week of 29 th February will be Tuesday, when the number of excess day is an exact multiple of 7. Therefore our problem is to find those positive integer K for which the number of excess days in $4K$ years after the year 2000 is an exact multiple of 7 and which $4K < 100$. The number of excess days in $4K$ years is $5K$. This is a multiple of 7 when $K = 7, 14, 21, \ldots$ *i.e.*, in $4 \times 7, 4 \times 14, 4 \times 21, \ldots$ years, after the year 2000. Since, only 3 of these numbers are less then 2000, therefore in 21st century, there are only 3 years namely 2028 2056 and 2084 in which 29th Feb is a Tuesday.

Example 117. *The number* 3 *can be written as a sum of positive integers in* 4 *ways viz.,* $3, 2+1, 1, 1+2, 1+1+1$. *Show that any positive integer n can be so expressed in* 2^{n-1} *ways.*

Solution Consider the sum

$$1+1+1+1+\ldots+1$$

There being n terms in all. We can break this sum into one or more parts (n parts at the most) by either putting or not putting parenthesis after the $n-1$ '+' signs. This can be done in 2^{n-1} ways.

Example 118. *The* 64 *squares of an* 8×8 *chess board are filled with positive integers in such a ways that each integer is the average of the integers on the neighbouring squares. Show that all the* 64 *integer entries are in fact equal.*

Solution Choose the square (call it A) which is filled with the smallest of all the positive integers (call it K, say) filled in the 64 squares. Consider all its neighbours. If any one of them is filled with a positive integer greater than K, then the average of all the integers in the squares occupying the neighbouring positions will be greater than K. This is a contradiction. Hence, all the neighbouring squares are filled with the same integer K.

Let us apply this procedure to all the squares in the row in which A lies, first beginning with the squares immediately preceding A (if A is not in first column) and reaching the first column and then starting with the square immediately to the right of A (if A is not in last column) reaching the last column. By this process, we find that all the squares in the row to which A belongs and the row immediately above and immediately below are filled with K. We now apply this procedure to all the elements in the row above A (if there be one such row) and then to the row below A (if there be one such row). We find that all the rows can be exhausted in a finite number of steps and that all the 64 squares are first with the same positive integer K.

Hence, all 64 integers entries are equal.

Let us Practice

Let us Practice

Level 1

1. How many numbers of 4 digits can be formed with the digits 1,2,3,4,5, no digits being repeated.
2. How many numbers each lying between 100 and 1000 can be formed with the digits 2,3,4,0, 8,9, no digit being repeated ?
3. How many numbers, of 9 digit numbers, which have all different digits ?
4. Find the sum of all the 4 digits numbers that can be formed with the digits 0,2,3 and 5.
5. There are 20 books, of which 4 are single volume and the other are books of 8, 5 and 3 volumes respectively. In how many ways can all these books be arranged on a shelf, so that volumes of the same book are not separated?
6. A library has 5 copies of one book, 4 copies each of 2 books, 6 copies each of 3 books and single copies each of 8 books. In how many ways, can all the books be arranged, so that copies of the same books are always together?
7. 6 papers are set in an examination, 2 of them in Mathematics. In how many different orders can the papers be given, if two Mathematics papers are not successive ?
8. In how many ways 18 white and 19 black be arranged so that all the 18 white balls are not be together. It is given that balls of same colour are identical.
9. In a class of 10 students, there are 3 girls, in how many ways can they be arranged in a row such that no two of the three girls are consecutive ?
10. In how many ways, the letter of the word 'DIRECTON' be arranged, so that their vowels are never together ?
11. How many words can be formed with the letter of the word 'VICE-CHANCELLOR' so that the vowels are together ?
12. Find the number of different permutations of the letters of the word 'BANANA'.
13. How many number of arrangements of the letters of the word 'BENEVOLENT' How many of them end in L ?
14. The letters of the word 'OUGHT' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'TOUGH' in this dictionary.
15. In how many ways, 5 boys and 4 girls can be seated at a round table in the following case
    (i) when there is no restriction.
    (ii) all the 4 girls sit together.
    (iii) all the 4 girls don't sit together.
    (iv) no two girsl sit together.
16. A man has 8 children to take them to a zoo. He takes three of them at a time to the zoo as often as he can without taking the same 3 children together more than once. How many times will he have to go to zoo? How many times a particular child will go ?
17. Out of 7 men and 4 ladies a committee of 5 is to be found . In how many ways can this be done so as to include at least 3 ladies ?
18. In an examination the question paper contain three different sections A, B and C containing 4, 5 and 6 questions respectively. In how many ways, a candidate can make a selection of 7 questions selecting at' least two question from each selection ?
19. From 8 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many way can this be done so as to include at least one lady ?
20. In an election for 3 seats there are 6 candidates. A voter can not vote for more than 3 candidates . In how many ways can he vote ?
21. In an election, number of candidate exceeds the number to be elected by 2. A man can vote in 56 ways. Find the number of candidates.
22. At an election a voter can vote for any number of candidates not greater than the number to be chosen. There are 10 candidates and 5 members are to be chosen. Find the numbers of ways in which a voter may vote.
23. A bag contain 5 red, 4 green and 3 blue balls of the same are supposed to be distinct (not alike). In how many ways

(i) some balls can be drawn from the bag ?
(ii) some balls containing at least one red and one green ball can be drawn ?

24. Find the number of selection of at least one red ball from 4 red and 3 green balls, if the balls of the same are different.

25. There are 3 books of Mathematics, 4 of Science and 5 of Literature. How many different collections can be made, such that each collection consist
(i) one book of each subject
(ii) at least one book of each subject
(iii) at least one book of Literature

26. All the 7-digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once and not divisible by 5, are arranged in the increasing order. Find the 2000th number in this list. **(RMO 2000)**

27. Find the number of positive integers x which satisfy the condition $\left[\frac{x}{99}\right]=\left[\frac{x}{101}\right]$ (Here, $[z]$ denotes, for any real z, the largest integer not exceeding z; *e.g.*, $[7/4]=1$.) **(RMO 2001)**

28. In n is an integer greater than 7, prove that $\binom{n}{7}-\left[\frac{n}{7}\right]$ is divisible by 7.
[Here, $\binom{n}{7}$ denotes the number of ways of choosing 7 objects from among n objects; also, for any real number x, $[x]$ denotes the greatest integer not exceeding x.] **(RMO 2003)**

29. Find the number of ordered triples (x, y, z) of non-negative integers satisfying the conditions:
(i) $x \le y \le z$; (ii) $x+y+z \le 100$. **(RMO 2003)**

30. Prove that the number of triples (A, B, C) where, A, B, C are subsets of $\{1, 2, \dots, n$ such that $A \cap B \cap C = \phi$, $A \cap B \ne \phi$, $B \cap C \ne 0$ is $7^n - 2 \cdot 6^n + 5^n$. **(RMO 2004)**

31. Determine all triples (a, b, c) of positive integers such that $a \le b \le c$ and
$a+b+c+ab+bc+ca=abc+1$. **(RMO 2005)**

32. Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (*e.g.*, 31545, 34155 are two such numbers.) **(RMO 2005)**

33. How many 6-digit numbers are there such that:
(a) the digits of each number are all from the set $\{1, 2, 3, 4, 5\}$
(b) any digit that appears in the number appears at least twice?
(*e.g.*, 225252 is an admissible number, while 222133 is not.) **(RMO 2007)**

34. Three non-zero real numbers a, b, c are said to be in harmonic progression, if $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$.
Find all three-term harmonic progressions a, b, c of strictly increasing positive integers in which $a=20$ and b divides c. **(RMO 2008)**

35. Find the number of all integer-sided isosceles obtuse angled triangles with perimeter 2008. **(RMO 2008)**

36. Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits 0, 1, 2, 3 occurs at least once in them. **(RMO 2008)**

37. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit. **(RMO 2009)**

38. For each integer $n \ge 1$, define $a_n=\left[\frac{n}{[\sqrt{n}]}\right]$, where $[x]$ denotes the largest integer not exceeding x, for any real number x. Find the number of all n in the set $\{1, 2, 3, \dots, 2010\}$ for which $a_n > a_{n+1}$. **(RMO 2010)**

39. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral, multiple of the reciprocal of the third integer. **(RMO 2010)**

40. Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8. **(RMO 2010)**

Level 2

1. The sum $\frac{1}{1!9!}+\frac{1}{3!7!}+\frac{1}{5!5!}+\frac{1}{7!3!}+\frac{1}{9!1!}$ can be written in the form $\frac{2^a}{b!}$, where a and b are positive integers. Find the ordered pair (a, b). (**Note** The ! marks are "factorial" symbols.)

2. There are 1994 employees in the office. Each of them knows 1600 others of them. Prove that we can find 6 employees, each of them knowing all 5 others.

3. Find the number of permutations $(p_1, p_2, \ldots, p_6)$ of 1, 2,....,6 such that for any k, $1 \le k \le 5$, $(p_1, p_2, \ldots, p_k)$ does not form a permutation of 1, 2,..., k.

   i.e., $p_1 \ne 1$ (p_1, p_2) is not a permutation of 1, 2 etc.

4. For non-negative integers n, r the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$, if $n < r$. Prove the identity $\sum_{d=1}^{\infty} \binom{n-r+1}{d}\binom{r-1}{d-1} = \binom{n}{r}$ for all integers n, r with $1 \le r \le n$.

5. Show that, where $k + n \le m$,
$$\sum_{i=0}^{n} \binom{n}{i}\binom{m}{k+i} = \binom{m+n}{n+k}.$$

6. An "n-society" is a group of n girls and m boys. Show that there exist numbers n_0 and m_0 such that every $n_0 - m_0$ society contains a subgroup of 5 boys and 5 girls in which all of the boys know all of the girls or none of the boys knows none of the girls.

7. (a) Find the three-digit numbers equal to the sums of the factorials of their digits.

   (b) Find all whole numbers equal to the sums of the squares of their digits.

8. Suppose the n^2 numbers 1, 2, 3,..., n^2 are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row (from left to right) and each column (from top to bottom) are in increasing order. Denote by a_{jk} the number in jth row and kth column. Suppose b_j is the maximum possible number of entries that can occur as a_{jj}, $1 \le j \le n$.

   Prove that $b_1 + b_2 + b_3 + \ldots + b_n \le \frac{n}{3}(n^2 - 3n + 5)$.

   (**Example** In the case $n = 3$, the only numbers which can occur as a_{22} are 4, 5 or 6 so that $b_2 = 3$.) **(INMO 2002)**

9. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points? **(INMO 2002)**

10. Find all 7-digit numbers formed by using only the digits 5 and 7 and divisible by both 5 and 7. **(INMO 2003)**

11. In a lottery, tickets are given nine-digit numbers using only the digits 1, 2, 3. They are also coloured red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours. Suppose the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green. Determine, with proof, the colour of the ticket bearing the number 123123123. **(INMO 2003)**

12. Prove that the number of 5-tuples of positive integers (a, b, c, d, e) satisfying the equation $abcde = 5(bcde + acde + abde + abce + abcd)$ is an odd integer. **(INMO 2004)**

13. All possible 6-digit numbers, in each of which the digits occur in non-increasing order (from left to right *e.g.*, 877550) are written as a sequence in increasing order. Find the 2005th number in this sequence. **(INMO 2005)**

14. Some 46 squared are randomly chosen from a 9×9 chess board and are coloured red. Show that there exists a 2×2 block of 4 squares of which at least three are coloured red. **(INMO 2006)**

15. Let $\sigma = (a_1, a_2, a_3, \ldots, a_n)$ be a permutation of (1, 2, 3,...,n). A pair (a_i, a_j) is said to correspond to an inversion of σ, if $i < j$ but $a_i > a_j$. (**Example** In the permutation (2, 4, 5, 3, 1), there are 6 inversions corresponding to the pairs (2, 1), (4, 3), (4, 1), (5, 3), (5, 1), (3, 1). How many permutations of (1, 2, 3,...,n), ($n \ge 3$), have exactly two inversions? **(INMO 2007)**

16. How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set {1, 2, 3, 4} and the six expressions $a_j^2 - a_j a_{j+1} + a_{j+1}^2$ for j = 1, 2, 3, 4, 5, 6 (where a_7 is to be taken as a_1) are all equal to one another? **(INMO 2010)**

17. All the points in the plane are coloured using three colours. Prove that there exists a triangle with vertices having the same colour such that *either* it is isosceles or its angles are in geometric progression. **(INMO 2010)**

18. Suppose five of the nine vertices of a regular nine-sided polygon are arbitrarily chosen. Show that one can select four among these five such that they are the vertices of a trapezium. **(INMO 2011)**

Solutions

Solutions

Level 1

1. Number of digits, $n = 5$

Number of places to be filled, $r = 4$.

Number of numbers of 4 digits out of the 5 digits (1,2,3,4,5) = Number of permutations of 5 things taken 4 at a time.

$$= {}^5P_4 = \frac{5!}{(5-4)!} = \frac{5!}{1!} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$$

$$= 120$$

2. Number of different digits, $n = 6$. Since number lying between 100 and 1000 are of 3 digits.

Total number of numbers of 3 digits formed with the digits 2,3,4,0,8,9, is

$${}^6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120 \quad \text{...(i)}$$

But of these 120 numbers, there are some numbers which begin with (zero) and which are not our purpose.

Now for the numbers beginning with zero number of digits remain to be used is (2,3,4,8,9) $= 5 = n$ and number of places remain to be filled up, $r = 2$.

Hence, number of numbers of 3 digits beginning with 0 (zero)

$$= {}^5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20 \quad \text{... (ii)}$$

Required number of numbers between 100 and 1000 formed with the digits 2,3,4,0,8,9 = number of numbers of 3 digits formed with the digits 2,3,4,0,8,9

$$= 120 - 20 = 100$$

3. Since no digits are given, so that we consider all the digits 0,1,2,3,4,5,6,7,8,9.

Number of digit, $n = 10$

For the numbers of digits, number of places to be filled up, $r = 9$.

Number of numbers of 9 digits formed with the digits 0,1,2,3,4,5,6,7,8,9, is ${}^{10}P_9$.

But in the ${}^{10}P_9$ numbers, there are some numbers which begin with 0.

For the number of numbers beginning with zero number of places to be filled up $= r = 8$
Number of digit remain to be utilised $= n = 9$.
Number of number of 9 digits beginning with $0 = {}^9P_8$.

Number of numbers of 9 digits

$$= {}^{10}P_9 - {}^9P_8 = \frac{10!}{1!} - \frac{9!}{1!}$$

$$= 10 \cdot 9! - 9! = (10-1)9!$$

$$= 9 \cdot 9!$$

4. Keeping 0 at unit (or tens or hundreds) place number of places to be filled up $= r = 3$.
Number of digits remain to be utilised $= n = 3$.
Hence, number of numbers of 4 digits formed with the digits 0,2,3 and 5 in which 0 may comes at unit place is ${}^3P_3 = \frac{3!}{0!} = 6$

i.e., 0 may come at unit or tens or hundreds place in 6 number of times.

Also keeping 2 (3 or 5) at unit (or tens or hundred) place. Number of places to be filled up $= r = 3$.

Number of digits remain (0,3,5) to be utilised of 4 digits $= n = 3$.

Hence number of numbers of 4 digits, such that 2 comes at unit place $= {}^3P_3$. But out of these ${}^3P_3$, numbers there are some number, which begin with zero. To find the number of these numbers keeping 0, at thousand place, 2 at unit place then number of places to be filled up $= r = 2$.

Number of digits remain to be utilised

$$= n = 2.$$

Numbers of numbers of 4 digits beginning with zero and ending with 2 is

$${}^2P_2 = \frac{2!}{0!} = 2! = 2.$$

Number of numbers such that 2(3 or 5) comes at unit place = ${}^3P_3 - 2 = 6 - 2 = 4$.

Hence, number of numbers, such that 2(3 or 5) comes at unit place = 4. *i.e.*, 2 (3 or 5) may comes at unit place in 4 number of times, 2 (3 or 5) may come at 10's place in 4 number of times 2 (3 or 5) may come at hundred place in 4 number of times. Number of digits remain to be filled up, $r = 3$. Number of digits remain to be utilised $n = 3$. Number of numbers of 4 digits such that 2(3 or 5) comes at 1000 place $= {}^3P_3 = 6$.

i.e., 2(3 or 5) may come at 1000 place 6 number of times.

6 number of times.

Sum of digits at unit place

$$= [0 \times 6 + (2 + 3 + 5) \times 4] \times 1$$

Sum of digits at tens place

$$= [0 \times 6 + (2 + 3 + 5) \times 4] \times 10$$

Sum of digits at hundred place

$$= [0 \times 6 + (2 + 3 + 5) \times 4] \times 100$$

Sum of digits at thousand place

$$= [0 \times 6 + (2 + 3 + 5) \times 4] \times 1000$$

Hence, sum of all number of 4 digits formed with digits 0,2,3,5 is 44440.

5. Let A, B, C, D be 4 books, each of single volume. Let the books E, F and G have volumes
$(E_1 \dots E_8)$, $(F_1 \dots F_5)$, $(G_1 \dots G_3)$ respectively

Number of place to be filled up = 7

Number of books $= n = 7$.

Number of arrangement of these books is

$${}^7P_7 = \frac{7!}{0!} = 7!$$

Also the 8 volumes $(E_1, \dots, E_8)$ of book E can be arranged among themselves in 8! ways, 5 volumes $(F_1, \dots, F_5)$ of book, F can be arranged in 5! ways and 3 volumes $(G_1, \dots, G_3)$ of book G can be arranged among themselves in 3! ways.

Total number of arrangements = 7!8!5!3!.

6. Let $(A_1, \dots, A_5)$ be 5 copies of one book A, $(B_1, \dots, B_4)$ *and* $(C_1, \dots, C_4)$ *be 4 copies each of 2 books* B and C. Let $(D_1, \dots, D_6)$, $(E_1, \dots, E_6)$, $(F_1, \dots, F_6)$ be 6 copies each of 3 books, D, $E, \dots, N$ be the single copies of 8 other books.

Total number of books = 14.

Now, for arrangements of these books number of places to be filled up $= r = 14$.

Number of books $= n = 14$.

Arrangements of these 14 books is 14!

Now for arrangements of $(A_1, \dots, A_5)$ 5 copies of the book A among themselves, we see that number of places to be filled up $= r = 5$

Number of things $= n = 5$.

But all the copies of the same book are identical. Hence, number of arrangement of 5 copies of the book A is

$$\frac{{}^5P_5}{5!} = \frac{5!}{5!} = 1$$

Number of arrangements of the copies of the book $B = \frac{4!}{4!}$ of book $C = \frac{4!}{4!}$ of books $D = \frac{6!}{6!}$ of book $E = \frac{6!}{6!}$ of book $F = \frac{6!}{6!}$.

Total number of arrangements of all the 14 books is

$$14! + \frac{5!}{5!} \times \frac{4!}{4!} \times \frac{4!}{4!} \times \frac{6!}{6!} \times \frac{6!}{6!} \times \frac{6!}{6!} = 14! + 1.$$

7. Let $P_1, P_2, P_3, P_4, P_5^m$ and P_6^m be 6 papers, where P_5^m and P_6^m are 2 papers of Maths. Number of arrangements of the 4 papers on which there is no restriction in a line is

$${}^4P_4 = 4! \quad (n = 4; r = 4)$$

Two Mathematics papers on these 5 places can be arranged in ${}^5P_2$ ways.

Number of arrangements of the 6 papers, if 2 Mathematics papers are not successive is

$$4! \times {}^5P_2 = 480$$

8. Total number of arrangements of 37 ball without any restriction in a line

$$= \frac{{}^{37}P_{37}}{18!19!} = \frac{37!}{18!19!}$$

Also keeping 18 white balls together $B_1, \dots, B_{19}$ Number of different balls = 20.

There can be arranged in $\frac{{}^{20}P_{20}}{19!}$ ways

Since, 19 are alike.

Also, the white balls among themselves can be arranged in $\frac{^{18}P_{18}}{18!} = \frac{18!}{18!}$ ways.

Number of arrangement of these 37 balls, so that all the 18 white balls are together

$$= \frac{^{20}P_{20}}{19!} \times \frac{18!}{18!} = \frac{20!}{19!}$$

So, required number of arrangements of the 37 balls, so that all 18 white balls are not together

$$= \frac{37!}{18!19!} - \frac{20!}{19}$$

9. Since in a class of 10 students, 3 are girls, so that number of boys = 7 and number of girls = 3.

Now, number of 7 boys in a line on which there is no restriction $= {}^7P_7 = 7!$ $(n = 7; r = 7)$

Also since no two girls sit together, so that number of places for the 3 girls = 8.

Hence, number of arrangements of the 3 girls according to the condition = $^8P_3$.

Hence, required number of arrangements of 10 student's (7 boys +3 girls) so that no two girls are together $= 7! \times {}^8P_3 = 7! \times \frac{8!}{5!}$

10. Total number of letters in the word DIRECTOR, is $n = 8 (D, I, R, E, C, T, O, R)$

Number of places $= r = 8$, two are alike. Hence number of, arrangement of the letters of the word DIRECTOR without any restriction is $\frac{^8P_8}{2!} = \frac{8!}{2!}$

Now keeping three vowels (I, E, O) together number of letters $= r = 6 \{D, R, C, T, R (I, E, O)\}$ and number of places $r = 6$.

Hence number of arrangements of these 6 is $\frac{^6P_6}{2!} = \frac{6!}{2!}$

But the three vowels among themselves can be arranged in 3! ways.

Hence, number of arrangements of letters of the word DIRECTOR, such that the three vowels are together $= \frac{6!}{2!} \times 3!$

Hence, the required arrangement of the letters of the word DIRECTOR such that the three vowels never come together is

$$\frac{8!}{2!} - \frac{6!}{2!} \times 3! = \frac{6!}{2!}(8 \times 7 - 3!)$$

$$= \frac{720}{2}(56 - 6)$$

$$= 360 \times 50 = 18000$$

11. Keeping the vowels together $V, C, C, H, N, C\ L, L, R\ (I, E, A, E, O)$

Number of letters (things) = 10

number of places to be filled up = 10

Hence, number of arrangements

$$= \frac{^{10}P_{10}}{3! \times 2!} = \frac{10!}{3! \times 2!}$$

Also for the vowels (I, E, A, E, O), number of vowels = 5 and number of places for the 5 vowels is $= r = 5$

Hence, vowels among themselves can be arranged in

$$= \frac{^5P_5}{2!} = \frac{5!}{2!} = 60$$

Hence, total number of arrangements or total number of words formed with the letters of the word VICE-CHANCELLOR so that the vowels are together is

$$\frac{10!}{3!2!} \times 60 = 5 \times 10!$$

12. There are $(r = 6)$ letters (B, A, N, A, N, A), since all the letters are to be taken, so that number of places to be filled up $= r = 6$.

Hence 3A's are alike and 2N's are alike.

Hence number of different permutations of the letters of the word BANANA is

$$= \frac{^6P_6}{3!2!} = \frac{6!}{3!2!} = \frac{6 \times 5 \times 4 \times 3!}{3!2!} = 60$$

13. Number of letters $= n = 10$ in the word BENEVOLENT of which 3 E's alike 2 N's are alike. Also number of places to be filed up $= r = 10$

Hence number of arrangements

$$= \frac{^{10}P_{10}}{3!2!} = \frac{10!}{3!2!} = 302400$$

But of these 302400 one word (arrangement is BENEVOLENT itself).

Hence, number of rearrangements

$$= 302400 - 1$$
$$= 302399$$

Now for the 2nd part putting L at the end number of letters remain to be utilised $= n = 9$ of which $3E$'s are alike and $2N$'s are alike.

Hence, number of word or arrangements' formed out of the letter of the word BENEVOLENT, ending in L is

$$\frac{{}^9P_9}{3!2!} = \frac{9!}{3! \times 2!}$$
$$= 30240$$

14. Number of letters in the word OUGHT

$$= n = 5.$$

But writing the letters alphabetically G, H, O, T, U Now for the words beginning with G number of places to be filled up $= r = 4$ and number of letters utilised $= n = 4$

Hence, number of words beginning with $G = {}^4P_4 = 4!$

Similarly, number of words beginning with $H = 4!$

Similarly, number of words beginning with $O = 4!$

Again for the words beginning with TG... number of places to be filled up $= r = 3$ and number of letters to be utilised $= n = 3$

Hence, number of words beginning with $TG = {}^3P_3 = 3! = 6$

Similarly, number of words beginning with $TH = 6$

Again for the words beginning with TOG.. number of places remain to be filled up $= r = 2$ and number of letter remain to be utilised $= n = 2$.

Hence, number of words beginning with $TOG = {}^2P_2 = 2!$

Number of words beginning with $TOH = 2!$

Now, the words beginning with TOU and $TOUGH$ come.

Hence, rank of the word TOUGH in the dictionary

$$= 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1$$
$$= 89\text{th}$$

15. (i) When there is no restriction, we have total numbers of boys and girls $= 5 + 4 = 9$, keeping one out of 9 fixed, remaining 8 can be seated at 9 round table in ${}^8P_8 = 8!$ ways.

(ii) In this case we first arrange the boys, for keeping one boy fixed remaining 4 boys can be seated in 4! ways. Also as all the 4 girls are to sit together we select one region and then 4 girls can be seated in this region in 4! ways.

Hence, number of ways in which 5 boys and 4 girls can be seated at a round table, so that all the 4 girls sit together is

$$4! \cdot {}^5C_1 \cdot 4! = 5 \times 576 = 2880$$

(iii) From case (i) we see that total numbers of arrangements 5 boys and 4 girls at a round table is $= 8! = 40320$ and from case (ii) number of arrangements of 5 boys and 4 girls at a round table, so that all the 4 girls are not together

$$= 40320 - 2880$$
$$= 37440$$

(iv) For, no two girls sit together, we first arrange 5 boys keeping one boy fixed remaining 4 boys can be arranged in 4! ways.

Since, no two girls are to sit together, so that number of place for the girls (between every two boys) is 5 and hence girls can be arranged according to the condition of the problem in 5! ways.

Hence required number of arrangements in this case $= 4! \cdot 5!$

$$= 2880$$

16. Number of place = number of children taken at a time $= r = 3$.

Hence, 3 children out of 8 can be selected in

$${}^8C_3 = \frac{8!}{3!5!} = 56 \text{ ways}$$

Hence, the man has to go to the zoo 56 times.

In the 2nd part of the problem, a particular child is to be included. Keeping the particular child aside number of remaining children

$$= n - 1 = 8 - 1 = 7$$

And number of places to be filled up

$$r = 3 - 1 = 2$$

Hence, number of selection of 3 children out of 8 children including a particular child

$$= {}^7C_2 \times {}^1C_1$$

$$= \frac{7!}{5!2!} = \frac{7 \times 6 \times 5!}{5!2!}$$

$$= 21$$

Hence, a particular child will go to the zoo 21 number of times.

17. Committees of 5 consisting of at least 3 ladies can be made in the following ways.

Committees of 5 consisting of 3 ladies and man can be made in ${}^4C_3 \cdot {}^7C_2$ ways.

Committees of 5 consisting of 4 ladies and 1 man can be made in ${}^4C_4 \cdot {}^7C_1$ ways.

Hence, committees of 5 can be formed in

$$= {}^4C_3 \cdot {}^7C_2 + {}^4C_4 \cdot {}^7C_1$$

$$= 4 \times 21 + 1 \times 7 = 84 + 7 = 91$$

18. Three groups A, B and C contain respectively 4, 5 and 6 question now to make a selection of 7 question, selecting at least two question from each section, a candidate will have to make up his choice in the following ways.

He can answer 2 questions from group A 2 question from group B and 3 questions from group C in ${}^4C_2 \cdot {}^5C_3 \cdot {}^6C_2$ ways.

He can answer 2 questions from group A and 3 questions from group B and 2 questions from group C in ${}^4C_2 \cdot {}^5C_2 \cdot {}^6C_3$ ways.

He can answer 3 question from group A 2 question from group B and 2 question from group C in ${}^4C_3 \cdot {}^5C_2 \cdot {}^6C_2$ ways.

Hence, total number of way to answer all the question

$$= {}^4C_2 \cdot {}^5C_2 \cdot {}^6C_3 + {}^4C_2 \cdot {}^5C_3 \cdot {}^6C_2 + {}^4C_3 \cdot {}^5C_2 \cdot {}^6C_2$$

$$= 6 \times 10 \times 20 + 6 \times 10 \times 15 + 4 \times 10 \times 15$$

$$= 1200 + 900 + 600 = 2700$$

19. Committees of 5 consisting of at least one lady can be made in the following ways.

Committees of 5 consisting of 2 ladies and 3 gentelmen can be made in ${}^4C_2 \cdot {}^8C_3$ ways.

Committees of 5 consisting of 3 ladies and 2 Gentelmen can be made in ${}^4C_3 \cdot {}^8C_2$.

Committees of 5 consisting of 4 ladies and and 1 Gentelmen Can be made in ${}^4C_4 \cdot {}^8C_1$ ways of committees of 5 consisting of 1 ladies and 4 gentelmen can be made in ${}^4C_1 \cdot {}^8C_4$ ways.

Hence, total number of committees.

$$= {}^4C_1 \cdot {}^8C_4 + {}^4C_2 \cdot {}^8C_3 + {}^4C_3 \cdot {}^8C_2 + {}^4C_4 \cdot {}^8C_1$$

$$= \frac{4 \times 8!}{4!4!} + \frac{4!}{2!2!} \times \frac{8!}{3!5!} + \frac{4!}{3!1!} \times \frac{8!}{2!6!} + 1 \times \frac{8!}{1!7!}$$

$$= \frac{4 \times 8 \times 7 \times 6 \times 5 \times 4!}{4!4!} + \frac{2 \times 3 \times 4}{4} \times \frac{8 \times 7 \times 6 \times 5!}{3! \times 5!} + \frac{4 \times 3!}{3!1!} \times \frac{8 \times 7 \times 6!}{2!6!} + \frac{1 \times 8 \times 7!}{1! \times 7!}$$

$$= 280 + 336 + 112 + 8 = 736$$

20. Since there are 3 seats and a voter can not vote for more than 3 candidates.

Hence, the voter can vote for 1 candidate out of 6 candidates in ${}^6C_1$ ways. Voter can vote for 2 candidates out of 6 candidates in ${}^6C_2$ ways.

And the vote can vote for 3 candidates out of 6 candidates in ${}^6C_3$ ways.

Finally total number of ways to vote

$$= {}^6C_1 + {}^6C_2 + {}^6C_3 = 6 + 15 + 20 = 41$$

21. Let the number of candidates be n. Therefore number of members to be elected $= n - 2$ and hence one can vote atmost for $n - 2$ candidates.

Hence, total number of ways in which one can vote

$$= {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_{n-2} = 56$$

or $${}^nC_0 + ({}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_{n-2}) + {}^nC_{n-1} + {}^nC_n = 1 + 56 + n + 1$$

$$(\because {}^nC_0 = {}^nC_n = 1 \text{ and } {}^nC_{n-1} = n)$$

or $$2^n = n + 58 \qquad \ldots(i)$$

Now only $n = 6$ satisfies (i) so that number of candidates $= n = 6$.

22. According to the problem, a voter has to vote for at least one candidates, and at most 5 candidates, the number of ways in which the voter may vote is

$$= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5$$

$$= 10 + 45 + 120 + 210 + 252 = 637$$

23. (i) Since, balls of same colour are also different. So that we have total number of different balls. $= 5 + 4 + 3 = 12$

In this case we have to select some balls (*i.e.*, zero or some or all 12 balls).

Now, one ball out of 12 can be selected in ${}^{12}C_1$ ways.

2 bals out of 12 can be selected in ${}^{12}C_2$ ways.

Similarly, 12 balls out of 12 can be selected in ${}^{12}C_{12}$ ways.

Hence, number of selection of some balls

$$= {}^{12}C_1 + {}^{12}C_2 + \dots + {}^{12}C_{12}$$
$$= ({}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2 + \dots + {}^{12}C_{12}) - 1$$
$$= 2^{12} - 1$$

(ii) Some balls containing at least one red and one green balls are to be selected.

Now 1 or some red balls can be selected in

$${}^5C_1 + \dots + {}^5C_5 = 2^5 - 1 = 31 \text{ ways}$$

Also 1 or some green balls out of 4 green balls can be selected in

$${}^4C_1 + {}^4C_2 + \dots + {}^4C_4 = 2^4 - 1 = 15 \text{ ways.}$$

Now zero or some blue balls out of 3 blue balls can be selected in

$${}^3C_0 + {}^3C_1 + \dots + {}^3C_3 = 2^3 \text{ ways.}$$

Hence, total number of selections of balls containing at least one red and one green balls.

$$= 31 \times 15 \times 2^3 = 31 \times 15 \times 8$$
$$= 3720$$

24. Since, at least one red ball must be drawn, so that number of selectin of 1 or some or all 4 red balls (all different)

$$= {}^4C_1 + {}^4C_2 + \dots + {}^4C_4$$
$$= 2^4 - 1 = 15$$

Also the number of selection of zero, some green balls from 3 different green balls

$$= {}^3C_0 + {}^3C_1 + \dots + {}^3C_3 = 2^3 = 8$$

Hence, total number of required selection

$$= 15 \times 8 = 120$$

25. (i) In this case, we have to select one book out of 3 books of Mathematics in ${}^3C_1$ ways, one book out of books of Science in ${}^4C_1$ ways

and one book out of 5 books of literature in ${}^5C_1$ ways.

Hence, number of different collections of one (exactly one) book of each subject

$$= {}^3C_1 \cdot {}^4C_1 \cdot {}^5C_1 = 3 \cdot 4 \cdot 5 = 60$$

(ii) Here we have to select at least one book of each subject out of 3 books of Mathematics, 4 books of Science and 5 books of Literature. Here it is not given how many books are to be selected. Hence, we have to select some books containing one book of each subject out of 3 books of Mathematics, 4 book of Science and 5 of Literature, which can be selected in

$$({}^3C_1 + {}^3C_2 + {}^3C_3)({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) \times ({}^5C_1 + {}^5C_2 + \dots + {}^5C_5)$$
$$= (2^3 - 1)(2^4 - 1)(2^5 - 1)$$

(iii) In this case, at least one book of literature is to be selected. Hence, we have to select 0 or 1, or 2 or 3 books from 3 books of mathematics, 0 or 1 or 2 or 3 or 4 books of science; 1 or 2 or 3 or 4 or 5 books from 5 books of literature which can be selected in

$$({}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3)({}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4)({}^5C_1 + {}^5C_2 + \dots + {}^5C_5)$$
$$= 2^3 . 2^4 (2^5 - 1) = 128(2^5 - 1)$$

26. The number of 7-digit numbers with 1 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once is $6! = 720$. But 120 of these end in 5 and hence are divisible by 5. Thus, the number of 7-digit numbers with 1 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once but not divisible by 5 is 600. Similarly, the number of 7-digits numbers with 2 and 3 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once but not divisible by 5 is also 600 each. These account for 1800 numbers. Hence, 2000th number must have 4 in the left most place.

Again the number of such 7-digit numbers beginning with 41, 42 and not divisible by 5 is $120 - 24 = 96$ each and these account for 192 numbers. This shows that 2000th number in the list must begin with 43.

The next 8 numbers in the list are 4312567, 4312576, 4312657, 4312756, 4315267, 4315276, 4315627 and 4315672. Thus, 2000th number in the list is 4315672.

27. We observe that $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = 0$, if and only if $x \in \{1, 2, 3, \ldots, 98\}$ and there are 98 such numbers. If we want $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = 1$, then x should lie in the set $\{101, 102, \ldots, 197\}$, which accounts for 97 numbers. In general, if we require $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = k$, where $k \geq 1$, then x must be in the set $\{101k, 101k + 1, \ldots, 99(k+1) - 1\}$ and there are $99 - 2k$ such numbers. Observes that this set is not empty only if $99(k+1) - 1 \geq 101k$ and this requirement is met only if $k \leq 49$. Thus, the total number of positive integers x for which $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$ is given by

$$98 + \sum_{k=1}^{49} (99 - 2k) = 2499$$

Remark

For any $m \geq 2$ the number of positive integers x such that $\left[\frac{x}{m-1}\right] = \left[\frac{x}{m+1}\right]$ is $\frac{m^2 - 4}{4}$ if m is even and $\frac{m^2 - 5}{4}$, if m is odd.

28. We have, $\binom{n}{7} = \frac{n(n-1)(n-2)\ldots(n-6)}{7!}$

In the numerator, there is a factor divisible by 7 and the other six factors leave the remainders 1, 2, 3, 4, 5, 6 in some order when divided by 7.

Hence, the numerator may be written as

$$7k \cdot (7k_1 + 1) \cdot (7k_2 + 2) \ldots (7k_6 + 6).$$

Also, we conclude that $\left[\frac{n}{p}\right] = k$, as in the set $\{n, n-1, \ldots, n-6\}$, $7k$ is the only number which is a multiple of 7. If the given number is called Q, then

$$Q = 7k \cdot \frac{(7k_1 + 1)(7k_2 + 2)\ldots(7k_6 + 6)}{7!} - k$$

$$= k\left[\frac{(7k_1 + 1)\ldots(7k_6 + 6) - 6!}{6!}\right]$$

$$= \frac{k[7t + 6! - 6!]}{6!}$$

$$= \frac{7tk}{6!}$$

We know that, Q is an integer and so $6!$ divides $7tk$. Since, GCD $(7, 6!) = 1$, even after cancellation there is a factor of 7 still left in the numerator. Hence, 7 divides Q, as desired.

29. We count by brute force considering the cases $x = 0, x = 1, \ldots, x = 33$. Observe that the least value x can take is zero and its largest value is 33.

<u>$x = 0$</u>. If $y = 0$, then $z \in \{0, 1, 2, \ldots, 100\}$; if $y = 1$, then $z \in \{1, 2, \ldots, 99\}$; if $y = 2$, then $z \in \{2, 3, \ldots, 98\}$ and so on. Finally if $y = 50$, then $z \in \{50\}$. Thus, there are altogether $101 + 99 + 97 + \ldots + 1 = 51^2$ possibilities

<u>$x = 1$</u>. Observe that $y \geq 1$. If $y = 1$, then $z \in \{1, 2, \ldots, 98\}$; if $y = 2$, then $z \in \{2, 3, \ldots, 97\}$; if $y = 3$, then $z \in \{3, 4, \ldots, 96\}$ and so on. Finally if $y = 49$, then $z \in \{49, 50\}$. Thus, there are altogether $98 + 96 + 94 + \ldots + 2 = 49 \cdot 50$ possibilities.

General case. Let x be even, say, $x = 2k$, $0 \leq k \leq 16$. If $y = 2k$, then $z \in \{2k, 2k+1, \ldots, 100 - 4k\}$, if $y = 2k + 1$, then $z \in \{2k+1, 2k+2, \ldots, 99 - 4k\}$; if $y = 2k + 2$, then $z \in \{2k+2, 2k+3, \ldots, 99 - 4k\}$ and so on. Finally, if $y = 50 - k$, then $z \in \{50 - k\}$. There are altogether

$(101 - 6k) + (99 - 6k) + (97 - 6k) + \ldots + 1$
$= (51 - 3k)^2$ possibilities.

Let x be odd, say, $x = 2k + 1$, $0 \leq k \leq 16$. If $y = 2k + 1$, then $z \in \{2k+1, 2k+2, \ldots, 98 - 4k\}$; if $y = 2k + 2$, then $z \in \{2k+2, 2k+3, \ldots, 97 - 4k\}$; if $y = 2k + 3$, then $z \in \{2k+3, 2k+4, \ldots, 96 - 4k\}$; and so on.

Finally, if $y = 49 - k$, then $z \in \{49 - k, 50 - k\}$. There are altogether

$$(98 - 6k) + (96 - 6k) + (94 - 6k) + \ldots + 2 = (49 - 3k)(50 - 3k)$$

possibilities.

The <u>last two cases</u> would be as follows :

<u>$x = 32$</u>: if $y = 32$, then $z \in \{32, 33, 34, 35, 36\}$; if $y = 33$, then $z \in \{33, 34, 35\}$; if $y = 34$, then $z \in \{34\}$; altogether $5 + 3 + 1 = 9 = 3^2$ possibilities.

$\underline{x = 33:}$ if $y = 33$, then $z \in \{33, 34\}$; only $2 = 12$ possibilities.

Thus the total number of triples, say T, is given by,

$$T = \sum_{k=0}^{16} (51 - 3k)^2 + \sum_{k=0}^{16} (49 - 3k)(50 - 3k).$$

Writing this in the reverse order, we obtain

$$T = \sum_{k=1}^{17} (3k)^2 + \sum_{k=0}^{17} (3k - 2)(3k - 1)$$

$$= 18 \sum_{k=1}^{17} k^2 - 9 \sum_{k=1}^{17} k + 34$$

$$= 18\left(\frac{17 \cdot 18 \cdot 35}{6}\right) - 9\left(\frac{17 \cdot 18}{2}\right) + 34$$

$$= 30,787.$$

Thus the answer is 30787.

Aliter

It is known that the number of ways in which a given positive integer $n \geq 3$ can be expressed as a sum of three positive integers x, y, z (that is, $x + y + z = n$), subject to the condition $x \leq y \leq z$ is $\left\{\frac{n^2}{12}\right\}$, where $\{a\}$ represents the integer closest to a. If zero values are allowed for x, y, z then the corresponding count is $\left\{\frac{(n+3)^2}{12}\right\}$, where now $n \geq 0$.

Since in our problem $n = x + y + z \in \{0, 1, 2, \ldots, 100\}$, the desired answer is

$$\sum_{n=0}^{100} \left\{\frac{(n+3)^2}{12}\right\}.$$

For $n = 0, 1, 2, 3, \ldots, 11$, the corrections for $\{\}$ to get the nearest integers are

$$\frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}, \frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}$$

So, for 12 consecutive integer values of n, the sum of the corrections is equal to

$$\left(\frac{3 - 4 - 1 - 0 - 1 - 4 - 3}{12}\right) \times 2 = \frac{-7}{6}.$$

Since, $\frac{101}{12} = 8 + \frac{5}{12}$, there are 8 sets of 12 consecutive integers in $\{3, 4, 5, \ldots, 103\}$ with 99, 100, 101, 102, 103 still remaining. Hence the total correction is

$$\left(\frac{-7}{6}\right) \times 8 + \frac{3 - 4 - 1 - 0 - 1}{12} = \frac{-28}{3} - \frac{1}{4} = \frac{-115}{12}.$$

So the desired number T of triples (x, y, z) is equal to

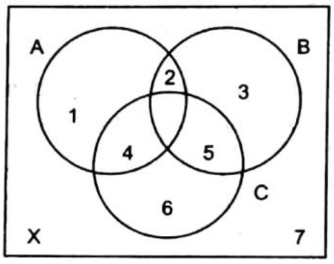
$$T = \sum_{n=0}^{100} \frac{(n+3)^2}{12} - \frac{115}{12}$$

$$= \frac{(1^2 + 2^2 + 3^2 + \ldots + 103^2) - (1^2 + 2^2)}{12} - \frac{115}{12}$$

$$= \frac{103 \cdot 104 \cdot 207}{6 \cdot 12} - \frac{5}{12} - \frac{115}{12}$$

$$= 30787.$$

30.



Let $X = \{1, 2, 3, \ldots, n\}$. We use Venn diagram for sets A, B, C to solve the problem. The regions other than $A \cap B \cap C$ (which is to be empty) are numbered 1, 2, 3, 4, 5, 6, 7 as shown in the figure; *e.g.*, 1 corresponds to $A / (B \cup C) - A \cap B^c \cup C^c$, corresponds to $A \cap B / C = A \cap B \cap C^c$, 7 corresponds to $X / (A \cup B \cup C) = A^c \cap B^c \cap C^c$, since $A \cap B \cap C = \phi$.

Firstly the number of ways of assigning elements of X to the numbers regions without any condition is 7^n. Among these there are cases in which 2 or 5 or both are empty. The number of distributions in which 2 is empty is 6^n. Likewise the number of distributions in which 5 is empty is also 6^n. But then we have subtracted twice the number of distributions in which both the regions 2 and 5 are empty. So, to compensate we have to add the number of distributions in which both 2 and 5 are empty. This is 5^n. Hence, the desired number of triples (A, B, C) in $7^n - 6^n - 6^n + 5^n = 7^n - 2 \cdot 6^n + 5^n$.

31. Putting $a - 1 = p, b - 1 = q$ and $c - 1 = r$, the equation may be written in the form

$$pqr = 2(p + q + r) + 4,$$

where p, q, r are integers such that $0 \leq p \leq q \leq r$. Observe that $p = 0$ is not possible, for then $0 = 2(p + q) + 4$ which is impossible in non-negative integers. Thus, we may write this in the form

$$2\left(\frac{1}{pq}+\frac{1}{qr}+\frac{1}{rp}\right)+\frac{4}{pqr}=1$$

If $p \geq 3$, then $q \geq 3$ and $r \geq 3$. Then, left side is bounded by $6/9 + 4/27$ which is less than 1. We conclude that $p = 1$ or 2.

Case I Suppose $p = 1$. Then, we have $qr = 2(q + r) + 6$ or $(q-2)(r-2) = 10$. This gives $q - 2 = 1, r - 2 = 10$ or $q - 2 = 2$ and $r - 2 = 5$ (recall $q \leq r$). This implies $(p, q, r) = (1, 3, 12)$, $(1, 4, 7)$.

Case II If $p = 2$, the equation reduces to $2qr = 2(2 + q + r) + 4$ or $qr = q + r + 4$. This reduces to $(q-1)(r-1) = 5$. Hence, $q - 1 = 1$ and $r - 1 = 5$ is the only solution. This gives $(p, q, r) = (2, 2, 6)$.

Reverting back to a, b, c, we get three triples: $(a, b, c) = (2, 4, 13), (2, 5, 8), (3, 3, 7)$.

32. Any such number should be both divisible by 5 and 3. The last digit of a number divisible by 5 must be either 5 or 0. Hence, any such number falls into one of the following seven categories:

(i) $abc15$

(ii) $ab150$

(iii) $ab155$

(iv) $a15b0$

(v) $a15b5$

(vi) $15ab0$

(vii) $15ab5$

Here, a, b, c are digit. Let us count how many numbers of each category are there.

(i) In this case $a \neq 0$ and the 3-digit number abc is divisible by 3, and hence one of the numbers in the set {102, 105, ..., 999}. This gives 300 numbers.

(ii) Again a number of the form $ab150$ is divisible by 15 if and only if the 2-digit number ab is divisible by 3. Hence, it must be from the set {12, 15, ..., 99}. There are 30 such numbers.

(iii) As in (ii), here are again 30 numbers.

(iv) Similar to (ii); 30 numbers

(v) Similar to (ii), 30 numbers.

(vi) We can begin the analysis of the number of the form $15ab0$ as in (ii). Here, again ab as a 2-digit number must be divisible by 3, but $a = 0$ is also permissible. Hence, it must be from the set {00, 03, 06,...,99}. There are 34 such numbers.

(vii) Here again there are 33 numbers; ab must be from the set {01, 04, 07,...,97}.

Adding all these we get

$300 + 30 + 30 + 30 + 30 + 34 + 33 = 487$ numbers.

However this is not the correct figure as there is over counting. Let us see how much over counting is done by looking at the intersection of each pair of categories. A number in (i) obviously cannot lie in (ii), (iv) or (vi) as is evident from the last digit. There cannot be a common number in (i) and (iii) as any two such numbers differ in the 4th digit. If a number belongs to both (i) and (v), then such a number of the form $a1515$. This, is divisible by 3 only for $a = 3, 6, 9$. Thus there are 3 common numbers in (i) and (ii). A number which is both in (i) and (vii) is of form $15c15$ and divisibility by 3 gives $c = 0, 3, 6, 9$; thus we have 4 numbers common in (i) and (vii). That exhaust all possibilities with (i).

Now (ii) can have common numbers with only categories (iv) and (vi). There are no numbers common between (ii) and (vi) as evident from 3rd digit. There is only one number common to (ii) and (vi), namely 15150 and this is divisible by 3. There is nothing common to (iii) and (v) as can be seen from the 3rd digit. The only number common to (iii) and (vii) is 15155 and this is not divisible by 3. It can easily be inferred that no number is common to (iv) and (vi) by looking at the 2nd digit. Similarly no number is common to (v) and (vii). Thus there are $3 + 4 + 1 = 8$ numbers which are counted twice.

We conclude that the number of 5-digit numbers which contain the block 15 and divisible by 15 is $487 - 8 = 479$.

33. Since each digit occurs at least twice, we have following possibilities

1. Three digits occur twice each. We may choose three digits from {1, 2, 3, 4, 5} in $\binom{5}{3} = 10$ ways. If each occurs exactly twice, the number of such admissible 6-digit numbers is

$$\frac{6!}{2!2!2!} \times 10 = 900$$

2. Two digits occur three times each. We can choose 2 digits in $\binom{5}{2} = 10$ ways

Hence, the number of admissible 6-digit numbers is $\frac{6!}{3!3!} \times 10 = 200$

3. One digit occurs four times and the other twice. We are choosing two digits again, which can be done in 10 ways. The two digits are interchangeable. Hence, the desired number of admissible 6-digit numbers is
$$2 \times \frac{6!}{4!2!} \times 10 = 300$$

4. Finally all digits are the same. There are 5 such numbers.

Thus, the total number of admissible numbers is $900 + 200 + 300 + 5 = 1405$.

34. Since $20, b, c$ are in harmonic progression, we have
$$\frac{1}{20} + \frac{1}{c} = \frac{2}{b},$$
which reduces to $bc + 20b - 40c = 0$. This may also be written in the form $(40 - b)(c + 20) = 800$.

Thus, we must have $20 < b < 40$ or equivalently, $0 < 40 - b < 20$. Let us consider the factorisation of 800 in which one term is less than 20
$$\begin{aligned}(40 - b)(c + 20) &= 800 = 1 \times 800 = 2 \times 400\\ &= 4 \times 200\\ &= 5 \times 160 = 8 \times 100\\ &= 10 \times 80 = 16 \times 50\end{aligned}$$
We thus get the pairs

$(b, c) = (39, 780), (38, 380), (36, 180), (35, 140), (32, 80), (30, 60), (24, 30)$.

Among these 7 pairs, we see that only 5 pairs (39, 780), (38, 380), (36, 180), (35, 140), (30, 60) fulfill the condition of divisibility: b divides c. Thus, there are 5 triples satisfying the requirement of the problem.

35. Let the sides be x, x, y, where x, y are positive integers. Since we are looking for obtuse angled triangles, $y > x$. Moreover, $2x + y = 2008$ shows that y is even. But $y < x + x$, by triangle inequality. Thus, $y < 1004$. Thus, the possible triples are $(y, x, x) = (1002, 503, 503)$, $(1000, 504, 504)$, $(998, 505, 505)$ and so on. The general form is $(y, x, x) = (1004 - 2k, 502 + k, 502 + k)$, where $k = 1, 2, 3, \ldots, 501$.

But the condition that the triangle is obtuse leads to $(1004 - 2k)^2 > 2(502 + k)^2$.

This simplifies to $\quad 502^2 + k^2 - 6(502)k > 0$

Solving this quadratic inequality for k, we see that
$$k < 502(3 - 2\sqrt{2}) \text{ or } k > 502(3 + 2\sqrt{2}).$$
Since, $k \leq 501$, we can rule out the second possibility. Thus $k < 502(3 - 2\sqrt{2})$, which is approximately 86.1432. We conclude that $k \leq 86$. Thus, we get 86 triangles
$$(y, x, x) = (1004 - 2k, 502 + k, 502 + k),$$
$$k = 1, 2, 3, \ldots, 86.$$
The last obtuse triangle in this list is (832, 588, 588). (It is easy to check that $832^2 - 588^2 - 588^2 = 736 > 0$, where as $830^2 - 589^2 - 589^2 = -4942 < 0$.)

36. We observe that $0 + 1 + 2 + 3 = 6$. Hence, the remaining two digits must account for the sum 4. This is possible with $4 = 0 + 4 = 1 + 3 = 2 + 2$. Thus we see that the digits in any such 6-digit number must be from one of the collections:

{0, 1, 2, 3, 0, 4}, {0, 1, 2, 3, 1, 3} or {0, 1, 2, 3, 2, 2}.

Consider the case in which the digits are from the collection {0, 1, 2, 3, 0, 4}. Here 0 occurs twice and the digits 1, 2, 3, 4 occur once each. But 0 cannot be the first digit. Hence, the first digit must be one of 1, 2, 3, 4. Suppose we fix 1 as the first digit. Then, the number of 6-digit numbers in which the remaining 5 digits are 0, 0, 2, 3, 4 is $5!/2! = 60$. Same is the case with other digits: 2, 3, 4. Thus the number of 6-digit numbers in which the digits 0, 1, 2, 3, 0, 4 occur is $60 \times 4 = 240$.

Suppose the digits are from the collection {0, 1, 2, 3, 1, 3}. The number of 6-digit numbers beginning with 1 is $5!/2! = 60$. The number of those beginning with 2 is $5!/(2!)(2!) = 30$ and the number of those beginning with 3 is $5!/2! = 60$. Thus the total number in this case is $60 + 30 + 60 = 150$. Alternately, we can also count it as follows: the number of 6-digit numbers one can obtain from the collection {0, 1, 2, 3, 1, 3} with 0 also as a possible first digit is $6!/(2!)(2!) = 180$; the numbers of 6-digit number one can obtain from the collection {0, 1, 2, 3, 1, 3} in which 0 is the first digit is $5!/(2!)(2!) = 30$. Thus the

number of 6-digit numbers formed by the collection $\{0, 1, 2, 3, 1, 3\}$ such that no number has its first digit 0 is $180 - 30 = 150$. Finally look at the collection $\{0, 1, 2, 3, 2, 2\}$. Here the number of of 6-digit numbers in which 1 is first digit is $5!/3! = 20$; the number of those having 2 as the first digit is $5!/2! = 60$; and the number of those having 3 as the first digit is $5!/3! = 20$. Thus the number of admissible 6-digit numbers here is $20 + 60 + 20 = 100$. This may also be obtained using the other method of counting:

$$6!/3! - 5!/3! = 120 - 20 = 100.$$

Finally the total number of 6-digit numbers in which each of the digits 0, 1, 2, 3 appears at least once is

$$240 + 150 + 100 = 490.$$

37. Let X denotes the set of all 3-digit natural numbers; let O be those numbers in X having only odd digits; and E be those numbers in X having only even digits. Then $X/(O \cup E)$ is the set of all 3-digit natural numbers having at least one odd digit and at least one even digit. The desired sum is therefore

$$\sum_{x\in X} x - \sum_{y\in O} y - \sum_{x\in E} z.$$

It is easy to compute the first sum

$$\sum_{x\in X} x = \sum_{j=1}^{999} j - \sum_{k=1}^{99} k = \frac{999\times 1000}{2} - \frac{99\times 100}{2}$$

$$= 50 \times 9891 = 494550$$

Consider the set O. Each number in O has its digits from the set $\{1, 3, 5, 7, 9\}$. Suppose the digit in unit's place is 1. We can fill the digit in ten's place in 5 ways and the digit in hundred's place in 5 ways. Thus there are 25 numbers in the set O each of which has 1 in its unit's place. Similarly, there are 25 numbers whose digit in unit's place is 3; 25 having its digit in unit's place as 5; 25 with 7 and 25 with 9. Thus the sum of the digits in unit's place of all the numbers in O is

$$25\,(1 + 3 + 5 + 7 + 9) = 25 \times 25 = 625$$

A similar argument shows that the sum of digits in ten's place of all the numbers in O is 625 and that in hundred's place is also 625. Thus the sum of all the numbers in O is

$$625\,(10^2 + 10 + 1) = 625 \times 111 = 69375$$

Consider the set E. The digits of numbers in E are from the set $\{0, 2, 4, 6, 8\}$, but the digit in hundred's place is never 0. Suppose the digit in unit's place is 0. There are $4 \times 5 = 20$ such numbers. Similarly, 20 numbers each having digits 2, 4, 6, 8 in their unit's place. Thus, the sum of the digits in unit's place of all the numbers in E is

$$20\,(0 + 2 + 4 + 6 + 8) = 20 \times 20 = 400$$

A similar reasoning shows that the sum of the digits in ten's place of all the numbers in E is 400, but the sum of the digits in hundred's place of all the numbers in E is $25 \times 20 = 500$. Thus the sum of all the numbers in E is

$$500 \times 10^2 + 400 \times 10 + 400 = 54400$$

The required sum is

$$494550 - 69375 - 54400 = 370775$$

38. Let us examine the first few natural numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9. Here we see that $a_n = 1, 2, 3, 2, 2, 3, 3, 4, 3$. We observe that $a_n \le a_{n+1}$ for all n except when $n + 1$ is a square in which case $a_n > a_{n+1}$. We prove that this observation is valid in general.

Consider the range

$$m^2, m^2 + 1, m^2 + 2, \ldots, m^2 + m,$$
$$m^2 + m + 1, \ldots, m^2 + 2m.$$

Let n take values in this range so that $n = m^2 + r$, where $0 \le r \le 2m$. Then we see that $[\sqrt{n}] = m$ and hence

$$\left[\frac{n}{[\sqrt{n}]}\right] = \left[\frac{m^2 + r}{m}\right] = m + \left[\frac{r}{m}\right]$$

Thus a_n takes the value

$$\underbrace{m, m, m, \ldots, m}_{m \text{ times}},$$
$$\underbrace{m + 1,\, m + 1, m + 1, \ldots, m + 1}_{m \text{ times}}, m + 2,$$

in this range. But when $n = (m + 1)^2$, we see that $a_n = m + 1$. This shows that $a_{n-1} > a_n$ whenever $n = (m + 1)^2$. When we take n in the set $\{1, 2, 3, \ldots, 2010\}$, we see that the only squares are $1^2, 2^2, \ldots, 44^2$ (since $44^2 = 1936$ and $45^2 = 2025$) and $n = (m + 1)^2$ is possible for only 43 values of m. Thus $a_n > a_{n+1}$ for 43 values of n. (These are $2^2 - 1, 3^2 - 1, \ldots, 44^2 - 1$.)

39. Let x, y, z be three distinct positive integers satisfying the given conditions.

We may assume that $x < y < z$. Thus we have three relations:

$$\frac{1}{y}+\frac{1}{z}=\frac{a}{x},$$

$$\frac{1}{z}+\frac{1}{x}=\frac{b}{y},$$

$$\frac{1}{x}+\frac{1}{y}=\frac{c}{z},$$

for some positive integers a, b, c. Thus

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z},$$

$$\frac{a+1}{x}=\frac{b+1}{y}=\frac{c+1}{z}=r,$$

say. Since $x<y<z$, we observe that $a<b<c$. We also get

$$\frac{1}{x}=\frac{r}{a+1}, \frac{1}{y}=\frac{r}{b+1}, \frac{1}{z}=\frac{r}{c+1}$$

Adding these, we obtain

$$r=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{r}{a+1}+\frac{r}{b+1}+\frac{r}{c+1}$$

or $$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=1 \qquad \ldots\text{(i)}$$

Using $a<b<c$, we get

$$1=\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}<\frac{3}{a+1}$$

Thus, $a<2$. We conclude that $a=1$. Put this in the relation (i), we get

$$\frac{1}{b+1}+\frac{1}{c+1}=1-\frac{1}{2}=\frac{1}{2}$$

Hence, $b<c$ gives $\dfrac{1}{2}<\dfrac{2}{b+1}$

Thus, $b+1<4$ or $b<3$. Since $b>a=1$, we must have $b=2$. This gives

$$\frac{1}{c+1}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6},$$

or $c=5$. Thus $x:y:z=a+1:b+1:c+1=2:3:6$. Thus the required numbers with the least sum are 2, 3, 6.

Aliter We first observe that $(1, a, b)$ is not a solution whenever $1<a<b$. Otherwise we should have $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{l}{1}=l$ for some integer l.

Hence, we obtain $\dfrac{a+b}{ab}=l$ showing that $a|b$ and $b|a$. Thus $a=b$ contradicting $a\neq b$. Thus the least number should be 2. It is easy to verify that (2, 3, 4) and (2, 3, 5) are not solutions and (2, 3, 6) satisfies all the conditions. (We may observe (2, 4, 5) is also not a solution.) Since $3+4+5=12>11=2+3+6$, it follows that (2, 3, 6) has the required minimality.

40. We divide the even 4-digit numbers having non-zero digits into 4 classes those ending in 2, 4, 6, 8.

(A) Suppose a 4-digit number ends in 2. Then the second right digit must be odd in order to be divisible by 4. Thus the last 2 digits must be of the form 12, 32, 52, 72 or 92. If a number ends in 12, 52 or 92, then the previous digits must be even in order not to be divisible by 8 and we have 4 admissible even digits. Now the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways and we get $9\times4\times3=108$ such numbers. If a number ends in 32 or 72, then the previous digit must be odd in order not to be divisible by 8 and we have 5 admissible odd digits. Here again the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways, and we get $9\times5\times2=90$ such numbers. Thus the number of 4-digit numbers having non-zero digits, ending in 2, divisible by 4 but not by 8 is $108+90=198$.

(B) If the number ends in 4, then the previous digit must be even for divisibility by 4. Thus the last two digits must be of the form 24, 44, 54, 84. If we take numbers ending with 24 and 64, then the previous digit must be odd for non-divisibility by 8 and the left most digit can be any non-zero digit. Here we get $9\times5\times2=90$ such numbers. If the last two digits are of the form 44 and 84, then previous digit must be even for non-divisibility by 8. And the left most digit can take 9 possible values. We thus get $9\times4\times2=72$ numbers. Thus the admissible numbers ending in 4 is $90+72=162$.

(C) If a number ends with 6, then the last two digits must be of the form 16, 36, 56, 76, 96. For numbers ending with 16, 56, 76, the previous digit must be odd. For numbers ending with 36, 76, the previous digit must be even. Thus we get her $(9\times5\times3)+(9\times4\times2)=135+72=207$ numbers.

(D) If a number ends with 8, then the last two digits must be of the form 28, 48, 68, 88. For numbers ending with 28, 68, the previous digit must be even. For numbers ending with 48, 88, the previous digit must be odd. Thus we get $(9 \times 4 \times 2) + (9 \times 5 \times 2) = 72 + 90 = 162$ numbers

Thus the number of 4-digit numbers, having non-zero digits, and divisible by 4 but not by 8 is $198 + 162 + 207 + 162 = 729$.

Aliter If we take any four consecutive even numbers and divide them by 8, we get remainders 0, 2, 4, 6 in some order. Thus there is only one number of the form $8k + 4$ among them which is divisible by 4 but not by 8. Hence, if we take four even consecutive numbers

$$1000a + 100b + 10c + 2,$$
$$1000a + 100b + 10c + 4,$$
$$1000a + 100b + 10c + 6,$$
$$1000a + 100b + 10c + 8,$$

there is exactly one among these four which is divisible by 4 but not by 8. Now we can divide the set of all 4-digits even numbers with non-zero digits into groups of 4 such consecutive even numbers with a, b, c non-zero. And is each group, there is exactly one number which is divisible by 4 but not by 8. The number of such groups is precisely equal to $9 \times 9 \times 9 = 729$, since we can vary a, b, c in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Level 2

1. If the sum is x, then

$$x(10!) = {}_{10}C_1 + {}_{10}C_3 + {}_{10}C_3 + {}_{10}C_5 + {}_{10}C_7 + {}_{10}C_9.$$

Since the sum of the odd-positioned binomial coefficients is equal to the sum of the even-positioned ones in any line of Pascal's triangle (this should be known to students, and is established by expanding $(1 - 1)^n$), the above sum is half of ${}_{10}C_0 + {}_{10}C_1 + \dots + {}_{10}C_{10}$ (which equals 2^{10}). Thus this sum is 2^9, and $x = 2^9/10!$. We have $(a, b) = (9, 10)$.

2. Let E denote the set of these 1994 employees. For each $x \in E$, let $S(x)$ denote the set of all employees whom x does not know. Then, by assumption, $|S(x)| = 393$ for all $x \in E$. Let a and b be any two employees who know each other. Since

$$|S(a) \cup S(b)| \le 2 \times 393 = 786 < 1992,$$

$\exists c \in E$ such that a, b and c form a triple of mutual acquaintances, since

$$|S(a) \cup S(b) \cup S(c)| \le 3 \times 393 = 1179 < 1991,$$

$\exists d \in E$ such that, a, b, c and d form a quadruple of mutual acquaintances.

Since

$$|S(a) \cup S(b) \cup S(c) \cup S(d)| \le 4 \times 393 = 1572 < 1990,$$

$\exists e \in E$ such that, a, b, c, d and e form a quintuple of mutual acquaintances.

Finally, since

$$|S(a) \cup S(b) \cup S(c) \cup S(d) \cup S(e)| \le 5 \times 393 = 1965 \times 1989,$$

$\exists f \in E$ such that a, b, c, d, e and f from a sextuple of mutual acquaintances.

3. Let M_k denote the number of permutations $(p_1, \dots, p_k)$ of 1, 2, ..., k such that for any $i < k, (p_1, \dots, p_1)$ is not a permutation of 1, 2..., i. Then, $(n - k)$ M_k is the number of permutations $(p_1, \dots, p_n)$ of 1, 2, ... , n in which k is the least integer such that $(p_1, \dots, p_k)$ is a permutation of 1, 2,..., k. Hence,

$$\sum_{k=1}^{n-1} M_k(n-k)!$$

is the total number of permutations of (1, 2, ..., n) in which there is a $k < n$ such that $(p_1, \dots, p_k)$ is a permutation of 1, 2,..., k.

Hence,

$$M_n = n! \sum_{k=1}^{n-1} M_k(n-k)!$$

We need M_6 clearly

$$M_1 = 1$$
$$M_2 = 2! - 1 = 1$$
$$M_3 = 3! - (2 \cdot 1 + 1 \cdot 1) = 3$$
$$M_4 = 4! - (3 + 2 + 6) = 13$$
$$M_5 = 5! - (13 + 2 \cdot 3 + 6 \cdot 1 + 24) = 71$$

$$M_6 = 6! - (71 + 2 \cdot 13 + 6 \cdot 3 + 24 \cdot 1 + 120 \cdot 1)$$
$$= 461$$

4. We use a combinatorial argument to establish the obviously equivalent identity

$$\sum_{d=1}^{\infty} \binom{n-r+1}{d}\binom{r-1}{r-d} = \binom{n}{r} \quad \ldots(*)$$

where $k = \min(r, n-r+1)$. It clearly suffices to demonstrate that the left hand side of (i) counts the number of ways of selecting r objects from n distinct object (without replacements). Let $|S_2| = r-1$. For each fixed $d = 1, 2, \ldots, k$, any selection of d object from $S_1 (S/S_2)$ together with any selection of $r-d$ objects from S_2 would yield a selection of r objects from S. The total number of such electrons is $\binom{n-r+1}{d}\binom{r-1}{r-d}$. Conversely, each selection of r objects from S clearly much arise in this manner.

Summing over $d = 1, 2, \ldots$ (*) follows.

5. With $k + n \le m$, we represent $\binom{m+n}{n+k}$ by taking j element from n, and the remaining $(n+k) - j$ elements from m, for every j $0 \le j \le n$. Thus

$$\binom{m+n}{n+k} = \sum_{j=0}^{n} \binom{n}{j}\binom{m}{n+k-j}$$
$$= \sum_{i=0}^{n} \binom{n}{n-i}\binom{m}{k+i}$$
$$= \sum_{i=0}^{n} \binom{n}{i}\binom{m}{k+i},$$

where we substituted $i = n - j$, and used $\binom{n}{i} = \binom{n}{n-a}$

6. We will show that we can take $n_0 = 9$. For $n_0 \ge 9$, observe that for each girl there must be at least 5 boys whom she does not know. We associate to each girl an ordered pair, the first element of which is a subset of 5 of, the boys all of whom she knows or all of whom she does not know, and the second element of which is 0 or 1 according as she knows the boys or not.

There are $\binom{9}{5} \times 2 = 252$ such pairs. Invoking the Pigeonhole Principle, if $m_0 \ge 4 \times 252 + 1 = 1009$, at least 5 girls must be assigned the same ordered pair, producing 5 girls and 5 boys for which each girl knows each boy, or no girl knows any of the boys.

7. (a) Let us denote the digits in the hundreds, tens and ones places of the sought-for number N as x, y, and z respectively; then we have $N = 100x + 10y + z$. The condition of the problem yields the relation

$$100x + 10y + z = x! + y! + z!$$

Since, $7! = 5040$ is a four-digit number, none of the digits of the number N can exceed 6. Consequently, the number N itself does not exceed 700, whence it follows that none of its digits can exceed 5 (because $6! = 720 > 700$). Further, at least one digit of the number N is equal to 5 because N is a three-digit number and $3 \cdot 4! = 72 \times 100$. It is clear that x cannot be equal to 5 since we have $3 \cdot 5! = 360 < 500$. It also follows that x cannot exceed 3. Further, we can assert that x does not exceed 2 since $3! + 2 \cdot 5! = 246 < 300$. Further, the number 255 does not satisfy the condition of the problem, and if only one digit of the sought-for number is equal to 5, then x cannot exceed 1 because $2! + 5! + 4! = 146 < 200$. Moreover, since $1! + 5! + 4! = 145 < 150$ we conclude that y cannot exceed 4; consequently z is equal to 5 because at least one of the digits of the number N must be equal to 5. Thus, we have $x = 1$, $4 \ge y > 0$ and $z = 5$, which allows us to easily find the single solution of the problem: $n = 145$.

(b) The sought-for number N cannot consist of more three digits because even $4 \cdot 9^2 = 324$ is a three-digit number. This allows us to write $N = 100x + 10y + z$ where x, y and z are the digits of the number N; here x can be equal to 0 and it is even possible that x and y are simultaneously equal to 0.

The condition of the problem implies $100x + 10y + z = x^2 + y^2 + z^2$ whence

$$(100 - x)x + (10 - y)y = z(z-1) \quad \ldots(*)$$

From the last equality if follows that $x = 0$ because, if otherwise, the number on the left-hand side of the equality would not be less than 90 (in case $x \ge 1$ we have $100 - x \ge 90$ and $(10 - y)y \ge 0$) whereas the

number on the right-hand side would not be greater than $9 \cdot 8 = 72$ (since $z \le 9$). Consequently, Eq. (*) has the form $(10-y)y = z(z-1)$. It can easily be verified that the last equality cannot be fulfilled for any positive integers z and y not exceeding 9 unless $y \ne 0$. If $y = 0$ we have a single solution: it is obvious that in this case $z = 1$.

Thus, the only number satisfying the condition of the problem is $N = 1$.

8. Since a_{jj} had to exceed all the numbers in the top left $j \times j$ submatrix (excluding itself), and since there are $j^2 - 1$ entries, we must have $a_{jj} \ge j^2$. Similarly, a_{jj} must not exceed each of the numbers in the bottom right $(n-j+1)\times(n-j+1)$ submatrix (other than itself) and there are $(n-j+1)^2 - 1$ such entries giving $a_{jj} \le n^2 - (n-j+1)^2 + 1$. Thus we see that

$a_{jj} \in \{j^2, j^2+1, j^2+2, \ldots, n^2-(n-j+1)^2+1\}$.

The number of elements in this set is $n^2 - (n-j+1)^2 - j^2 + 2$. This implies that

$$b_j \le n^2 - (n-j+1)^2 - j^2 + 2$$
$$= (2n+2)j - 2j^2 - (2n-1)$$

It follows that

$$\sum_{j=1}^{n} b_j \le (2n+2)\sum_{j=1}^{n} j - 2\sum_{j=1}^{n} j^2 - n(2n-1)$$
$$= (2n+2)\left(\frac{n(n+1)}{2}\right) - 2\left(\frac{n(n+1)(2n+1)}{6}\right) - n(2n-1)$$
$$= \frac{n}{3}(n^2 - 3n + 5)$$

which is the required bound.

9. Any set of 100 lines in the plane can be partitioned into a finite number of disjoint sets, say $A_1, A_2, A_3, \ldots, A_k$, such that

(i) Any two lines is each A_j are parallel to each other, for $1 \le j \le k$ (provided, of course, $|A_j| \ge 2$)

(ii) for $j \ne l$, the lines in A_j and A_l are not parallel.

If $|A_j| = m_j$, $1 \le j < k$, then the total number of points of intersection is given by $\sum_{1 \le j \le l < k} m_j m_l$, as no three lines are concurrent,. Thus we have to find positive integers $m_1, m_2, \ldots, m_k$ such that

$$\sum_{j=1}^{k} m_j = 100, \sum m_j m_l = 2002,$$

for an affirmative answer to the given question.

We observe that

$$\sum_{j=1}^{k} m_j^2 = \left(\sum_{j=1}^{k} m_j\right)^2 - 2\left(\sum m_j m_l\right)$$
$$= 100^2 - 2(2002) = 5996$$

Thus we have to choose $m_1, m_2, \ldots, m_k$ such that

$$\sum_{j=1}^{k} m_j = 100, \sum_{j=1}^{k} m_j^2 = 5996$$

We observe that $[\sqrt{5996}] = 77$. So, we may take $m_1 = 77$, so that

$$\sum_{j=2}^{k} m_j = 23, \sum j = 2^k m_j^2 = 67$$

Now, we may choose

$$m_2 = 5, m_3 = m_4 = 4,$$
$$m_5 = m_6 = \ldots = m_{14} = 1$$

Finally, we can take

$$k = 14, (m_1, m_2, \ldots, m_{14})$$
$$= (77, 5, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

proving the existence of 100 lines with exactly 2002 points of intersection.

10. Clearly, the last digit must be 5 and we have to determine the remaining 6 digits. For divisibility by 7, it is sufficient to consider the number obtained by replacing 7 by 0; *e.g.*, 5775755 is divisible by 7, if and only if 5005055 is divisible by7. Each such number is obtained by adding some of the numbers from the set {50, 500, 5000, 50000, 500000, 5000000} along with 5. We look at the remainders of these when divided by 7; they are {1, 3, 2, 6, 4, 5}. Thus it is sufficient to check for those combinations of remainders which add up to a number of the from $2 + 7k$, since the last digit is already 5. These are {2}, {3, 6}, {4, 5}, {2, 3, 4}, {1, 3, 5}, {1, 2, 6}, {2, 3, 5, 6}, {1, 4, 5, 6} and {1, 2, 3, 4, 6}. These correspond to the numbers 7775775, 7757575, 5577775, 7575575, 5777555, 7755755, 5755575, 5557755, 755555.

11. The following sequence of moves lead to the colour of the ticket bearing the number 123123123

Line No.	Ticket No.	Colour	Reason
1	122222222	red	Given
2	222222222	green	Given
3	313113113	blue	Lines 1 & 2
4	231331331	green	Lines 1 & 3
5	331331331	blue	Lines 1 & 2
6	123123123	red	Lines 4 & 5

If 123123123 is reached by some other root, red colour must be obtained even along that root. For if for example 123123123 gets blue from some other root, then the following sequence leads to a contradiction

List No.	Ticket No.	Colour	Reason
1	122222222	red	Given
2	123123123	blue	Given
3	231311311	green	Lines 1 & 2
4	211331311	green	Lines 1 & 2
5	332212212	red	Lines 4 & 2
6	113133133	blue	Lines 3 & 5
7	331331331	green	Lines 1 & 2
8	222222222	red	Lines 6 & 7

Thus the colour of 22222222 is red contradicting that it is green.

12. We write the equation in the form

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}=\frac{1}{5}$$

The number of five tuple (a,b,c,d,e) which satisfy the given relation and for which $a \neq b$ is even, because for if (a,b,c,d,e) is a solution, then so is (b,a,c,d,e) which is distinct from (a,b,c,d,e). Similarly the number of five tuples which satisfy the equation and for which $c \neq d$ is also even. Hence it suffices to count only those five tuples (a,b,c,d,e) for which $a=b, c=d$. Thus the equation reduces to

$$\frac{2}{a}+\frac{2}{c}+\frac{1}{e}=\frac{1}{5}$$

Here again the tuple (a,a,c,c,e) for which $a \neq c$ is even because we can associate different solution (c,c,a,a,e) to this five tuple. Thus it suffices to consider the equation

$$\frac{4}{a}+\frac{1}{e}=\frac{1}{5}$$

and show that the number of pairs (a, e) satisfying this equation is odd.

This reduces to

$$ae = 20e + 5a$$

or $$(a-20)(e-5)=100$$

But observe that

$$\begin{aligned}100 &= 1\times 100 = 2\times 50\\ &= 4\times 25 = 5\times 20\\ &= 10\times 10 = 20\times 5 = 25\times 4\\ &= 50\times 2 = 100\times 1\end{aligned}$$

Note that no factorisation of 100 as product of two negative numbers yield a positive tuple (a,e). Hence we get these 9 solutions. This proves that the total number of five tuples (a,b,c,d,e) satisfying the given equation is odd.

13. I. Consider a 6-digit number whose digits from left to right are in non-increasing order. If 1 is the first digit of such a number, then the subsequent digits exceed 1. The set of all such numbers with initial digit equal to 1 is {100000, 110000, 111000, 111100, 111110, 111111}

There are elements in this set.

Let us consider 6-digit number with initial digit 2. Starting form 200000, we an up to 222222. We count these numbers as follows

200000 -	211111	: 6
220000 -	221111	: 5
2220000 -	222111	: 4
222200 -	222211	: 3
222220 -	222221	: 2
222222 -	222222	: 1

The number of such numbers is 21. Similarly, we count numbers with initial digit 3; the sequence starts from 300000 and with 333333. We have

300000	- 322222	: 21
330000	- 332222	: 15
333000	- 333222	: 10
333300	- 333322	: 6
333330	- 333332	: 3
333333	- 333333	: 1

we obtain the total number of numbers starting from 3 equal to 56. Similarly,

400000	- 433333	: 56
440000	- 443333	: 35
444000	- 444333	: 20
444400	- 444433	: 10
444440	- 444443	: 1
444444	- 444444	: 1
		126
500000	- 544444	: 126
550000	- 554444	: 70
555000	- 555444	: 35
555500	- 555544	: 15
555550	- 555554	: 5
555555	- 555555	: 1
		252
600000	- 655555	: 252
660000	- 665555	: 126
666000	- 666555	: 56
666600	- 666655	: 21
666660	- 666665	: 6
666666	- 666666	: 1
		142
700000	- 766666	: 462
770000	- 776666	: 210
777000	- 777666	: 84
777700	- 777766	: 28
777770	- 777776	: 7
777777	- 777777	: 1
		792

Thus, the number of 6-digit numbers where digits are non-digits are non-increasing starting from 100000 and ending with 777777 is

$$792 + 462 + 252 + 126 + 56 + 21 + 6 = 1715$$

Since $2005 - 1715 = 290$, we have to consider only 290 numbers in the sequence with initial digit 8. We have

800000	- 855555	: 252
860000	- 863333	: 35
864000	- 864110	: 3

II. It is known that the number of ways of choosing r objects from n different types of objects (with repetitions allowed) is $\binom{n+r-1}{r}$. In particular, if we want to write r-digit numbers using n digits allowing for repetitions with the additional condition that the digits appear in non-increasing order, we see that this can be done in $\binom{n+r-1}{r}$ ways.

Now we group the given numbers into different classes and write the number of ways in which each class can be obtained. To keep track we also write the cumulative sums of the number of numbers so obtained. Observe that the numbers themselves are written in ascending order. So, we exhaust numbers beginning with 1, then beginning with 2 and so on.

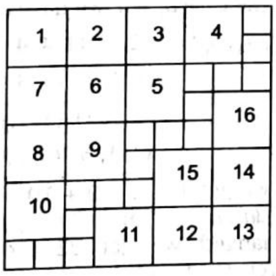
Numbers	Digits used other than the fixed part	n	r	$\binom{n+r-1}{r}$	Cumulative sum
beginning with 1	1, 0	2	5	$\binom{6}{5} = 6$	6
2	2, 1, 0	3	5	$\binom{7}{5} = 21$	27
3	3, 2, 1, 0	4	5	$\binom{8}{5} = 56$	83
4	4, 3, 2, 1, 0	5	5	$\binom{9}{5} = 126$	209

Numbers	Digits used other than the fixed part	n	r	$\binom{n+r-1}{r}$	Cumulative sum
5	5, 4, 3, 2, 1, 0	6	5	$\binom{10}{5} = 252$	461
6	6, 5, 4, 3, 2, 1, 0	7	5	$\binom{11}{5} = 462$	923
7	7, 6, 5, 4, 3, 2, 1, 0	8	5	$\binom{12}{5} = 792$	1715
form 800000 to 855555	5, 4, 3, 2, 1, 0	6	5	$\binom{10}{5} = 252$	1967
form 860000 to 863333	3, 2, 1, 0	4	4	$\binom{7}{5} = 35$	2002

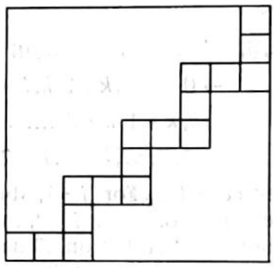
The next three 6-digit numbers are 864000, 864100, 864110.

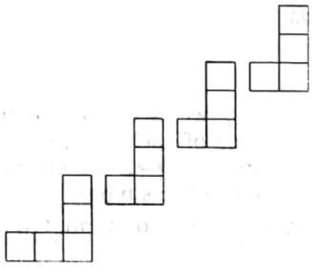
Hence, the 2005th number in the sequence is 864110.

14. Consider a partition of 9×9 chessboard using sixteen 2×2 block of 4 squares each and remaining seventeen single squares as shown in the figure below.



If any one of these 16 big squares contain 3 red squares then we are done. On the contrary, each may contain at most 2 red squares and these account for at most $16 \cdot 2 = 32$ red squares. Then there are 17 single squares connected in zig-zag fashion. It looks as follows





We split this again in to several mirror images of L-shaped figures as shown above. There are four such forks. If all the five units squares of the first fork are red, then we can get a 2×2 square having three red squares. Hence, there can be at most four unit squares having red colour. Similarly, there can be at most three red squares from each of the remaining three forks. Together we get

$4 + 3 \cdot 3 = 13$ red squares. These together with 32 from the big squares account for only 45 red squares. But we know that 46 squares have red colour. The conclusion follows.

15. In a permutation of $(1, 2, 3, \ldots, n)$, two inversions can occur in only one of the following two ways

(A) Two disjoint consecutive pairs are interchanged

$$(1, 2, 3, j-1, j, j+1, j+2, \ldots, k-1, k, k+1, k+2, \ldots, n)$$
$$\to (1, 2, \ldots, j-1, j+1, j, j+2, \ldots, k-1, k+1, k, k+2, \ldots, n)$$

(B) Each block of three consecutive integers can be permuted in any of the following 2 ways;

$$(1, 2, 3, \ldots k, k+1, k+2, \ldots, n) \to (1, 2, \ldots, k+2, k, k+1, \ldots, n);$$
$$(1, 2, 3, \ldots, k, k+1, k+2, \ldots, n) \to (1, 2, \ldots, k+1, k+2, k, \ldots, n).$$

Consider case (A). For $j = 1$, there are $n-3$ possible values of k; for $j = 2$, there are $n-4$ possibilities for k and so on. Thus, the number of permutations with two inversions of this type is

$$1 + 2 + \ldots + (n-3) = \frac{(n-3)(n-2)}{2}$$

In case (B), we see that there are $n-2$ permutations of each type, since k can take values from 1 to $n-2$. Hence, we get $2(n-2)$ permutations of this type.

Finally, the number of permutations with two inversions is

$$\frac{(n-3)(n-2)}{2} + 2(n-2) = \frac{(n+1)(n-2)}{2}.$$

16. Without loss of generality we may assume that a_1 is the largest among $a_1, a_2, a_3, a_4, a_5, a_6$. Consider the relation

$$a_1^2 - a_1a_2 + a_2^2 = a_2^2 - a_2a_3 + a_3^2.$$

This leads to

$$(a_1 - a_3)(a_1 + a_3 - a_2) = 0$$

Observe that $a_1 \geq a_2$ and $a_3 > 0$ together imply that the second factor on the left side is positive. Thus,

$$a_1 = a_3 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Using this and the relation

$$a_3^2 - a_3a_4 + a_4^2 = a_4^2 - a_4a_5 + a_5^2,$$

We conclude that $a_3 = a_5$ as above. Thus, we have

$$a_1 = a_3 = a_5 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Let us consider the other relations. Using

$$a_2^2 - a_2a_3 + a_3^2 = a_3^2 - a_3a_4 + a_4^2,$$

we get $a_2 = a_4$ or $a_2 + a_4 = a_3 = a_1$. Similarly, two more relations give either $a_4 = a_6$ or $a_4 + a_6 = a_5 = a_1$; and either $a_6 = a_2$ or $a_6 + a_2 = a_1$. Let us give values to a_1 and count the number of six-tuples in each case.

(A) Suppose $a_1 = 1$. In this case all a_j's are equal and we get only one six-tuple $(1, 1, 1, 1, 1, 1)$.

(B) If $a_1 = 2$, we have $a_3 = a_5 = 2$. We observe that $a_2 = a_4 = a_6 = 1$ or $a_2 = a_4 = a_6 = 2$. We get two more six-tuples $(2, 1, 2, 1, 2, 1)$, $(2, 2, 2, 2, 2, 2)$.

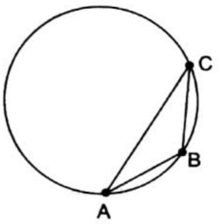
(C) Taking $a_1 = 3$, we see that $a_3 = a_5 = 3$. In this case we get nine possibilities for (a_2, a_4, a_6);
$(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)$.

(D) In the case $a_1 = 4$, we have $a_3 = a_5 = 4$ and
$(a_2, a_4, a_6) = (2, 2, 2), (4, 4, 4),$
$(1, 1, 1), (3, 3, 3), (1, 1, 3),$
$(1, 3, 1), (3, 1, 1), (1, 3, 3),$
$(3, 1, 3), (3, 3, 1)$.

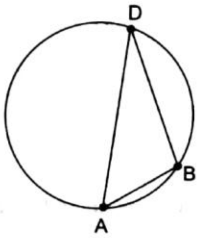
Thus, we get $1 + 2 + 9 + 10 = 22$ solutions. Since (a_1, a_3, a_5) and (a_2, a_4, a_6) may be interchanged we get 22 more six-tuples. However, there are 4 common among these, namely, $(1, 1, 1, 1, 1, 1)$, $(2, 2, 2, 2, 2, 2)$, $(3, 3, 3, 3, 3, 3)$ and $(4, 4, 4, 4, 4, 4)$. Hence, the total number of six-tuples is $22 + 22 - 4 = 40$.

17. Consider a circle of positive radius in the plane and inscribe a regular heptagon *ABCDEFG* in it. Since the seven vertices of this heptagon are coloured by three colours, some three vertices have the same colour, by pigeonhole principle. Consider the triangle formed by these three vertices. Let us call the part of the circumference separated by any two consecutive vertices of the heptagon an *arc*. The three vertices of the same colour are separated by *arcs* of length l, m, n as we move

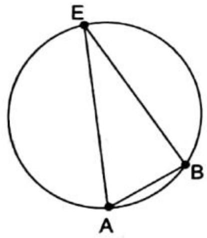
say, counter-clockwise, along the circle, starting from a fixed vertex among these three, where $l + m + n = 7$. Since, the order of l, m, n does not matter for a triangle, there are four possibilities: $1 + 1 + 5 = 7$; $1 + 2 + 4 = 7$; $1 + 3 + 3 = 7$; $2 + 2 + 3 = 7$. In the first, third and fourth cases, we have isosceles triangles. In the second case, we have a triangle whose angles are in geometric progression. The four corresponding figures are shown below.



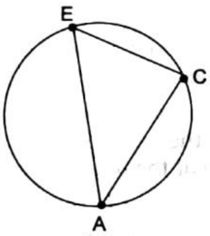
(i)



(ii)



(iii)



(iv)

In (i), $AB = BC$; in (iii), $AE = BE$; in (iv), $AC = CE$; and in (ii), we see that $\angle D = \pi/7$, $\angle A = 2\pi/7$, and $\angle B = 4\pi/7$, which are in geometric progression.

18. Suppose four distinct points P, Q, R, S (in that order on the circle) among these five are such that $\widehat{PQ} = \widehat{RS}$. Then, $PQRS$ is an isosceles trapezium, with $PS \parallel QR$. We use this in our argument.

- If four of the five points chosen are adjacent, then we are through as observed earlier. (In this case four points A, B, C, D are such that $\widehat{AB} = \widehat{BC} = \widehat{CD}$.) See Fig 1.
- Suppose only three of the vertices are adjacent, say A, B, C (see Fig 2.). Then, the remaining two must be among E, F, G, H. If these two are adjacent vertices, we can pair them with A, B or B, C to get equal arcs. If they are not adjacent, then they must be either E, G or F, H or E, H. In the first two cases, we can pair them with A, C to get equal arcs. In the last case, we observe that $\widehat{HA} = \widehat{CE}$ and $AHEC$ is an isosceles trapezium.

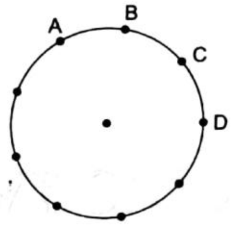


Fig. 1

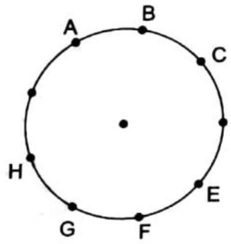


Fig. 2

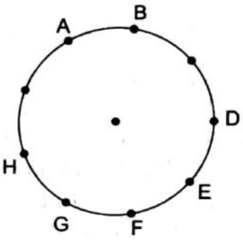


Fig. 3

- Suppose only two among the five are adjacent, say A, B. Then, the remaining three are among D, E, F, G, H. (See Fig 3.) If any two of these are adjacent, we can combine them with A, B to get equal arcs. If no two among these three vertices are adjacent, then they must be D, F, H. In this case $\widehat{HA} = \widehat{BD}$ and $AHDB$ is an isosceles trapezium.

  Finally, if we choose 5 among the 9 vertices of a regular nine-sides polygon, then some two must be adjacent. Thus, any choice of 5 among 9 must fall in to one of the above three possibilities.

Aliter Here, is another solution used by many students. Suppose you join the vertices of the nine-sides regular polygon. You get $\binom{9}{2} = 36$ line segments. All these fall in to 9 sets of parallel lines. Now, using any 5 points, you get $\binom{5}{2} = 10$ line segments. By pigeonhole principle, two of these must be parallel. But, these parallel lines determine a trapezium.

Unit 5 Geometry

Unit 5

Geometry

Congruent Triangles

Two triangles are congruent if and only if one of them can be made to superpose on the other so as to cover it exactly.

Some Elementary Theorem of Congruency (Without proof) which are often used

1. **Side Angle Side (SAS) Congruence Theorem**

   Two triangles are congruent, if two side and the included angle of one are equal to the corresponding sides and the included angle of the other triangle.

2. **Angle Side Angle (ASA) Congruence Theorem**

   Two triangles are congruent, if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

3. **Angle Angle Side (AAS) Congruence Theorem**

   If any two angles and a non-included side of one triangle are equal to the corresponding angles side of another triangle, then the two triangles are congruent.

4. **Side Side Side (SSS) Congruence Theorem**

   Two triangles are congruent, if the three side of one triangle are equal to the corresponding three side of the other triangle.

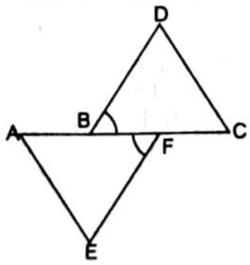
5. **Right Angle Hypotenuse Side (RHS) Congruence Theorem**

   Two right triangles are congruent, if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other triangle.

Note Angles opposite to equal sides of a triangle are equal.

Example 1 *In the given figure, $AB = CF$, $EF = BD$; $\angle AFE = \angle DBC$.*

Prove that $\Delta AFE \cong \Delta CBD$.



Solution $\therefore$ $AB = CF$

$\therefore$ $AB + BF = BF + FC$

$\Rightarrow$ $AF = CB$ ...(i)

In $\Delta\, AFE$ and $\Delta\, CBD$

$AF = CB$ [from Eq. (i)]

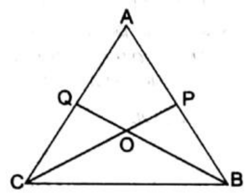
$EF = BD$ [given]

$\angle\, AFE = \angle\, DBC$ [given]

$\therefore$ $\Delta\, AFE \cong \Delta CBD$ [By SAA congruence rule]

Example 2 *P and Q are two points on equal sides AB and AC of an isosceles $\Delta\, ABC$ such that $AP = AQ$.*

Prove that $PC = QB$.



Solution $\because$ $AP = AQ$ and $AB = AC$

$\therefore$ $AB - AP = AC - AQ$

$\Rightarrow$ $PB = QC$ ...(i)

In Δs PBC and QBC, we have

$PB = QC$ [by Eq. (i)]

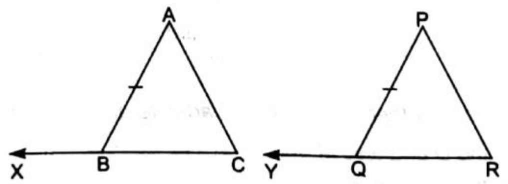
$BC = BC$ [Common]

$\angle\, PBC = \angle\, PCB$ [$\because$ $AB = AC$]

$\therefore$ $\Delta\, PBC \cong \Delta QBC$ [By SAS congruence rule]

$\therefore$ $PC = QB$

Example 3 *In $\Delta\, ABC$ and $\Delta\, PQR$, $AB = PQ$, $BC = QR$; CB and RQ are extended to X and Y respectively. $\angle\, ABX = \angle\, PQY$. Prove that $\Delta\, ABC \cong \Delta\, PQR$.*



Solution $\because$ $\angle ABX = \angle PQY$

$180° - \angle ABC = 180° - \angle PQR$ [$\because$ $\angle\, ABX + \angle\, ABC = 180°$;

$\angle\, PQY + \angle\, PQR = 180°$ (linear pair)]

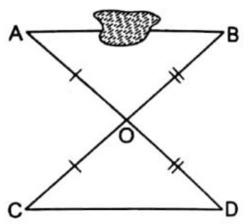
$\angle\, ABC = \angle\, PQR$ ...(i)

In $\Delta\ ABC$ and $\Delta\ PQR$

$$AB = PQ \qquad \text{(given)}$$
$$\angle\ ABC = \angle\ PQR \qquad \text{[from Eq. (i)]}$$
$$BC = QR \qquad \text{(given)}$$
$$\Delta\ ABC \cong \Delta\ PQR \qquad \text{[from Eq. (i)]}$$

[By SAS congruence rule]

Example 4 *Hari wishes to determine the distance between two objects A and B, but there is an obstacle between these two objects which prevent him from making a direct measurement. He devises an ingenlous way to overcome this difficulty. First he fixes a pole at a convenient point O so that from O, both A and B are visible. Then he fixes another pole at the point D on the line AO (produced) such that AO = DO. In a similar way he fixes a third pole at the point C on the line BO (produced) such that BO = CO. Then he measures CD which is equal to 170 cm. Prove that the distance between the objects A and B is also 170 cm.*

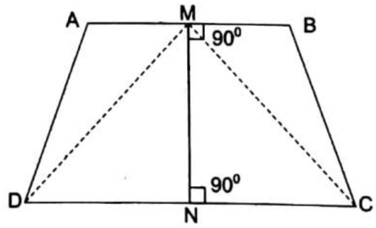


Solution In $\Delta s\ AOB$ and $\Delta\ COD$

$$OA = OD \qquad \text{(given)}$$
$$\angle\ AOB = \angle\ COD \qquad \text{(vertically opposite angles)}$$
$$OB = OC \qquad \text{(given)}$$
$$\therefore \qquad \Delta\ AOB \cong \Delta COD$$
$$\Rightarrow \qquad AB = CD \qquad \text{(c.p.c.t.)}$$
$$\Rightarrow \qquad AB = 170 \text{ cm} \qquad [\because\ CD = 170 \text{ cm}]$$

Hence proved.

Example 5 *The line segment joining the mid point M and N of opposite sides AB and DC of quadrilateral ABCD is perpendicular to both these sides.*



Prove that the other sides of the quadrilateral are equal.

Solution Join M and D and also M and C

in $\Delta\ CMN$ and $\Delta\ DMN$

$$DN = NC [N \text{ is mid point of } CD]$$
$$\angle\ DNM = \angle\ CNM = 90°$$
$$MN = MN \qquad \text{(common)}$$
$$\Delta\ CMN \cong \Delta\ DMN \qquad \text{[By SAS congruence rule]}$$

$\Rightarrow \quad CM = DM \quad$ ...(i)

$\angle DMN = \angle CMN$

and $\quad \angle AMN = \angle BMN \quad$ [90° each (given)]

$\Rightarrow \quad \angle AMN - \angle DMN = \angle BMN - \angle CMN$

$\Rightarrow \quad \angle AMD = \angle CMB \quad$ ...(ii)

In $\quad \Delta AMD$ and ΔCMB

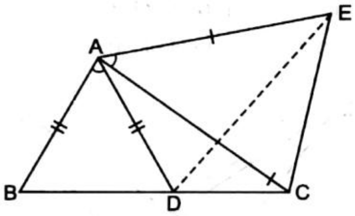
$AM = MB \quad$ [$\because$ M is mid point of AB]

$\angle AMD = \angle CMB \quad$ [from Eq. (ii)]

$DM = MC \quad$ [from Eq. (i)]

$\therefore \quad \Delta AMD \cong \Delta CMB \Rightarrow AD = BC$

Example 6 *In the given figure,*
$AC = AE$; $AB = AD$, $\angle BAD = \angle EAC$. Prove that $BC = DE$



Solution Join DE

$\angle BAD = \angle EAC$

$\Rightarrow \quad \angle BAD + \angle DAC = \angle EAC + \angle DAC \Rightarrow \angle BAC = \angle DAE \quad$ ...(i)

In ΔABC and ΔADE

$AB = AD \quad$ (given)

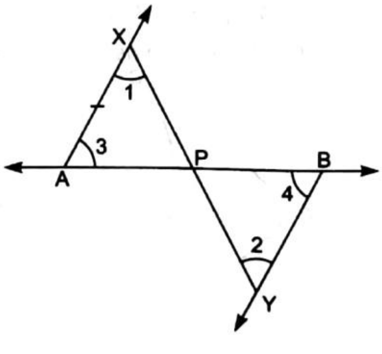
$\angle BAC = \angle DAE \quad$ [from Eq. (i)]

$AC = AE$

$\Delta ABC \cong \Delta ADE \quad$ [By SAS congruence rule]

$\Rightarrow \quad BC = DE \quad$ [c.p.c.t]

Example 7 *AB is a line segment. AX and BY are two equal line segments drawn opposite sides of line AB such that $AX \parallel BY$.*



If line segments AB and XY intersect each other at point P. Prove that
(a) $\Delta\ APX \cong \Delta\ BPY$
(b) line segments AB and XY bisect each other at P.

Solution (a) $\because AX \parallel BY$ and XY is a transversal

$\angle 1 = \angle 2$ (Alternate interior angles) ...(i)

$\because$ $AX \parallel BY$ and AB is a transversal

$\angle 3 = \angle 4$ (Alternate interior angles) ...(ii)

In $\Delta\ APX$ and $\Delta\ PBY$

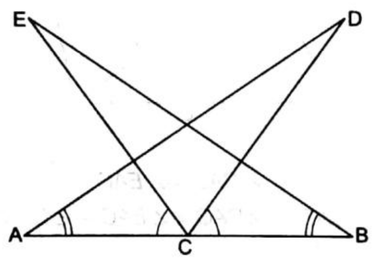
$\angle 1 = \angle 2$ [from Eq. (i)]
$AX = BY$ (given)
$\angle 3 = \angle 4$ [By Eq. (ii)]
$\therefore$ $\Delta\ APX \cong \Delta\ BPY$ (ASA rule)

(b) $\because$ $\Delta\ APX \cong \Delta\ BPY$

$AP = PB$ and $PX = PY$ [c.p.c.t.]

$\therefore$ AB and XY bisect each other at P.

Example 8 *In the figure, C is the mid point of AB, $\angle\ BAD = \angle\ CBE$, $\angle\ ECA = \angle\ DCB$. Prove that*

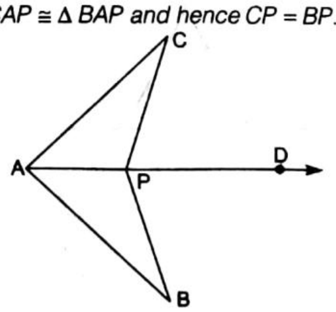


(a) $\Delta\ DAC \cong \Delta\ EBC$ *(b) $DA = EB$*

Solution In $\Delta\ DAC$ and $\Delta\ EBC$

$AC = CB$ [$\because$ C is mid point of AB]
$\angle\ CAD = \angle\ CBE$ (given)
$\angle\ ACD = \angle\ BCE$
[$\because \angle\ ACD = 180° - \angle\ DCB$, $\angle\ BCE = 180° - \angle\ ECA$ and $\angle\ DCB = \angle\ ECA$]
$\Delta\ DAC \cong \Delta\ EBC$ [By AAS rule]
$DA = EB$ (c.p.c.t.)

Example 9 *In the figure, $\angle\ CPD = \angle\ BPD$, AD is bisector of $\angle\ BAC$. Prove that $\Delta\ CAP \cong \Delta\ BAP$ and hence $CP = BP$.*



Solution

$$\angle BPD = \angle CPD$$

$\Rightarrow$ $$180° - \angle BPA = 180° - \angle CPA$$

$\Rightarrow$ $$\angle BPA = \angle CPA \quad \text{...(i)}$$

In $\Delta s\ BAP$ and ΔCAP, we have

$$\angle BAP = \angle CAP \quad [\because AD \text{ is bisector of } \angle BAC]$$

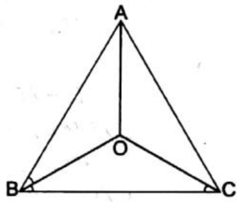
$$AP = AP \quad \text{(common)}$$

$$\angle BPA = \angle CPA \quad \text{[from Eq. (i)]}$$

$\therefore$ $$\Delta BAP \cong \Delta CAP \quad \text{[By ASA congruence rule]}$$

$\therefore$ $$BP = CP \quad \text{(By c.p.c.t.)}$$

Example 10 *In $\Delta\ ABC$, $AB = AC$. Bisectors of angles B and C intersect at point O. Prove that $BO = CO$ and the ray AO is bisector of $\angle BAC$.*



Solution In $\Delta\ ABC$,

$$AB = AC$$

$\Rightarrow$ $$\angle B = \angle C \quad [\because \text{Angle opposite to equal sides are equal}]$$

$\Rightarrow$ $$\frac{1}{2}\angle B = \frac{1}{2}\angle C$$

$\Rightarrow$ $$\angle ABO = \angle ACO \quad \text{...(i)}$$

[$\because$ OB and OC are bisectors of $\angle B$ and C respectively]

$\therefore$ $$\angle OBC = \frac{1}{2}\angle B$$

and $$\angle OCB = \frac{1}{2}\angle C$$

$\Rightarrow$ $$OB = OC \quad \text{...(ii)}$$

[$\because$ sides opposite to equal angles are equal]

In $\Delta\ ABO$ and $\Delta\ ACO$

$$AB = AC \quad \text{(given)}$$

$$\angle ABO = \angle ACO \quad \text{[from Eq. (i)]}$$

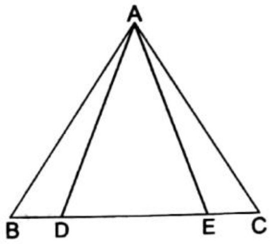
$$OB = OC \quad \text{[from Eq. (ii)]}$$

$$\Delta\ ABO \cong \Delta\ ACO \quad \text{[By SAS rule]}$$

$\Rightarrow$ $$\angle BAO = \angle CAO \quad \text{[c.p.c.t.]}$$

$\Rightarrow$ AO is the bisector of $\angle BAC$.

Example 11 *In the given figure, $AD = AE$. D and E are points on BC such that $BD = EC$. Prove that $AB = AC$.*



Solution In $\Delta\ ADE$

$$AD = AE$$
$$\angle ADE = \angle AED$$
$$\Rightarrow \quad 180° - \angle ADE = 180° - \angle AED$$
$$\Rightarrow \quad \angle ADB = \angle AEC \quad \ldots(i)$$

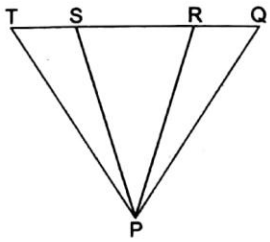
In $\Delta\ ABD$ and $\Delta\ ACE$

$$AD = AE \quad \text{(given)}$$
$$\angle ADB = \angle AEC \quad \text{[from Eq. (i)]}$$
$$BD = EC$$
$$\Delta\ ABD \cong \Delta\ ACE \quad \text{[By ASA rule]}$$
$$\Rightarrow \quad AB = AC \quad \text{[c.p.c.t.]}$$

Example 12 *In figure,*

$$PS = PR,$$
$$\angle TPS = \angle QPR$$

Prove that $PT = PQ$



Solution In $\Delta\ PRS$

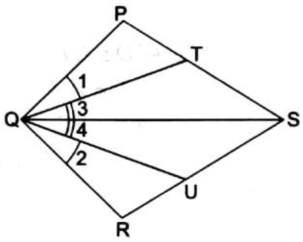
$$PS = PR$$
$$\Rightarrow \quad \angle PRS = \angle PSR \quad [\because \text{Angle opposite to equal sides are equal}]$$
$$\Rightarrow \quad 180° - \angle PRS = 180° - \angle PSR$$
$$\Rightarrow \quad \angle PRQ = \angle PST \quad \ldots(i)$$

In $\Delta\ PST$ and $\Delta\ PRQ$

$$\angle TPS = \angle QPR \quad \text{(given)}$$
$$PS = PR \quad \text{(given)}$$

$$\angle PST = \angle PRQ \quad \text{[from Eq. (i)]}$$
$$\Delta PST \cong \Delta PRQ \quad \text{[By ASA rule]}$$
$$\Rightarrow \quad PT = PQ$$

Example 13 *PQRS is a quadrilateral. T and U are points on PS and RS respectively such that $PQ = RQ$. $\angle PQT = \angle RQU$ and $\angle TQS = \angle UQS$. Prove that $QT = QU$.*



Solution In Δ *PQS* and *RQS*,

$$QS = QS \quad \text{(common)}$$
$$\angle PQS = \angle RQS \quad [\because \angle PQS = \angle 1 + \angle 3 = \angle 2 + \angle 4 = \angle RQS]$$
$$PQ = QR$$
$$\therefore \quad \Delta PQS \cong \Delta RQS \quad \text{[By SAS rule]}$$
$$\therefore \quad \angle QPS = \angle QRS$$
$$i.e., \quad \angle QPT = \angle QRU$$

In Δ *PQT* and Δ *QRU*,

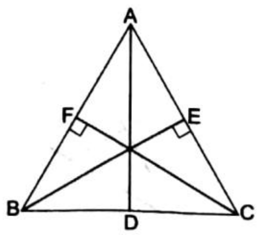
$$\angle 1 = \angle 2$$
$$\angle QPT = \angle QRU$$
$$PQ = QR$$
$$\therefore \quad \Delta PQT \cong \Delta QRU \quad \text{(By c.p.c.t.)}$$
$$\therefore \quad QT = QU$$

Concept

Some Properties of An Isosceles Triangle (without proof)

1. If two angles of a triangle are equal, then sides opposite to them are also equal.
2. If the altitude from one vertex of a triangle bisects the opposite side, then the triangle is isosceles.
3. In an isosceles triangle , altitude from the vertex bisects the base.
4. If the bisector of the vertical angle of a triangle bisects the base of the triangle , then the triangle is isosceles.

Example 1 *The altitudes of Δ ABC, AD, BE and CF are equal. Prove that Δ ABC is an equilateral triangle .*



Solution In right angled triangles BCE and BCF

$BC = BC$ (common)

$BE = CF$ (given)

$\therefore \quad \Delta BCE \cong \Delta BCF$ [By RHS rule]

$\therefore \quad \angle B = \angle C$ [c.p.c.t]

$\Rightarrow \quad AC = AB$ ...(i)

[$\because$ side opposite to equal angles are equal]

Similarly, $\Delta ABD \cong \Delta ABE$

$\therefore \quad \angle B = \angle A$

$\Rightarrow \quad AC = BC$ ...(ii)

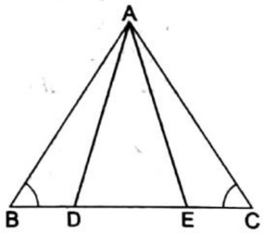
[$\because$ side opposite to equal angles are equal]

From Eqs. (i) and (ii),

$$AB = BC = AC$$

$\therefore$ ΔABC is an equilateral triangle.

Example 2 *In figure, $AB = AC$. D and E are points on BC such that $BD = EC$ (or $BE = CD$). Prove that $AD = AE$.*



Solution $\because \quad AB = AC$

$\therefore \quad \angle ABC = \angle ACB$

$\angle B = \angle C$

In ΔABD and ΔAEC

$AB = AC$ (given)

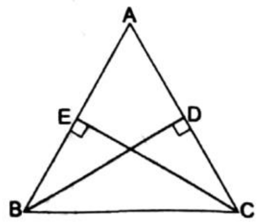
$BD = EC$ (given)

$\angle B = \angle C$ (Proved)

$\therefore \quad \Delta ABD \cong \Delta AEC$

$\therefore \quad AD = AE$ [c.p.c.t.]

Example 3 *ΔABC is isosceles with $AB = AC$. Prove that altitudes BD and CE of triangle are equal.*



Solution In $\Delta\, ABC$

$$\angle ABC = \angle ACB \qquad [\because AB = AC]$$

i.e., $\angle CBE = \angle BCD$

In Δ *s* BCE and $\Delta\, BCD$

$$\angle CBE = \angle BCD \qquad \text{[from Eq. (i)]}$$

$$\angle CEB = \angle CDB \qquad (90^\circ \text{ each})$$

$$\angle BCE = \angle CBD$$

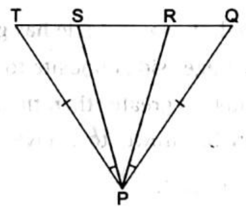
$[\because \angle BCE = 90^\circ - \angle B;\ \angle CBD = 90^\circ - \angle C,\ \angle B = \angle C]$

$$BC = BC \text{ (common)}$$

$\therefore$ $\Delta\, BCD \cong \Delta\, BCE$ [By AAS rule]

$\therefore$ $BD = CE$ [c.p.c.t.]

Example 4 *In the figure, $PQ = PT$, $\angle TPS = \angle QPR$. Prove that $\Delta\, PRS$ is isosceles*



Solution In $\Delta\, PQT$

$$PQ = PT$$

$\therefore$ $\angle TQP = \angle QTP$

In $\Delta\, PQR$ and $\Delta\, PST$

$$PQ = PT$$

$$\angle TQP = \angle QTP$$

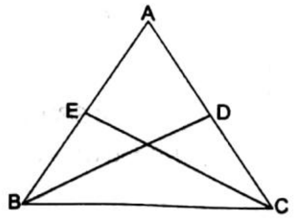
$$\angle QPR = \angle TPS \qquad \text{(given)}$$

$\therefore$ $\Delta\, PQR \cong \Delta\, PST$ (By AAS rule)

$\therefore$ $PS = PR$ [c.p.c.t.]

$\therefore$ $\Delta\, PRS$ is isosceles.

Example 5 *$\Delta\, ABC$ is an isosceles triangle with $AB = ACBD$ and CE are two medians of triangle. Prove that $BD = CE$*



Solution In $\Delta\ ABD$ and $\Delta\ AEC$

$AB = AC$ (given)

$AD = AE$

[$\because$ D is mid point of $AC \therefore AD = DC$ *i.e.,* $AD = \frac{1}{2}AC$]

$\angle BAC = \angle CAB$ (common)

[$\because E$ is mid point of AB, $\therefore$ $AE = EB$

i.e., $AE = 1/2\ AB$,

$\because$ $AB = AC \therefore AD = AE$]

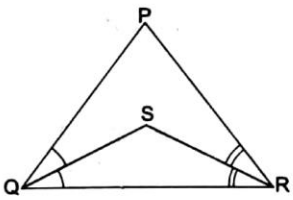
$\therefore$ $\Delta\ ABD \cong \Delta\ AEC$ (By SAS rule)

$\therefore$ $BD = CE$ [c.p.c.t.]

Some Inequality Relations in a Triangle (Without Proof)

1. If two sides of a triangle are unequal, the longer side has greater angle opposite to it.
2. In a triangle, the greater angle has longer side opposite to it.
3. The sum of any two sides of a triangle is greater than the third side.
4. Of all the line segments that can be drawn to a given line from a point not lying on it the perpendicular line segment is the shortest.

Example 1 *In figure $PQ > PR$. QS and RS are the bisectors of $\angle Q$ and $\angle R$ respectively. Prove that $SQ > SR$.*



Solution In $\Delta\ PQR$

$PQ > PR$ (given)

$\Rightarrow$ $\angle PRQ > \angle PQR$

[Angle opposite to greater side of a triangle is greater]

$\Rightarrow$ $\frac{1}{2}\angle PRQ$

$> \frac{1}{2}\angle PQR$

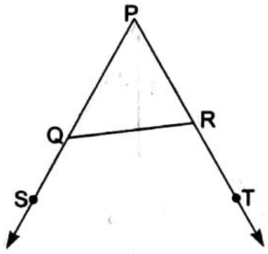
$\Rightarrow$ $\angle SRQ > \angle SQR$

[$\because$ RS and QS are bisectors of $\angle PRQ$ and $\angle PQR$ respectively]

$\Rightarrow$ $SQ > SR$

(Side opposite to greater angle is greater)

Example 2 *In figure PQ and PR are produced and* $\angle SQR < \angle TRQ$.



Prove that $PR > PQ$.

Solution

$$\left.\begin{aligned}\angle PQR + \angle SQR &= 180^\circ\\ \angle PRQ + \angle TRQ &= 180^\circ\end{aligned}\right\}\text{linear pair}$$

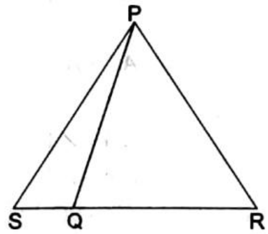
$\therefore$ $\angle PQR + \angle SQR = \angle PRQ + \angle TRQ$

But $\angle SQR < \angle TRQ$

$\therefore$ $\angle PQR < \angle PRQ$

$\therefore$ $PR > PQ$ [Side opposite to greater angle is greater]

Example 3 *Q is a point on side RS of* ΔPSR *such that* $PQ = PR$. *Show that* $PS > PQ$



Solution In $\Delta PQR, PQ = PR$

$\therefore$ $\angle PRQ = \angle PQR$ [Angles opposite to equal sides are equal] ...(i)

In $\Delta PSQ, SQ$ is produced to R

ext. $\angle PQR > \angle PSQ$ ...(ii)

[exterior angle of a triangle > each of interior opposite angles]

From Eqs. (i) and (ii)

$\angle PRQ = \text{ext. } \angle PQR > \angle PSQ$

i.e., $\angle PRQ > \angle PSQ$

i.e., $\angle PRS > \angle PSR$ [$\because \angle PRQ = \angle PRS$ and $\angle PSQ = \angle PSR$]

In ΔPSR,

$\angle PRS > \angle PSR$

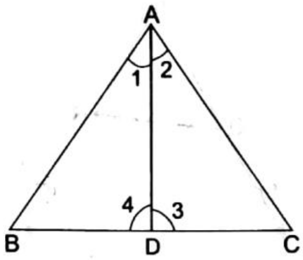
$\therefore$ $PS > PR$

[$\because$ side opposite to greater angle is greater]

$PR = PQ$

$\therefore$ $PS > PQ$

Example 4 *AD is the bisector of $\angle A$ of ΔABC, D lies on BC. Show that $AB > BD$ and $AC > CD$.*



Solution In ΔABC

AD is bisector of $\angle A$

$\therefore \quad \angle 1 = \angle 2 \quad$ ...(i)

But in ΔABD

ext. $\quad \angle ADC > \angle 1$

[exterior $\angle$ of a triangle is greater than each of interior opposite angles]

i.e., $\quad \angle 3 > \angle 1$

$\therefore \quad \angle 3 > \angle 2 \quad$ [from Eq. (i)]

$\therefore$ from ΔADC

$AC > AD$

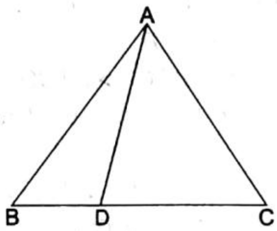
[$\because$ Side opposite to greater angle is greater]

Also $\quad \angle 4 > \angle 2 \quad$ [In ΔADC, $\angle 4$ is exterior angle]

$\angle 4 > \angle 1$

$AB > BD$ [$\because$ Side opposite to greater angle is greater]

Example 5 *In figure, $AB > AC$. D is any point on side BC of ΔABC. Show that $AB > AD$.*



Solution $\because \quad AB > AC$

$\therefore \quad \angle ACB > \angle ABC \quad$ ...(i)

[$\because$ Angle opposite to greater side is greater]

In ΔACD, CD is produced to B forming an exterior angle ADB

$\therefore \quad \angle ADB > \angle ACD$

[$\because$ exterior angle of a triangle is greater than each of interior opposite angles]

$\Rightarrow \quad \angle ADB > \angle ACB \quad$ [$\because \angle ACD = \angle ACB$] ...(ii)

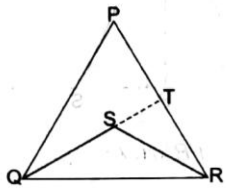
from Eqs. (i) and (ii)

$$\angle ADB > \angle ABC$$

$\Rightarrow$ $\angle ADB > \angle ABD$ [$\because \angle ABC = \angle ABD$]

$\Rightarrow$ $AB > AD$ [$\because$ side opposite to greater angle is greater]

Example 6 *S is any point in the interior of ΔPQR. Show that $SQ + SR < PQ + RP$*



Solution Produce QS to meet PR at T.

In ΔPQT

$$PQ + PT > QT$$

[$\because$ Sum of two sides of a triangle is greater than third side]

$\Rightarrow$ $PQ + PT > QS + ST$ [$\because QT = QS + ST$] ...(i)

In ΔRST

$$ST + TR > SR \quad \text{...(ii)}$$

On adding Egs. (i) and (ii) gives

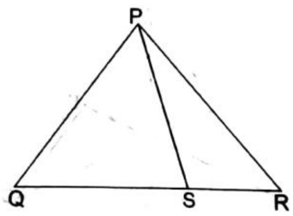
$$PQ + PT + ST + TR > SQ + ST + SR$$

$\Rightarrow$ $PQ + (PT + TR) > SQ + SR$

$\Rightarrow$ $PQ + PR > SQ + SR$

$\therefore$ $SQ + SR < PQ + PR$

Example 7 *S is any point on the side QR of ΔPQR. Show that $PQ + QR + RP > 2PS$.*



Solution In ΔPQS

$$PQ + QS > PS \quad \text{...(i)}$$

[$\because$ sum of two sides of a triangle is greater than third side]

In ΔPSR

$$RP + RS > PS \quad \text{...(ii)}$$

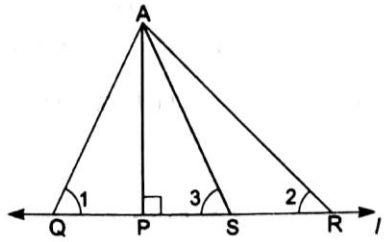
On adding Eqs. (i) and (ii) , we get

$$PQ + QS + RP + RS > 2PS$$

$$PQ + (QS + RS) + RP > 2PS$$

$PQ + QR + RP > 2PS$ [$\because QS + RS = QR$]

Example 8 *In figure, $AP \perp l$ i.e., AP is the shortest line segment that can be drawn from A to the line l. If $PR > PQ$. Show that $AR > AQ$*



Solution Cut off $PS = PQ$ from PR join AS.

In $\Delta\ APQ$ and $\Delta\ APS$

$$\angle APQ = \angle APS \quad \text{(each } 90°\text{)}$$
$$AP = AP \quad \text{(common side)}$$
$$PQ = PS$$

$\therefore$ $\Delta\ APQ \cong \Delta\ APS$

$AP = AS$ and $\angle 1 = \angle 3$ ...(i)

In ΔARS, $\angle 3 > \angle 2$ [exterior angle property] ...(ii)

from Eqs. (i) and (ii)

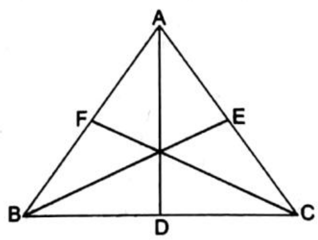
$$\angle 1 > \angle 2$$

from $\Delta\ AQR$ $\angle 1 > \angle 2$

$\therefore$ $AR > QR$

$\therefore$ $AR > AQ$

Example 9 *Prove that perimeter of a triangle is greater than sum of its three medians.*



Solution In a triangle ABC let, perimeters is equal to $AB + BC + AC$ and sum of medians is equal to $AD + BE + CF$. Now we know that sum of any two sides of a triangle is greater than the third side.

$\therefore$ $AB + AC > 2AD$ ... (i)

[$\because$ AD is median of $\Delta\ ABC$ through the point A]

Similarly,

$BC + BA > 2BE$ ...(ii)

and $CA + CB > 2CF$ ...(iii)

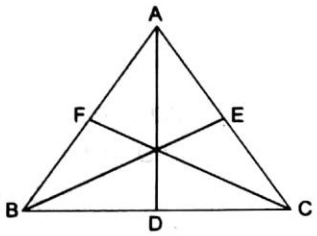
Adding Eqs. (i), (ii) and (iii), we get

$$(AB + AC) + (BC + BA) + (CA + CB) > 2$$
$$(AD + BE + CF)\, 2(AB + BC + AC) > 2\,(AD + BE + CF)$$

Hence, $AB + BC + AC > AD + BE + CF$

Example 10 *Show that the sum of the three altitudes of a triangle is less than the sum of the three sides of a triangle.*

Solution In a ΔABC let perimeter equal to $AB + BC + CA$ and sum of altitudes is equal to $AD + BE + CF$. Now since $AD \perp BC$



$\therefore$ $AD < AB,\ AD < AC$

$\therefore$ $AD + AD < AB + AC$

$2AD < AB + AC$ ...(i)

$\therefore$ $BE \perp CA$

$\Rightarrow$ $BE < BC,\ BE < BA$

$\Rightarrow$ $BE + BE < BC + BA$

$2BE < BC + BA$ ...(ii)

$\therefore$ $CF \perp AB$

$\therefore$ $CF < CA, CF < CB$

$\therefore$ $CF + CF < CA + CB$

$2\,CF < CA + CB$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$2\,(AD + BE + CF) < AB + AC + BC + BA + CA + CB$$
$$= 2\,AB + 2\,BC + 2\,CA = 2\,(AB + BC + CA)$$

Hence, $AD + BE + CF < AB + BC + CA$

Example 11 *$PR > PQ$, PS is the bisector of $\angle QPR$.*
Show that $x > y$.

Solution In ΔPQR, $PR > PQ\ \angle PQR > \angle PRQ$

[$\because$ Angle opposite to greater side is greater]

$\angle 1 = \angle 2$ [$\because$ PS is bisector of $\angle P$]

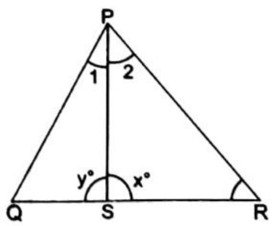
$\therefore$ $\angle PQR + \angle 1 > \angle PRQ + \angle 2$

$\Rightarrow$ $180° - y° > 180° - x°$

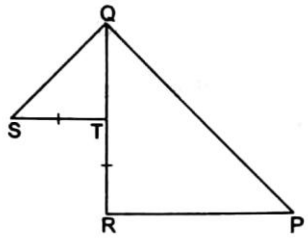
[$\because \angle PQS + \angle 1 + y° = 180°,\ \angle PRS + \angle 2 + x° = 180°$]

$\Rightarrow$ $-y° > -x°$

$\Rightarrow$ $y° < x°$ *i.e.*, $x° > y°$



Example 12 *T is a point on side QR of ΔPQR. S is a point such that $RT = ST$. Prove that $PQ + PR > QS$*



Solution In ΔRPQ

$$PQ + PR > RQ \qquad \text{...(i)}$$

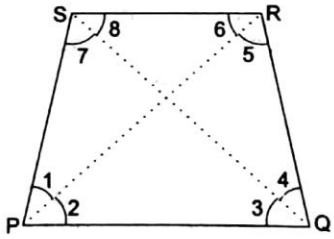
$$RQ = TR + TQ$$

$$= ST + TQ > QS \qquad \text{[from } \Delta STQ] \text{ ...(ii)}$$

from Eqs. (i) and (ii)

$$PQ + PR > QS$$

Example 13 *PQRS is a quadrilateral. PQ is its longest side. RS is the shortest side, prove that $\angle R > \angle P$ and $\angle S > \angle Q$.*



Solution Join PR and QS,

$\because$ PQ is longest side of quadrilateral $PQRS$

$\therefore$ In ΔPQR

$$PQ > QR$$

$\therefore$ $\angle 5 > \angle 2$ ... (i)

[$\because$ Angle opposite to greater side is greater]

$\because$ RS is smallest side of quadrilateral. $PQRS$

$\therefore$ $PS > RS$

$\therefore$ $\angle 6 > \angle 1$ ...(ii)

[$\because$ Angle opposite to greater side is greater]

On adding Eqs. (i) and (ii)

$$\angle 5 + \angle 6 > \angle 2 + \angle 1$$

$\Rightarrow$ $\angle R > \angle P$

Hence proved.

In ΔPQS, $PQ > PS$ [$\because$ PQ is longest side]

$\therefore$ $\angle 8 > \angle 3$ ...(iii)

In ΔSRQ, $RQ > RS$ [$\because$ RS is shortest side]

$\therefore$ $\angle 7 > \angle 4$ ...(iv)

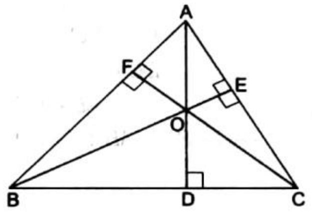
On adding Eqs. (iii) and (iv), we get

$$\angle 8 + \angle 7 > \angle 3 + \angle 4$$

$\Rightarrow$ $\angle S > \angle Q$

Hence proved.

Example 14 *In Δ ABC, AD$\perp$ BC; BE $\perp$ AC; CF $\perp$ AB. Prove that AD + BE + CF > AB + BC + CA.*



Solution In ΔABD, $AD < AB$ ...(i)

(Perpendicular line segment is shortest)

In ΔADC, $AD < AC$ ...(ii)

(Perpendicular line segment is shortest)

$2AD < AB + AC$ ...(iii)

[On adding Eqs. (i) and (ii)]

Similarly $2BE < AB + BC$ ... (iv)

$2CF < BC + AC$ ...(v)

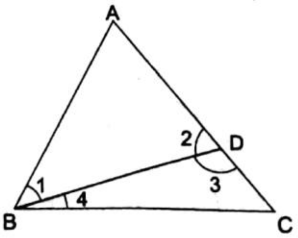
On adding Eqs. (iii), (iv) and (v), we have

$$2(AD + BE + CF) < 2(AB + BC + CA)$$

$\Rightarrow$ $AD + BE + CF < AB + BC + CA$

Example 15 *In Δ ABC, AC > AB*
Prove that AC − AB < BC, AC − BC < AB,
BC − AB < AC.

Solution From AC, cut $AD = AB$,



join $AB = AD$ [construction]

$\angle 2 = \angle 1$ ...(i)

[Base angle of an isosceles triangle]

$\angle 3 > \angle 1$ ...(ii)

[ext. angle of ΔABD]

$\angle 2 > \angle 4$ ...(iii)

[ext. angle of ΔBDC]

$\angle 3 > \angle 2$

[from Eqs. (i) and (ii)]

$\angle 3 > \angle 4$ [from Eqs. (ii) and (iii)]

In ΔBDC

$\because$ $\angle 3 > \angle 4$

$\therefore$ $BC > DC$

$\Rightarrow$ $DC < BC \Rightarrow AC - AD < BC$

$\Rightarrow$ $AC - AB < BC$ [$\because AD = AB$]

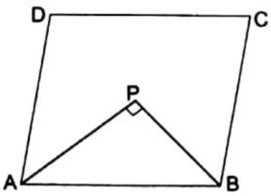
Similarly, $AC - BC < AB, BC - AB < AC$

Hence proved.

Properties of a Parallelogram

1. A diagonal of a parallelogram divides it into two congruent triangles.
2. In a parallelogram opposite sides are equal.
3. In a parallelogram the opposite angles are equal.
4. In a parallelogram the diagonals bisect each other.
5. In a quadrilateral, if opposite sides are equal, it is a parallelogram.
6. In a quadrilateral, if opposite angles are equal, it is parallelogram.
7. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
8. Each of the four sides of a rhombus is of same length.
9. Each of the angles of a square is a right angle and each of the four sides is of same length.
10. If the two diagonals of a parallelogram are equal, it is a rectangle.
11. The diagonal of a rhombus are perpendicular to each other.
12. If the diagonals of a parallelogram are perpendicular then it is a rhombus.
13. The diagonals of a square are equal and perpendicular to each other.

Example 1 *In a parallelogram, show that angle bisector of two adjacent angles intersect at right angles*



Solution In a parallelogram $ABCD$ $\angle A$ and $\angle B$ are adjacent angles. B sectors of $\angle A$ and $\angle B$ meet at the point P. Now

$\angle A + \angle B = 90°$ [sum of adjacent angles of a parallelogram]

$$\Rightarrow \quad \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90°$$

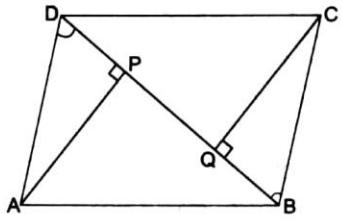
$$\Rightarrow \quad \angle PAB + \angle PBA = 90°$$

But $\quad \angle APB + \angle PBA + \angle PAB = 180°$ [sum of angles of a triangle]

$$\therefore \quad 90° + \angle APB = 180°$$

$$\angle APB = 90°$$

Example 2 *Let ABCD be a parallelogram. Let AP, CQ be the perpendicular from A and C on its diagonal BD. Prove that AP = CQ*



Solution From ΔAPD and ΔBQC

$AD = BC$ ($\because$ $ABCD$ is a parallelogram $\therefore$ $AD = BC$)

$\angle ADP = \angle CBQ$ [$\because$ $AD \parallel BC$ and BD is transversal]

$\angle APD = \angle CQB$ [each 90°]

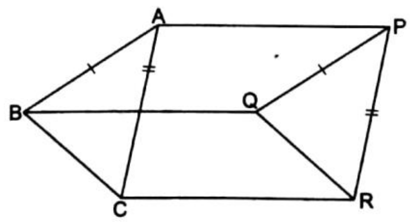
$\Delta APD \cong \Delta BQC$ [By AAS rule]

$\therefore \quad AP = CQ$

Example 3 $AB \parallel PQ, AB = PQ, AC \parallel PR$

$AC = PR$. *Prove that*

$BC \parallel QR$ and $BC = QR$.



Solution $\because \quad AB \parallel PQ, AB = PQ$

$\therefore \quad ABQP$ is a parallelogram

$\therefore \quad BQ = AP$ and $BQ \parallel AP$ ...(i)

$\because \quad AC \parallel PR, AB = PR$

$\therefore \quad ACRP$ is parallelogram

$\therefore \quad CR = AP$ and $CR \parallel AP$ ...(ii)

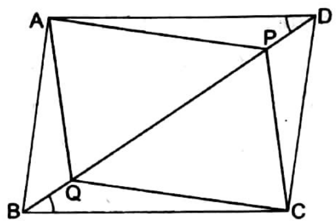
From Eqs. (i) and (ii)

$$BQ = CR \text{ and } BQ \parallel CR$$

$\therefore$ $BQRC$ is a parallelogram

$\therefore$ $BC \parallel QR$ and $BC = QR$

Example 4 *In a parallelogram ABCD, P and Q are two points taken on its diagonal BD such that DP = BQ. Prove that APCQ is a parallelogram.*



Solution In $\Delta\, APD$ and $\Delta\, CQB$

$DP = BQ$ (given)

$AD = BC$ [opposite sides of a parallelogram]

$\angle ADP = \angle CBQ$ (Alternate angles)

[$\because$ $AD \parallel BC$ and transversal BD meets them at D, P respectively]

$\therefore$ $\Delta\, APD \cong \Delta\, CQB$ [By SAS]

$AP = CQ$ [c.p.c.t.]

Similarly,

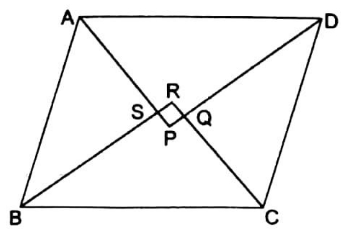
$CP = AQ$

$\therefore$ $APCQ$ is a parallelogram.

Example 5 *Prove that the angle bisector of a parallelogram forms a rectangle.*

Solution Let $ABCD$ is a parallelogram in which angle bisectors from a quadrilateral $PQRS$.

$\because$ AP is bisector of $\angle A$ and BR is bisector of $\angle B$ meeting at each other at point S.



$\therefore$ $\angle BAS + \angle ABS$

$$= \frac{1}{2}\angle A + \frac{1}{2}\angle B = \frac{1}{2} \times 180°$$

[$\because$ $AD \parallel BC$ and AB intersects them]

$\angle BAS + \angle ABS = 90°$

But $\angle BAS + \angle ABC + \angle ASB = 180°$

$\therefore$ $\angle ASB = 90°$

Thus $\angle RSP = 90^\circ$

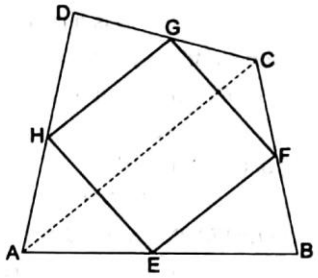
[$\because \angle RSP$ and $\angle ASB$ are vertically opposite angles]

Similarly, we prove that

$$\angle SRQ = \angle RQA = 90^\circ$$

Hence, $PQRS$ is a rectangle.

Example 6 *Prove that the figure formed by joining the mid points of the pairs of adjacent sides of a quadrilateral is a parallelogram.*



Solution $ABCD$ is a quadrilateral E, F, G, H are mid points of sides AB, BC, CD, DA respectively.

$PQRS$ is a parallelogram

Join A and C.

In $\Delta\, ABC, E$ and F are the mid points of sides AB and BC respectively

$$\therefore \qquad EF \parallel AC \text{ and } EF = \frac{1}{2} AC \qquad \ldots\text{(i)}$$

In $\Delta\, ADC, G$ and H are mid points of CD and AD respectively

$$HG \parallel AC \text{ and } RS = \frac{1}{2} AC \qquad \ldots\text{(ii)}$$

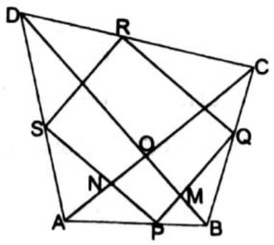
from Eqs. (i) and (ii), we get

$$EF \parallel HG \text{ and } EF \parallel RS$$

$\therefore$ $EFGH$ is a parallelogram.

Example 7 *The diagonals of a quadrilateral ABCD are perpendicular. Show that the quadrilateral formed by joining the mid points of its sides, is a rectangle.*

Solution A quadrilateral $ABCD$. AC and BD are its diagonals such that $AC \perp BD$



P, Q, R, S are mid points of sides AB, BC, CD, DA respectively.

Join PQ, QR, RS and SP

In $\Delta\ ABC$, P and Q are mid points of AB and BC respectively.

$\therefore \quad PQ \parallel AC$ and AC and $PQ = \frac{1}{2} AC$ ...(i)

In $\Delta\ ADC$, R and S are mid points of CD and AD respectively.

$\therefore \quad RS \parallel AC$ and $RS = \frac{1}{2} AC$ ...(ii)

$\therefore \quad PQ \parallel RS$ and $PQ = RS$ [from Eqs. (i) and (ii)]

$\therefore \quad PQRS$ is a parallelogram

Let the diagonals AC and BD meet at point O.

Now, in $\Delta\ ABD$, P is mid point of AB and S is mid point of AD.

$\therefore \quad PS \parallel BD$ i.e., $PN \parallel BD$

Also from Eq. (i), $QP \parallel AC$

$\therefore \quad PM \parallel NO$

In quadrilateral $PMON$, $PN \parallel MO$ and $PM \parallel NO$

$PMON$ is a parallelogram $\Rightarrow \angle MPN = \angle MON$

[$\because$ opposite angles of parallelogram are equal]

$\Rightarrow \quad \angle MPN = \angle BOA$ ($\because \angle BOA = \angle MON$)

$\Rightarrow \quad \angle MPN = 90°$ [$\because AC \perp BC, \therefore \angle BOA = 90°$]

$\Rightarrow \quad \angle QPS = 90°$

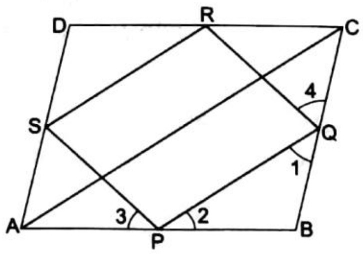
Hence $PQRS$ is a parallelogram with one angle

$\angle QPS = 90°$

$\therefore$ $PQRS$ is a rectangle.

Example 8 *ABCD is a rhombus. P, Q, R, S are mid points of AB, BC, CD, DA respectively. Prove that PQRS is a rectangle.*

Solution Join AC



In $\Delta\ ABC$, P and Q are mid points of AB and BC respectively

$\therefore \quad PQ \parallel AC$ and $PQ = \frac{1}{2} AC$ ...(i)

In $\Delta\ ADC$, R and S are mid points of CD and AD respectively

$\therefore \quad SR \parallel AC$ and $SR = \frac{1}{2} AC$ ...(ii)

from Eqs. (i) and (ii), we get

$$PQ \parallel SR \text{ and } PQ = SR$$

$\therefore$ $PQRS$ is a parallelogram.

Again $ABCD$ is a rhombus.

$\because$ $AB = BC$

[$\because$ All sides of a rhombus are equal]

$\therefore$ $\frac{1}{2}AB = \frac{1}{2}BC \Rightarrow PB = BQ$ ...(iii)

[$\because$ P and Q are mid points of AB, BC respectively]

In $\Delta PBQ, PB = PQ$

$\therefore$ $\angle 1 = \angle 2$ ...(iv)

[$\because$ Angle opposite to equal sides are equal]

$\because$ $ABCD$ is a rhombus.

$\therefore$ $AB = BC = CD = AD$

$\Rightarrow$ $AB = BC, CD = AD$

$\Rightarrow$ $\frac{1}{2}AB = \frac{1}{2}BC ; \frac{1}{2}CD = \frac{1}{2}AD$

$\therefore$ $AD = CQ, CR = AS$

In ΔAPS and ΔCQR

$AP = CQ$ [By Eqs. (iii)]

$AS = CR$ [By Eqs. (iv)]

$PS = QR$ [$\because$ $PQRS$ is parallelogram]

$\Delta APS \cong \Delta CQR$ [By SSS rule]

$\angle 3 = \angle 4$ [c.p.c.t.] ...(v)

Now, $\angle 3 + \angle SPQ + \angle 2 = 180°$

and $\angle 1 + \angle PQR + \angle 4 = 180°$

$\therefore$ $\angle 3 + \angle SPQ + \angle 2 = \angle 1 + \angle PQR + \angle 4$

$\Rightarrow$ $\angle SPQ = \angle PQR$ [$\because$ $\angle 1 = \angle 2, \angle 3 = \angle 4$] ...(vi)

Now transversal PQ cuts parallel lines SP and RQ at P, Q respectively.

$\therefore$ $\angle SPQ + \angle PQR = 180°$

$\Rightarrow$ $\angle SPQ + \angle SPQ = 180°$

$2\angle SPQ = 180°$

$\angle SPQ = 90°$

Hence, $PQRS$ is a parallelogram such that its one angle $\angle SPQ = 90°$.

Hence $PQRS$ is rectangle.

Similar Triangles (Ratio and Proportion)

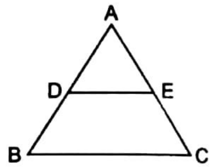
Two figures are said to be similar, if they are of same shape. For two polygons to be similar this requires that each angle of the one must be equal to each angle of the other and corresponding sides are proportional. In case of a triangle equality of angles turns out to be equivalent to proportionality of corresponding sides. Here are some facts regarding similar triangles.

1. In any triangle, a straight line drawn parallel to one of the sides divides the remaining sides proportionally and conversely.

   In the figure, if $DE \parallel BC$, then

   $$\frac{AD}{DB} = \frac{AE}{EC}$$

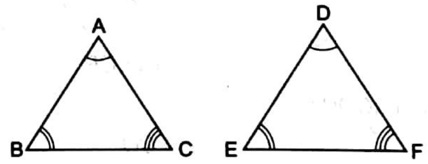
   and equivalently, $$\frac{AD}{AB} = \frac{AE}{EC}$$

   Conversely, if D and E are points in the sides AB and AC respectively, such that $\frac{AD}{DB} = \frac{AE}{EC}$, then $DE \parallel BC$

2. If in the two triangles, the sides of one triangle are proportional to those of the other then the corresponding angle of the two triangle s are equal.

   In figure, if $$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

   then $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$

3. If in two triangles the angles of one triangle are equal to those of the other, then side opposite to these angles are proportional. In the previous figure

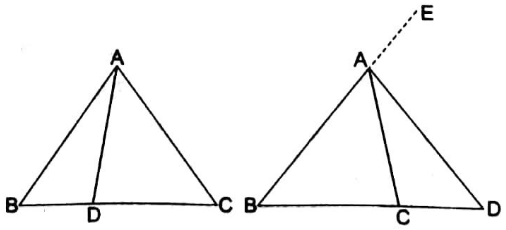
   if $$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$$

   then $$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

4. If in two triangles, one angle of one triangle is equal to one angle of the other triangle and the side containing these angles are proportional, then two triangle are similar.

   In the figure, if $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$, then $\Delta\, ABC$ and $\Delta\, DEF$ are similar.

5. The internal (respectively external) bisector of an angle of a triangle divides the opposite side internally (respectively externally) in the ratio of the sides containing the angle.

In figure (a), AD is the internal bisector of $\angle A$ Consequently $\frac{AB}{AC} = \frac{BD}{DC}$

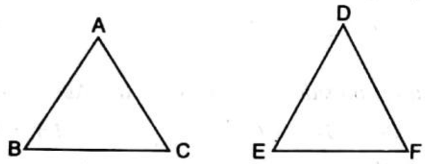
In figure (b), AD is the external bisector of $\angle A$ Consequently $\frac{AB}{AC} = \frac{BD}{DC}$

6. The areas of similar triangles are in the ratio of squares of the corresponding sides.

$$\Delta\, ABC \sim \Delta\, DEF$$

Correspondence being

$$A \leftrightarrow D,\ B \leftrightarrow E,\ C \leftrightarrow F$$



Consequently
$$\frac{[ABC]}{[DEF]} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2} = \frac{BC^2}{EF^2}$$

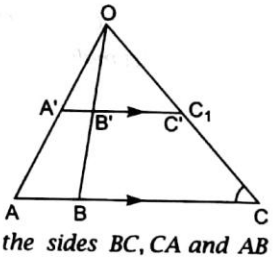
$[ABC]$ and $[DEF]$ denote the area of ΔABC and ΔDEF respectively.

Definitions

1. A line segment joining a vertex of a triangle to any point on the opposite side (the point may be on the extension of the opposite side also) is now called a Cevian.
   Altitudes, medians, angle bisectors are all Cevians.
2. Three points A, B, C are said to be collinear, if they lie on a straight line.
3. Three straight lines are said to be concurrent, if all three passes through a point.

Theorem 1 *If A, B, C and A', B', C' are point an two parallel lines such that $AB/A'B' = BC/B'C'$, then AA', BB', CC' are concurrent, if they are not parallel.*

Proof Let AA' and BB' meet at O when AA' and BB' not parallel. Join OC and let it cut $A'B'$ at C_1. By Basic proportionality theorem



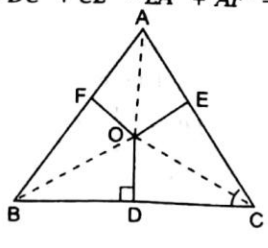
$$\frac{BC}{B'C_1} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$$

$$\Rightarrow \qquad B'C_1 = B'C'$$

$\Rightarrow C_1$ and C' coincide. Thus, CC' passes through O.

Theorem 2 *From a point O; OD, OE, OF are drawn perpendicular to the sides BC, CA and AB respectively of a ΔABC, then*

$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0$$



Proof $$BD^2 = OB^2 - OD^2$$

and $$DC^2 = OC^2 - OD^2$$

$\Rightarrow$ $$BD^2 - DC^2 = OB^2 - OC^2 \quad \text{...(i)}$$

Similarly,

$$CE^2 - EA^2 = OC^2 - OA^2 \quad \text{...(ii)}$$
$$AF^2 - FB^2 = OA^2 - OB^2 \quad \text{...(iii)}$$

On adding Eqs. (i), (ii) and (iii), we get

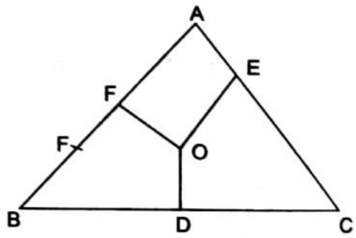
$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0$$

Theorem 3 *If D, E, F be points on sides BC, CA, AB of a Δ ABC such that*

$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0,$$

then perpendiculars at D, E, F to the respective sides are concurrent.

Proof Let the perpendiculars at D, E to BC, CA respectively meet at O. Let OF' be the perpendicular from O to AB. Using theorem 2 we have



$$BD^2 - DC^2 + CE^2 - EA^2 + AF'^2 - F'B^2 = 0 \quad \text{... (i)}$$

But $$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0 \quad \text{...(ii)}$$

$\Rightarrow$ $$AF'^2 - F'B^2 = AF^2 - FB^2$$

or $$(AF' + F'B)(AF' - FB') = (AF + FB)(AF - FB)$$

$\Rightarrow$ $$AB(AF' - F'B) = AB(AF - FB)$$

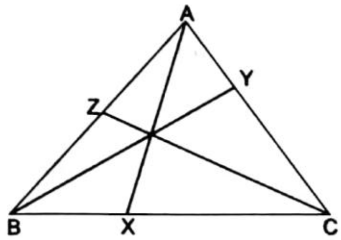
$\Rightarrow$ $$AF' - F'B = AF - FB \quad [\because AB \neq 0]$$

$$AF - AF' = FB - F'B$$
$$FF' = -FF'$$
$$2FF' = 0 \Rightarrow FF' = 0$$

is F, F' coincides

Theorem 4 *CEVAS' S THEOREM*

ABC is a triangle and AX, BY and CZ are three Concurrent cevians. Then,



$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

Proof Let the cevians meet at P. Let $[ABC]$ denote the area of Δ ABC

$\therefore$ $$[BPX] = \frac{1}{2} BX \cdot h$$

and $$[CPX] = \frac{1}{2} XC \cdot h$$

h = length of perpendicular from P to BC, then

$$\frac{[BPX]}{[CPX]} = \frac{\frac{1}{2} BX \cdot h}{\frac{1}{2} XC \cdot h} = \frac{BX}{XC} \quad \text{...(i)}$$

If h' = length of perpendicular from A to BC, then

$$\frac{[BAX]}{[CAX]} = \frac{\frac{1}{2} BX \cdot h'}{\frac{1}{2} XC \cdot h'} = \frac{BX}{XC} \quad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$\frac{BX}{XC} = \frac{[BAX]}{[CAX]} = \frac{[BPX]}{[CPX]} \quad \text{...(iii)}$$

from elementary algebra, if $\frac{a}{b} = \frac{c}{d}$, then each of the ratios $\frac{a}{b}, \frac{c}{d} = \frac{a-c}{b-d}$

So, from Eq. (iii) $$\frac{BX}{XC} = \frac{[BAX]-[BPX]}{[CAX]-[CPX]} = \frac{[APB]}{[APC]}$$

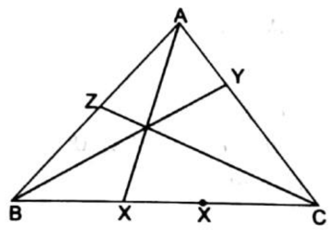
$\therefore$ $$\frac{BX}{XC} = \frac{[APB]}{[APC]}$$

Similarly, $$\frac{CY}{YA} = \frac{[BPC]}{[APB]} \quad \text{and} \quad \frac{AZ}{ZB} = \frac{[APC]}{[BPC]}$$

Multiplying, we get $$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

Theorem 5 *CONVERSE OF THEOREM* 4.

If 3 *Cevians* AX, BY, CZ *satisfy* $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$, *then they are concurrent.*



Proof Let BY and CZ meet at P. Let AP meet BC at X'. By ceva's theorem

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1 \quad \text{...(i)}$$

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1 \quad \text{...(ii)}$$

We have $$\frac{BX'}{X'C} = \frac{BX}{XC}$$

Adding 1 to both sides

$$\frac{BX'}{X'C} + 1 = \frac{BX}{XC} + 1$$

$$\frac{BX' + X'C}{X'C} = \frac{BX + XC}{XC}$$

$$\Rightarrow \quad \frac{BC}{X'C} = \frac{BC}{XC}$$

$$X'C = XC \Rightarrow X'C - XC = 0$$

$$\Rightarrow \quad XX' = 0$$

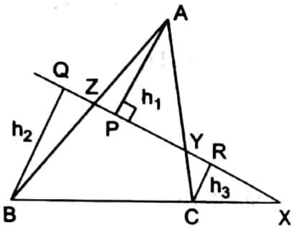
$\therefore$ X and X' Coincides. Thus, the 3 cevians are concurrent.

Theorem 6 *MENELAUS THEOREM*

If a transversal cuts the sides BC, CA, AB of a Δ ABC at X, Y, Z respectively.
Then,

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

Proof Let h_1, h_2, h_3 be lengths of $\perp$s AP, BQ, CR respective from A, B, C on the transversal.



Δ BQX and Δ CRX are similar.

$$\therefore \quad \frac{BX}{XC} = \frac{h_2}{h_3} \quad \ldots(i)$$

Similarly,

$$\frac{CY}{YA} = \frac{h_3}{h_1} \quad \ldots(ii)$$

and

$$\frac{AZ}{ZB} = \frac{h_1}{h_2} \quad \ldots(iii)$$

Hence,

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{h_2}{h_3} \cdot \frac{h_3}{h_1} \cdot \frac{h_1}{h_2} = 1$$

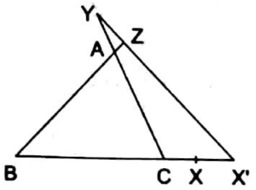
As per the directed lengths we have $\frac{BX}{XC}$ is negative. The other two ratios are positive.

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

Theorem 7 *(CONVERSE OF THEOREM 6.)*

If X, Y, Z are three points on each of the sides BC, CA, AB of Δ ABC or on their extensions such that $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$, then X, Y, Z are collinear.

Proof Let us give the proof of this theorem for a transversal which cuts all the 3 sides externally. X, Y, Z are points on the extensions of BC, CA and AB respectivey. Such that



$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1 \quad \ldots(i)$$

Produce YZ to meet BC extended at X'. As in the case of converse of ceva's theorem we shall prove that X, X' coincide. We have MENELAUS THEOREM.

$$\frac{BX'}{X'C}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=-1 \quad \text{...(ii)}$$

from Eqs. (i) and (ii), we get

$$\frac{BX}{XC}=\frac{BX'}{X'C}$$

Subtract 1 from both sides.

$$\frac{BX}{XC}-1=\frac{BX'}{X'C}-1$$

i.e., $$\frac{BX-XC}{XC}=\frac{BX'-X'}{X'C}$$

i.e., $$\frac{BC}{XC}=\frac{BC}{X'C}$$

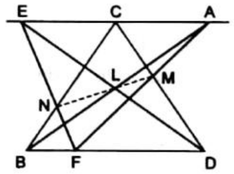
$\Rightarrow$ $$X'C=XC$$

or $$X'C-XC=0$$

i.e., $XX'=0$ or X, X' Coincides.

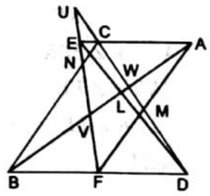
Theorem 8 *PAPPUS THEOREM*

If A, C, E are three points on one straight line. B, D, F on another and if the three lines AB, CD, EF meet respectively DE, FA and BC at L, M, N, then these three points L, M, N are collinear.



Proof Try yourself

For students convenience the adjacent diagram is given.



Use Menelaus theorem for $\Delta\, UVW$.

Considering the transversals *LDE*, *AMF*, *BCN*, *ACE*, *BDF*

Concept The square of the length of the internal bisector of $\angle A$ of $\Delta\, ABC$ is $\dfrac{4bcs\,(s-a)}{(b+c)^2}$.

Proof $\because$ *D* divides *BC* internally in $AB : AC$

$\therefore$ $$BD=\frac{ac}{b+c}\,;\; DC=\frac{bc}{b+c}$$

$\therefore$ $$AD^2=AB\cdot AC-BD\cdot DC$$

$$=bc-\frac{ac}{b+c}\cdot\frac{bc}{b+c}=\frac{bc}{(b+c)^2}\{(b+c)^2-a^2\}=\frac{4bcs\,(s-a)}{(b+c)^2}$$

Theorem 9 *STEINER LEHMUS PROBLEM*

Prove that, if two angle bisector of a triangle are equal, then triangle is isosceles.

Proof 1 Suppose BE and CF are internal bisectors of $\angle B$ and $\angle C$, are equal.

We know that,

$$BE = \frac{4ca\,(s-b)}{(c+a)^2},\ CF = \frac{4abs\,(s-c)}{(a+b)^2}$$

$$\therefore \quad \frac{4acs\,(s-b)}{(c+a)^2} = \frac{4abs\,(s-c)}{(a+b)^2}$$

$$\Rightarrow \quad c\,(a+b)^2(s-b) = b\,(c+a)^2\,(s-c)$$

$$\Rightarrow \quad s\,\{c\,(a+b)^2 - b\,(c+a)^2\} + bc\,\{(c+a)^2 - (a+b)^2\} = 0$$

$$\Rightarrow \quad s\,\{c\,(a^2+b^2) - b\,(c^2+a^2)\} + bc\,(c-b)\,(2a+b+c) = 0$$

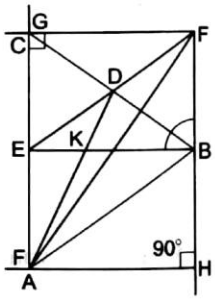
$$\Rightarrow \quad s\,(c-b)\,(a^2-bc) + bc\,(c-b)\,(2s+a) = 0$$

$$\Rightarrow \quad (c-b)\,\{bcs + a^2s + abc\} = 0$$

$$\Rightarrow \quad c = b$$

$$\therefore \quad bcs + a^2s + abs > 0$$

Proof 2 AD and BE are bisectors of $\angle A$ and $\angle B$ in $\Delta\, ABC$



They, intersect at K

$$AD = BE \text{ (given)}$$

Draw $\angle BEF$ and $\angle EBF$ are equal to $\angle BAD$ and $\angle ADB$ respectively. Draw AH and $FG \perp AC$ and FB

(a) $\Delta\, ADB \cong \Delta\, EBF$ [by ASA]

$$AB = EF$$

[$\because AD = BE\,;\ \angle DAB = \angle FEB,\ \angle ADB = \angle EBF$]

$$DB = BF$$

(b)

$$\angle AEF = \angle AEK + \angle KEF$$
$$= \angle AEK + \angle EAK$$
$$= \angle AKB = \angle KDB + \angle KBD$$
$$= \angle EBF + \angle EBA = \angle ABF$$

$$\therefore \quad \angle FEG = \angle ABH$$

(c) $\Delta\, ABH \cong \Delta\, FEG$ [By (a) and (b) and construction]

$$\therefore \quad AH = FG$$

$$BH = EG$$

(d) $\Delta\, AFG \cong \Delta\, FAH$ (hypotenuse and length)

$$AG = FH$$

(e) $AE = AG - GE = FH - HB$

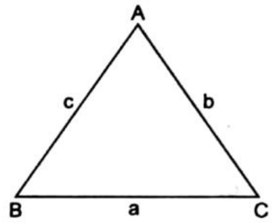
$$FB = BD \quad \text{[By (a)]}$$

(f) $\Delta\, ABE \cong \Delta\, BAD$ (SSS) $\Rightarrow \angle EAB = \angle DBA$

$$\Rightarrow \quad \angle A = \angle B$$

$$\Rightarrow \quad CB = CA$$

Basic Rules for a Triangle



In a Δ ABC, the angles are denoted by capital letters A, B and C; and the lengths of the sides opposite to these angles are denoted by small letters a, b and c respectively.

Simiperimeter of the triangle is

$$s = \frac{a+b+c}{2}$$

and its are is denoted by Δ.

I. Sine rule In any ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the radius of the circumcircle of the ΔABC.

II. Cosine rule In any ΔABC

$$a^2 = b^2 + c^2 - 2bc \cos A$$

or $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Similarly, $$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

and $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

III. Projection rule In any ΔABC

$$a = b \cos C + c \cos B$$
$$b = c \cos A + a \cos C$$
$$c = a \cos B + b \cos A$$

IV. Area of a triangle

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$$
$$= \frac{1}{2} ab \sin C = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = rs$$

where R and r are the radii of the circumcircle and the incircle of ΔABC respectively.

Theorem 1 *If D be a point on the side BC of a ΔABC such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$, then prove that*

(i) $(m+n) \cot \theta = m \cot \alpha - n \cot \beta$ (ii) $(m+n) \cot \theta = n \cot B - m \cot C$

Proof (i) Given, $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$

$\therefore$ $\angle ADB = (180° - \theta)$, $\angle BAD = \alpha$ and $\angle DAC = \beta$

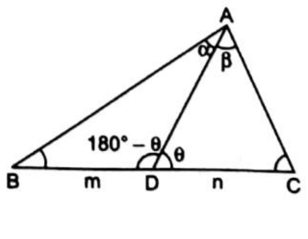
$\therefore$ $\angle ABD = 180° - (\alpha + 180° - \theta) = \theta - \alpha$

and $\angle ACD = 180° - (\theta + \beta)$

From ΔABD, $$\frac{BD}{\sin \alpha} = \frac{AD}{\sin(\theta - \alpha)}$$

From ΔADC, $$\frac{DC}{\sin \beta} = \frac{AD}{\sin[180° - (\theta + \beta)]}$$

or $$\frac{DC}{\sin \beta} = \frac{AD}{\sin(\theta + \beta)}$$



Dividing Eq. (i) by Eq. (ii), we get

$$\frac{BD\sin\beta}{DC\sin\alpha}=\frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$$

or $$\frac{m\sin\beta}{n\sin\alpha}=\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\sin\theta\cos\alpha-\cos\theta\sin\alpha}$$

or $$m\sin\theta\sin\beta\cos\alpha-m\cos\theta\sin\alpha\sin\beta$$
$$=n\sin\alpha\sin\theta\cos\beta+n\sin\alpha\cos\theta\sin\beta$$

or $$m\cot\alpha-m\cot\theta=n\cot\beta+n\cot\theta$$

[dividing both sides by $\sin\alpha\sin\theta\sin\beta$]

or $$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$$

(ii) Given, $\frac{BD}{DC}=\frac{m}{n}$ and $\angle ADC=\theta$

$\therefore$ $$\angle ADB=180^\circ-\theta$$
$$\angle ABD=B \text{ and } \angle ACD=C$$

$\therefore$ $$\angle BAD=180^\circ-(180^\circ-\theta+B)=\theta-B$$

and $$\angle DAC=180^\circ-(\theta+C)$$

Now from ΔABD, $$\frac{BD}{\sin(\theta-B)}=\frac{AD}{\sin B} \quad \text{...(i)}$$

and from ΔADC,

$$\frac{DC}{\sin[180^\circ-(\theta+C)]}=\frac{AD}{\sin C}$$

or $$\frac{DC}{\sin(\theta+C)}=\frac{AD}{\sin C} \quad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{BD}{DC}\cdot\frac{\sin(\theta+C)}{\sin(\theta-B)}=\frac{\sin C}{\sin B}$$

or $$\frac{m}{n}\frac{(\sin\theta\cos C+\cos\theta\sin C)}{(\sin\theta\cos B-\cos\theta\sin B)}=\frac{\sin C}{\sin B}$$

or $$m\sin\theta\cos C\sin B+m\cos\theta\sin C\sin B$$
$$=n\sin\theta\sin C\cos B-n\cos\theta\sin B\sin C$$

or $$m\cot C+m\cot\theta=n\cot B-n\cot\theta$$

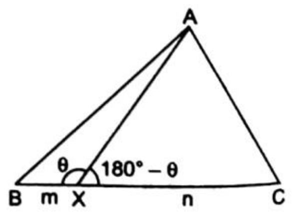
[dividing both sides by $\sin B\sin C\sin\theta$]

or $$(m+n)\cot\theta=n\cot B-m\cot C$$

Theorem 2 *STEWART'S THEOREM*

Let AX be a cevian of length p divide BC into segments $BX=m, XC=n$, then

$$a(p^2+mn)=b^2m+c^2n$$



Proof Using cosine rule for ΔABX and ΔACX

$$\cos\theta=\frac{p^2+m^2-c^2}{2pm} \quad \text{...(i)}$$

$$\cos(180-\theta)=\frac{p^2+n^2-b^2}{2pn}=-\cos\theta \quad \text{...(ii)}$$

Where $$\angle AXD=\theta \text{ and } \angle AXC=180^\circ-\theta$$

So, $$\frac{p^2+m^2-c^2}{2pm} = -\left[\frac{p^2+n^2-b^2}{2pm}\right]$$

$$m(p^2+n^2-b^2) = -n(p^2+m^2-c^2)$$

or $$p^2m + nm^2 - b^2m = -p^2n - nm^2 + c^2n$$

ie, $$b^2m + c^2n = p^2m + p^2n + mn^2 + nm^2$$

$$= p^2(m+n) + mn(m+n)$$

$\Rightarrow$ $$b^2m + c^2n = (m+n)(p^2+mn)$$

$\Rightarrow$ $$b^2m + c^2n = a(p^2+mn)$$

Example 1 *Find the length of the medians of a triangle.*

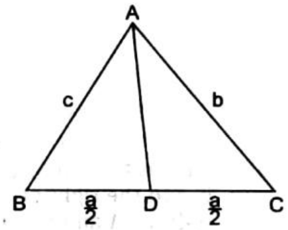
Solution Using Stewart's theorem

$$b^2\frac{a}{2} + c^2\frac{a}{2} = a\left(AD^2 + \frac{a}{2}\cdot\frac{a}{2}\right)$$

i.e., $$\frac{b^2+c^2}{2} = AD^2 + \frac{a^2}{4}$$

i.e., $$AD^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4}$$

i.e., $$AD = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$



Example 2 *If AD be the median through vertex A of a ΔABC, then prove that $AB^2 + AC^2 = 2(AD^2 + BD^2)$*

Solution Using Stewart's theorem, we have

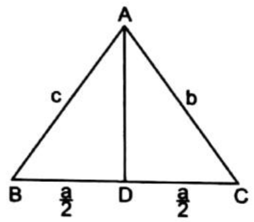
$$AD = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$

$\Rightarrow$ $$AD^2 = \frac{1}{4}(2b^2+2c^2-a^2)$$

$\Rightarrow$ $$2AD^2 = b^2 + c^2 - \frac{a^2}{2}$$

$$= AC^2 + AB^2 - 2BD^2$$

$\Rightarrow$ $$2(AD^2 + BD^2) = AC^2 + AB^2$$



Construction of Triangles

A triangle can be constructed in each of the following cases

$\Rightarrow$ When all the three sides are given

$\Rightarrow$ When one side and two angles are given

$\Rightarrow$ When two sides and the included angle are given

But there are many other cases when it is possible to construct the triangle.

Let us take a very special case when two sides and one anlge (other than the included anlge) are given.

To construct a ΔABC, when sides a and b and $\angle A$ are given.

There may exist no one or two triangles depending on the relation between the given parts.

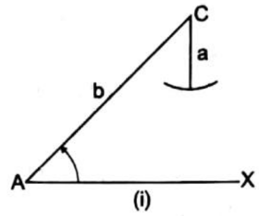
Now, there is possibility of having two triangles This case is called the Ambigous case.

We construct $\angle A$ and cut off $AC = b$ this fixes the vertex C.

With C as centre we draw an arc in order to locate (if possible) B.

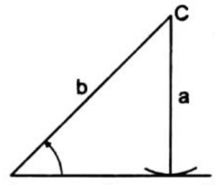
(I) If $A < 90°$, then several cases arise

(i) if $a < p$ (where $p = b \sin A$ is perpendicular from C on AX), then the arc does not cut AX and no triangle is possible.



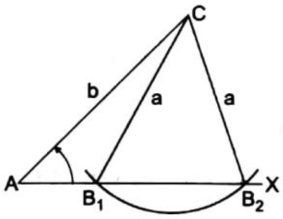
(i)

(ii) If $a = p$, then the arc touches AX. Therefore one triangle is possible and it is right angled.

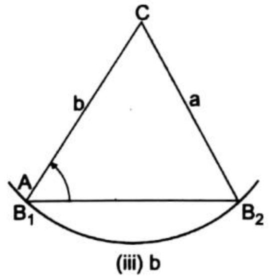


(ii)

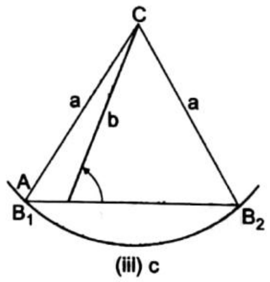
(iii) If $a > p$, then the arc cuts AX, at two points both these points lie to the right of A, if $a < b$. One of them lies to the right of A and other coincides with A if $a = b$ [See fig. (iii) b] and one of them lies to the right of A and other to left of A, if $a > b$ [See fig. (iii) c].



(iii) a



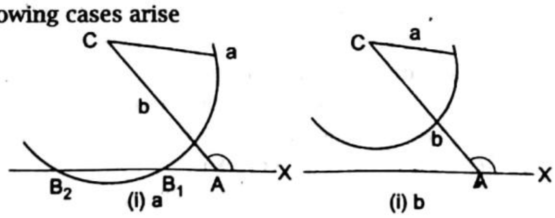
(iii) b



(iii) c

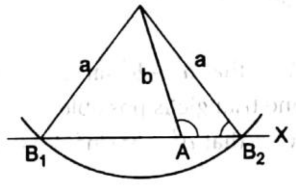
Thus, two triangles are possible if $a < b$ and only one triangle is possible.
If $a \geq b$. Because of the possibility of two triangles the case $a < b$, $a > b \sin A$.
A acute is called the Ambiguous case

(II) If $A > 90°$, then following cases arise



(i) a (i) b

(i) If $a \le b$, the arc does not cut AX at any point to the right of A and no triangle is possible.
(ii) If $a > b$, then the arc cuts AX at two points, only one triangle is possible.



Now we will discuss the case by using sine formula

(I) If $A < 90°$, then following cases arise

(i) If $a < b \sin A$, then from the formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{...(i)}$$

$\sin B > 1$ and consequently no solution is possible

(ii) If $a > b \sin A$, then Eq. (i) gives two values of B one of which is acute and the other obtuse if $a \le b$, then $A \le B$, so that only the acute value of B is permissible and consequently there is one solution.

If $a < b$, then $A < B$ so that both the values of B are possible and consequently there may be two solution.

(II) If $A > 90°$, then following cases arise

(i) If $a \le b$, then $A \le B$, so that B must also be an obtuse angle which is not possible. Hence, no solution is possible.

(ii) If $a > b$, then only the acute value of B is permissible. Only one triangle is possible.

Having determined B (whenever there exists a permissible value of B), We determine C by the formula

$$C = 180° - (A + B)$$

The remaining side c is then found in SAS case. In the Ambiguous case the value of C and c corresponding to the two values of B have to be found separately.

By use of logarithms

$$\log \sin B > 0 \Rightarrow \sin B > 1 \Rightarrow \text{No solution}$$

$$\log \sin B = 0 \Rightarrow \sin B = 1 \Rightarrow \angle B = 90°$$

$$\log \sin B < 0 \Rightarrow \sin B < 1. \text{ There are two values of } B, B_1 \text{ and } B_2$$

Remark

Next case can be determined by using Cosine formula.

If a, b, A are given, then cosine formula for a gives

$$a^2 = b^2 + c^2 - 2bc \cos A$$

or
$$c^2 - 2bc \cos A + b^2 - a^2 = 0 \quad \text{...(i)}$$

Solving Eq. (i) as a quadratic in c, we get

$$c = \frac{2b \cos A \pm \sqrt{\{4b^2 \cos^2 A - 4(b^2 - a^2)\}}}{2}$$

$$= b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A} \quad \text{...(ii)}$$

Since, c is the length of the side of a triangle therefore it must be +ve.

Two different possibilites arise

(i) $A < 90°$. If $A < 90°$, $\cos A$ is +ve so that $b\cos A$ is +ve.

Three cases arise.

(a) If $a < b \sin A$, then $a^2 < b^2 \sin^2 A$ so that $a^2 - b^2 \sin^2 A < 0$. The two values of c are imaginary and no triangle is possible.

(b) If $a = b \sin A$, then $a^2 = b^2 \sin^2 A$ so that $a^2 - b^2 \sin^2 A = 0$. There is only one value of c $(= b\cos A)$ from Eq. (ii) which is +ve. Only one triangle is possible.

(c) If $a > b \sin A$, then $a^2 > b^2 \sin^2 A$ so that $a^2 - b^2 \sin^2 A > 0$. In this case, Eq. (ii) gives two real and distinct values of c.

i.e., $\quad b\cos A + \sqrt{a^2 - b^2 \sin^2 A}$ is +ve

and $\quad b\cos A - \sqrt{a^2 - b^2 \sin^2 A}$ is +ve

If
$$b^2 \cos^2 A > a^2 - b^2 \sin^2 A$$
$$b^2 > a^2$$
$$b > a$$

Two triangles are possible if $b > a$ only one triangle is possible, if $b \leq a$

(ii) $A > 90°$. If $A > 90°$

$\cos A$ is – ve so that $b\cos A$ is – ve

The value of $b\cos A - \sqrt{a^2 - b^2 \sin^2 A}$ is - ve

The value of $b\cos A + \sqrt{a^2 - b^2 \sin^2 A}$ is +ve if

$$\sqrt{a^2 - b^2 \sin^2 A} > -b\cos A$$
$$a^2 - b^2 \sin^2 A > b^2 \cos^2 A$$
$$a^2 > b^2$$
$$a > b$$

We find that when $A > 90°$

No triangle is possible when $a \leq b$.

Only one triangle is possible when $a > b$.

Circumcircle

The perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrence is the centre of the circumcircle of the triangle. It is called the circumcentre of the triangle and its radius is called the circumradius of the triangle and is denoted by

$$R = \frac{abc}{4\Delta}$$

Where a, b, c are length of the sides of triangle

and $\quad \Delta =$ Area

Incircle

The internal bisectors of the angles of a triangle are concurrent. The point of concurrence is called the incentre of the triangle.

The circle with this point as centre and the length of the perpendicular from this point on any one of the sides of the triangle touches all the three sides of the triangle internally and is called the incircle of the triangle.

Its radius is called inradius of the triangle and is denoted by r.

$$r = \frac{\Delta}{s}$$

where $s =$ semiperimeter of the triangle

$\Delta =$ its area

Excircles

Given a $\Delta\ ABC$, there are four circles which touch all the three sides of the triangle. One of them is the incircle, which touches all the sides internally. The other three touch the side externally and are called excircles.

The excircle opposite A (respectively B, C) is the one whose centre lies on the internal bisector of $\angle A$ (respectively $\angle B, \angle C$).

The centre of the excircle opposite A (respectively B, C) is usually denoted by I_1 (respectively I_2, I_3)

Its radius is r_1 (respectively r_2 r_3).

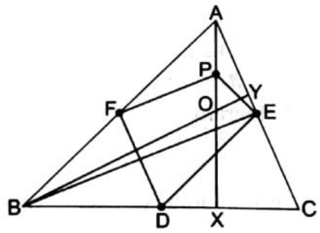
$$r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$$

Nine Point on Circle

Prove that in any triangle the mid point of the sides, the feet of perpendiculars from the vertices on the opposite sides and the mid points of the joins of the orthocentre to the vertices all lie on a circle.

Let X, Y be the feet of perpendiculars drawn from A and B on BC and CA respectively.

Let O be the orthocentre of the triangle. Let P be the mid point of AO.



In $\Delta\ ABO$, F is the mid point of AB.

P is the mid point of AO.

$\therefore$ $FB \parallel BO$

In $\Delta\ ABC$,

F is mid point of AB.

D is mid point of BC.

$\therefore$ $FD \parallel AC$

Now, $FP \parallel BY$, $FD \parallel AC$ and $BY \perp AC$

$\therefore$ $FP \perp FD$ i.e., $\angle DFP$ is a rigtht angle.

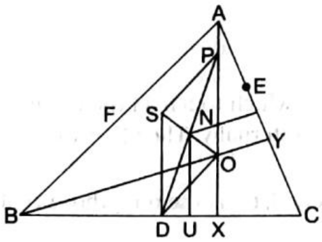
Similarly, $\angle PED$ is a right angle. Also $\angle PXD$ is a right angle.

$\therefore F, E$ and X all lie on a circle with PD as diameter so that P and X both lie on the circle through the points D, E, F.

Similarly, we can show that, if Z be the foot of perpendicular. from C on AB. Q and R be the mid points of BO and CO respectively, then Y, Z, Q, R also lie on this circle. Thus the nine points $D, E, F, P, Q, R, X, Y, Z$ all lie on a circle.

Theorem 1 *Prove that the nine point centre of a triangle is collinear with the circumcentre and the orthocentre and bisects the segment joining them. Also prove that radius of the nine point circle of a triangle is half the radius of the circumcircle.*

Proof Let S be the circumcentre of $\Delta\ ABC$.



$\because$ D and X lie on nine point circle.

$\therefore$ Its centre lies on the perpendicular bisector of DX. Let U be mid point of DX. Let the perpendicular from U on BC meet SO at N. Since SD, NU and OX are parallel and $DU = UX$.

$\therefore$ $SN = NO$ *i.e.*, N is mid point of SO.

Now, to show that N is centre of the nine point of circle, we have to observe that nine point centre must also lie on the perpendicular bisector of EY. This perpendicular will also meet SO in N. N is nine point centre.

It follows that the circumcentre, nine points centre and orthocentre are all collinear.

The nine point centre is the mid point of the segment joining the circumcentre and orthocentre.

Now, to show that the radius of nine point circle is half the circumradius.

Since PD is a diameter of the nine point circle so N is mid point of PD.

$\because$ SO and PD bisect each other at N

$\therefore$ S, D, O, P are vertices of of a parallelogram.

$\Rightarrow$ $SD = PO = AP$ [$\because$ P is mid point of AO]

Now, $SD \parallel AP$ and $SD = AP$

$\therefore$ S, D, P, A are the vertices of a parallelogram

Consequently $DP = SA$

But SA is the circumradius of $\Delta\ ABC$.

$\therefore$ Radius of the nine point circle is half the radius of circumcircle.

Theorem 2 *Prove that in any triangle the circumcentre, the centroid, the nine point centre and the orthocentre are all collinear.*

Proof Through P draw $PG' \parallel SO$ so as to meet AD in G'.

Let AD meet SO in G.

We will show that

$$AG' = G'G = GD$$

So as to conclude that G divides AD in $2:1$.

Consequently it is the centroid of $\Delta\ ABC$.

In $\Delta\ AGO$, P is mid point of AO and $PG' \parallel GO$.

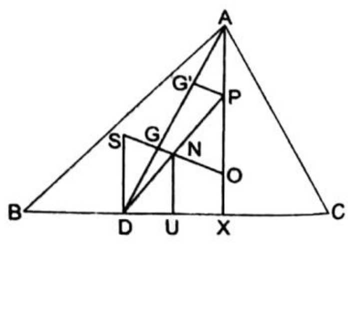
$\therefore$ G' is mid point of AG.

i.e., $AG' = G'G$.

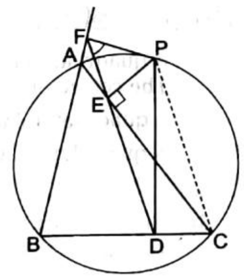
In $\Delta\ PDG'$, N is the mid point of PD and $NG \parallel PG'$.

$\therefore$ G is mid point of $G'D$ *i.e.*, $G'G = GD$.

Now, $AG' = G'G = GD$ so that $AG = \frac{2}{3} AD$



Simson's Line



Prove that the feet of perpendiculars drawn from a point on the circumcircle of a triangle on the sides are collinear.

Let D, E, F be the feet of perpendiculars drawn from a point P on the circumcircle of triangle ABC on the sides BC, CA, AB respectively.

We shall prove that the points D, E, F are collinear by showing that

$$\angle PED + \angle PEF = 180°$$

$$\angle PEA + \angle PFA = 180°$$

$\therefore$ The points P, E, A, F are concyclic.

Consequently

$$\angle PEF = \angle PAF \quad ...(i)$$

[Angles in the same segment]

$\because$ $\angle PEC = \angle PDC = 90°$

$\therefore$ P, E, D, C are concyclic.

$\therefore$ $\angle PED + \angle PCD = 180° \quad ...(ii)$

$\because$ P, A, B, C are concyclic.

$\therefore$ $\angle PAF = \angle PCB \quad ...(iii)$

from Eqs. (i) and (iii)

$$\angle PEF = \angle PCD \quad ...(iv)$$

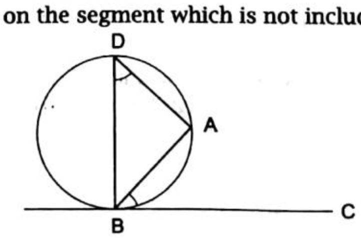
from Eqs. (ii) and (iv)

$$\angle PED + \angle PEF = 2 \text{ right angles}$$

Hence, D, E, F are collinear.

Some Important Properties Related to Circle

1. A chord subtends an angle of the same magnitude at a point at the circumference of each of the two segments into which it divides the circle.
2. A chord subtends at the centre an angle twice the magnitude of the angle it subtends at the circumferences.
3. A diameter subtends a right angle at any point of the circumference.
4. The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.
5. The perpendicular from the centre of a circle to a chord bisects the chord.
6. If a line segment AB subtends a right angle at C, then C lies on the circle on AB as diameter.
7. The tangent to a circle at a point theorem is perpendicular to the radius through that point.
8. **Alternate segment theorem** If AB is a chord of a circle. BC is the tangent to the circle at B, then $\angle ABC = \angle ADC$, where D lies on the segment which is not included between AB and BC.



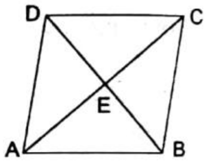
9. Tangents to a circle which are parallelogram touch the circle at the ends of a diameter which is perpendicular to them.
10. The two tangents to a circle drawn from a point outside it are of equal length.

Some Definitions

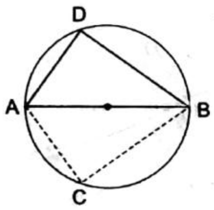
1. A quadrilateral which has a circle passing through all its four vertices is called a cyclic quadrilateral. The centre of the circle is called the centre of the quadrilateral.
2. A quadrilateral which has a circle touching all its four sides is called a circumscribed quadrilateral.
3. A quadrilateral which has both a circumcircle and an incircle is called a bicentric quadrilateral.

Note

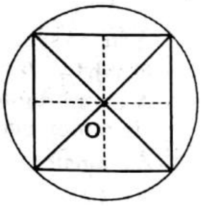
1. Any rectanlge is a cyclic quadrilateral. The diagonals are equal and bisect each other so that the circle with centre at the point of intersection of the diagonals and radius equal to half the length of a diagonal passes through all the vertices of the rectangle.
2. Assume that a parallelogram is a cyclic quadrilateral and O is the centre of the circumscribing circle. Let E be the point of intersection of the diagonals. As E is the midpoint of both the diagonals, OE is perpendicular to both AC and BD which is impossible unless O and E coincides. If O and E coincides BD and AC should be equal, so that $ABCD$ is a rectangle.
i.e., a parallelogram is a cyclic quadrilateral if and only if it is a rectangle.



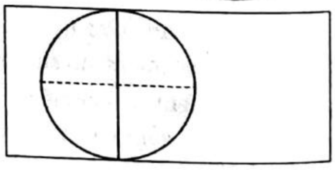
3. A cyclic quadrilateral need not always be a rectangle. Take a circle with centre O. AB is its diameter. Take two points C and D one on each of the two semicircles determined by AB such that $AC \neq BD$. Thus, $ACBD$ is a cyclic quadrilateral which is not a rectangle.



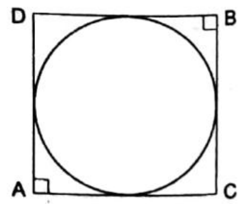
4. A square is a circumscribed quadrilateral. If O is the point of intersection of its diagonals, then O is equidistant from the sides (*i.e.*, the perpendicular distance of each side from O is the same *i.e.*, half the length of the side). So the circle with centre at O and radius equal to half the length of the side touches the side at their respective midpoints. Thus, a square is bicentric as it also have a circumcircle.



5. A rectangle which is not a square, is not a circumscribed quadrilateral since the opposite sides being parallel tangents of a circle, the line joining the points of contact is a diameter of the circle perpendicular to them so the diameter is equal to either of the sides of the rectangle. This is impossible unless the length and breadth of the rectangle are the same. So, a rectangle is a circumscribed quadrilateral if and only if it is a square.



6. There are non square quadrilaterals which are bicentric. To construct one such, take any circle. Draw pairs of perpendicular tangents from points A, B not lying on the same diameter of the circle. So, $ACBD$ is a resultant quadrilateral which is bicentric.



Cyclic Quadrilateral

Theorem 1 *If a quadrilateral is cyclic, then sum of each pair of opposite angles is* $180°$.

Proof $ABCD$ is a cyclic quadrilateral with O is its centre.
Now, one of the angles subtended by BD at the centre is reflex.

$$\angle DAB = \frac{1}{2} \text{ reflex} \angle DOB$$

$$\angle DCB = \frac{1}{2} \angle DOB$$

Hence, $$\angle DAB + \angle DCB = \frac{1}{2} \quad (\text{reflex } \angle DOB + \angle DOB)$$

$$= \frac{1}{2} \times 360° = 180°$$

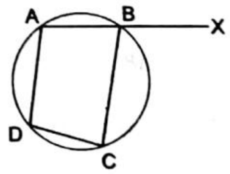
i.e., $$\angle A + \angle C = 180°$$

$\because$ $$\angle A + \angle B + \angle C + \angle D = 360°$$

$\therefore$ $$\angle B + \angle D = 180°$$

Corollary *The exterior angle of a cyclic quadrilateral is equal to the interior opposite angles.*

Proof Let $ABCD$ be cyclic.



Extend AB to X

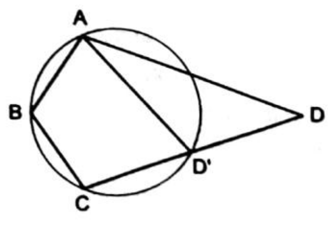
$$\angle ABC + \angle XBC = 180° \quad \text{(linear pair)}$$

$$\angle ABC + \angle ADC = 180°$$

$$\angle XBC = \angle ADC$$

Theorem 2 *If in a quadrilateral the sum of a pair of opposite angles is* $180°$, *then it is cyclic.*

Proof Let $ABCD$ be a quadrilateral with $\angle B + \angle D = 180°$.Let Q be the circle passing through the vertices A, B and C.



If it passes through D, then $ABCD$ is a cyclic quadrilateral, if Q does not pass through D. Let CD meet Q at D'. $ABCD'$ is cyclic.

By theorem 1, $\angle B + \angle D' = 180°$

But $\angle B + \angle D = 180°$

So that $\angle D = \angle D'$. But $\angle AD'C = \angle D'$ is the exterior angle of $\Delta ADD'$

$$\angle AD'C = \angle ADC + \angle DAD'$$

$\angle DAD' = 0$ *i.e.*, D and D' coincides.

$ABCD$ is cyclic.

Theorem 3 *Ptolemy's Theorem*

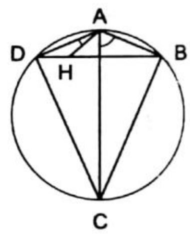
In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the pairs of opposite sides.

Proof We need to prove that

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

Now, $ABCD$ is cyclic quadrilateral choose a point on H on BD such that $\angle DAH = \angle BAC$

$$\angle ADH = \angle ADB = \angle ACB \quad \text{...(i)}$$



So, $\Delta HAD \sim \Delta BAC$

Hence, $\frac{DH}{BC} = \frac{AD}{AC}$ or $AD \cdot BC = AC \cdot DH$ ...(ii)

By (1), $\angle AHD = \angle ABC$

Hence, $\angle AHB = \angle ADC$ (supplementary angles)

Also $\angle ABH = \angle ABD = \angle ACD$

$\Delta HAB \sim \Delta DAC$

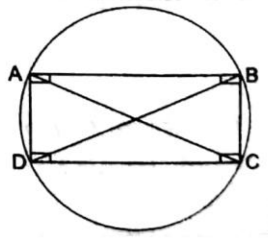
Hence, $\frac{HB}{DC} = \frac{AB}{AC}$

i.e., $HB \cdot AC = DC \cdot AB$ ...(iii)

From Eqs. (ii) and (iii)

$$AB \cdot DC + AD \cdot BC = AC \cdot (HB + HD)$$
$$= AC \cdot BD$$

Note If the quadrilateral is a rectangle (whose diagonals are equal), then it is cyclic.



In this case Ptolemy's theorem reduces to Pythagoras theorem.

$AC = BD$ (diagonals)

$AB = CD$ and $AD = BC$

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$
$$AC^2 = AB^2 + BC^2$$

Example 1 *ABCD is a trapezium.* $AB \parallel CD$ *and* $AD = BC$

It is given that $AB = a, CD = b, AD = BC = c$

Find the lengths of the diagonals.

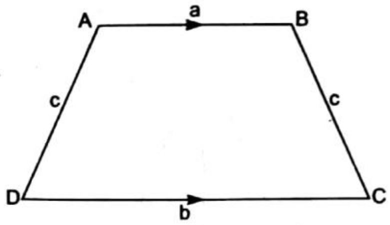
Solution $\angle ABC + \angle BCD = 180°$ $[\because AB \parallel CD]$

$$\angle ADC = \angle BCD$$

$$\angle ABC + \angle ADC = 180°$$

$\therefore$ $ABCD$ is cyclic [By theorem 2]

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$ [Ptolemy's theorem]



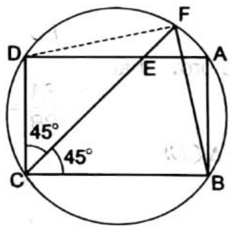
i.e., $ab + c^2 = AC \cdot BD$

But $AC = BD$

So that $AC^2 = ab + c^2$

or $AC = \sqrt{ab + c^2} = BD$

Example 2 *E is a point on the side AD of rectangle ABCD. So that DE = 6. Also DA = 8 ; DC = 6. If CE extended meets the circumcircle of the rectangle at F. Find the lengths of DF and FB.*



Solution $AC = BD = \sqrt{8^2 + 6^2} = 10$

$\angle DCE = \angle DEC = 45°$ [$\because CD = DE$]

$CE = \sqrt{DE^2 + CD^2} = 6\sqrt{2}$

Also $\angle FDB = \angle FCB = 45°$

$\angle DBF = \angle DCF = 45°$

In Δ BFD, $\angle FDB = \angle FBD = 45°$

So $FD = FB$

$DE \cdot EA = CE \cdot EF$

$6 \times 2 = 6\sqrt{2} \cdot EF$

$EF = \sqrt{2}$ and $CF = 7\sqrt{2}$

$CDFB$ is a cyclic quadrilateral

$CD \cdot FB + BC \cdot FD = CF \cdot BD$ [By Ptolemy's theorem]

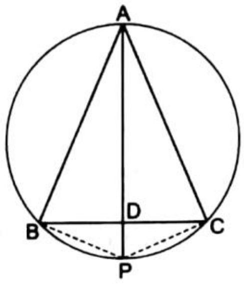
$(CD + BC)\, BF = 7\sqrt{2} \cdot 10$

$14 \times BF = 7\sqrt{2} \cdot 10$

$BF = 5\sqrt{2} = FD$

Example 3 *A line drawn from vertex A of an equilateral Δ ABC meets BC at D and the circumcircle at P. Prove that*

(a) $PA = PB + PC$ (b) $\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC}$



Solution $ABPC$ is cyclic quadrilateral

$$AB \cdot PC + AC \cdot PB = AP \cdot BC \quad \text{[from Theorem 3]}$$

But $AB = BC = CA$

So, $PA = PB + PC$ ...(i)

Dividing Eq. (i) By $PB \cdot PC$

We have $\frac{1}{PC} + \frac{1}{PB} = \frac{PA}{PB \cdot PC}$

It is enough to prove that

$$\frac{PA}{PB} = \frac{PC}{PD} \quad \text{...(ii)}$$

In Δ ABP and Δ CDP

$$\angle BAP = \angle DCP$$

$$\angle APB = \angle ACB = 60°$$

$$\angle APC = \angle ABC = 60°$$

Hence, $\angle APB = \angle APC = \angle DPC$

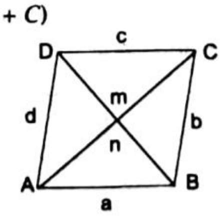
$\therefore$ $\Delta ABP \sim \Delta CDP$

So $\frac{PA}{PC} = \frac{PB}{PD}$

or $\frac{PA}{PB} = \frac{PC}{PD}$

Non-cyclic Quadrilaterals

Theorem 1 *ABCD is a quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = m$ and $BD = n$. Then,* $m^2 n^2 = a^2 c^2 + b^2 d^2 - 2abcd \cos(A + C)$



Proof Construct a $\Delta ABE \sim \Delta CAD$ on side AB so that $\angle ABE = \angle CAD$ and $\angle BAE = \angle ACB$.

Now, $$\frac{AB}{CA} = \frac{AE}{CD} = \frac{BE}{AD}$$

Hence, $$AE = \frac{ac}{m} \text{ and } BE = \frac{ad}{m} \quad \ldots(i)$$

Construct a $\Delta ADF \sim \Delta CAB$ so that

$$\angle ADF = \angle CAB \text{ and } \angle DAF = \angle ACD$$

We have $$\frac{AD}{CA} = \frac{AF}{CB} = \frac{DF}{AB}$$

Hence, $$AF = \frac{bd}{m}$$

and $$DF = \frac{da}{m} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get, $BE = DF$

Now, $$\angle EBD + \angle BDF = 3 + \angle ABD + \angle BDA + 4$$
$$= \angle ABD + \angle BDA + \angle BAD = 180°$$

So, $$BE \parallel DF \text{ and } BE = DF$$

Hence, $BEFD$ is a parallelogram

So $$EF = BD = n$$
$$\angle EAF = 1 + 2 + 3 + 4 = \angle A + \angle C$$

Using cosine rule in ΔEAF, we get

$$EF^2 = AE^2 + AF^2 - 2AE \cdot AF \cdot \cos \angle EAF$$

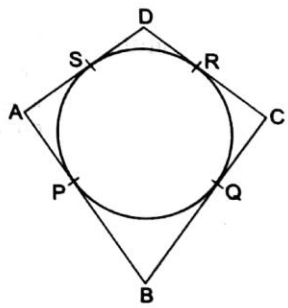
i.e., $$n^2 = \frac{a^2c^2}{m^2} + \frac{b^2d^2}{m^2} - \frac{2ac}{m^2} \cdot \frac{bd}{m} \cos(\angle A + \angle C)$$

i.e., $$m^2n^2 = a^2c^2 + b^2d^2 - 2abcd \cos(\angle A + \angle C)$$

Circumscribed Quadrilateral

Theorem 2 *Let ABCD be a quadrilateral with an incircle. Then, the sum of the opposite sides are equal*

i.e., $AB + CD = AD + BC$



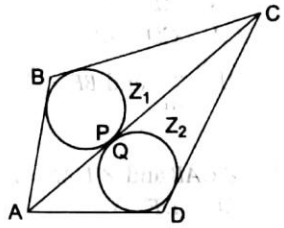
Proof Let the incircle touch the sides AB, BC, CD, DA at P, Q, R and S respectively,

Now, $$AP = AS$$
$$BP = BQ$$
$$CR = CQ$$
$$DR = DS \quad \ldots(i)$$

So $$AB + CD = AP + BP + CR + DR$$
$$= AS + BQ + CQ + DS = AS + DS + BQ + CQ$$
$$= AD + BC$$

Hence $$AB + CD = AD + BC$$

Example 1 *ABCD be a circumscribed quadrilateral, then prove that the circles inscribed in two Δ s ABC and Δ ADC touch each other.*



Solution Let the incircle of $\Delta\ ABC$ be Z_1 and of $\Delta\ ADC$ be Z_2.

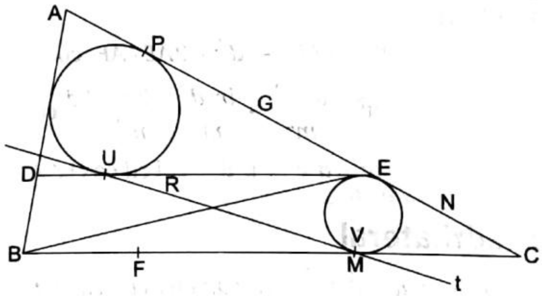
Since, Z_1 and Z_2 lie on either side of AC, if they touch each other, they touch a point on AC.

Let Z_1 touch AC at P and Z_2 touch AC at Q.

$$PQ = AQ - AP$$
$$= \frac{1}{2}(AD + AC - DC) - \frac{1}{2}(AB + AC - BC)$$
$$= \frac{1}{2}(AD + BC - AB - CD) = 0$$

Since, the quadrilateral is circumscribed.

Example 2 *Let the incircle of $\Delta\ ABC$ touch AB at D and let E be a point on the side AC. Prove that the incircles of $\Delta\ ADE$, $\Delta\ BCE$ and $\Delta\ BDE$ have common tangents.*



Solution Let the incircle Q of $\Delta\ ABC$ touch AB at D, BC at F and AC at G respectively. Let the incircle Q_1 of $\Delta\ ADE$ touch the sides EA, AD and DE at P, Q and R respectively. Let the incircle Z_2 of $\Delta\ BCE$ touch the sides BC, CE, EB at M, N and L respectively Let t be the common tangent of circles Z_1 and Z_2 meeting the lines DE, BE at S and T and touching Z_1 at U and Q_2 at V respectively.

We have to prove that t touches the incircle of $\Delta\ BED$. It is enough to prove that the quadrilateral $BDST$ is a circumscribed quadrilateral since the incircle of $BDST$ is in circle of $\Delta\ BDE$

$$BD + ST = BF + UV - SU - TV$$
$$= BF + PN - SU - TV$$

[$\because$ UV, PN lengths of direct common tangents between Q_1 and Q_2]

$$= BF + GP + GN - SR - TL$$
$$= BF + DQ + FM - SR - TL$$
$$= BM + DR - SR - TL$$
$$= BL - TL + DS = BT + DS$$

Area Axiom

Every polygonal regoin has an area, measured in square units

There is a standard square region of side one metre called a square metre. Which is unit of area denoted as m^2 or sq m and is a +ve real number.

Congruent Area Axiom

If $\Delta\ ABC$ and $\Delta\ PQR$ are two congruent angles, then area (region $\Delta\ ABC$) = area (region $\Delta\ PQR$).

i.e., two congruent regions have equal area.

Area Monotone Axiom

If R_1 , R_2 are two polygonal regions such that $R_1 \subset R_2$, then $ar\,(R_1) \le ar\,(R_2)$

Area Addition Axiom

If R_1 and R_2 are two polygonal regions, whose intersection is a finite number of points and line segments and $R = R_1 \cup R_2$, then

$$ar\,(R) = ar\,(R_1) + ar\,(R_2)$$

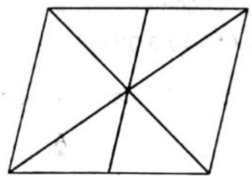
Let us take some facts about the area of parallelogram.

1. A diagonal of a parallelogram divides it into two triangles of equal area.
2. Parallelogram on the same base and between the same parallel are equal in area.
3. The area of a parallelogram is the product of its base and the corresponding altitude.

Now, let us take some examples.

Example 1 *The diagonals of a || gm ABCD intersects at O. A line through O meets AB in X and CD in Y. Show that*

$$ar\,(\text{quad. } AXYD) = \frac{1}{2}\,ar\,(||\,gm\ ABCD)$$



Solution

$$ar\,(\Delta\ ACD) = \frac{1}{2}\,ar\,(\text{parallelogram} ABCD) \qquad \text{...(i)}$$

[diagonal of a parallelogram divides it into two triangles of equal area]

In $\Delta\ AOX$ and $\Delta\ COY$, we have

$\angle\ AOX = \angle\ COY$ (vertically opposite angles)

$AO = CO$ [$\because$ diagonals of parallelogram bisect each other $\therefore$ O is mid point of AC]

$\angle\ OAX = \angle\ OCY$ [$\because$ $AB \,||\, DC$ and AC cuts them $\therefore \angle\ CAB = \angle\ ACD \Rightarrow \angle\ OAX = \angle\ OCY$]

$\Delta\ AOX \cong \Delta\ COY$ [ASA]

$\Rightarrow \quad ar\,(\ \Delta\ AOX) = ar\,(\Delta COY)$

$\Rightarrow \quad ar\,(\Delta\, AOX) + ar\,(\text{quad. } AOYD)$

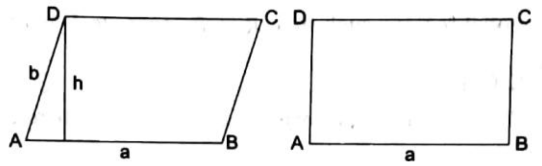
$= ar\,(\Delta\, COY) + ar\,(\text{quad. } AOYD)$

$\Rightarrow \quad ar\,(\text{quad. } AXYD) = ar\,(\Delta\, ACD)$ ...(ii)

from Eqs. (i) and (ii), we get

$$ar\,(\text{quad. } AXYD) = \frac{1}{2}\, ar\,(\text{parallelogram } ABCD)$$

Example 2 *Prove that of all parallelogram of which the sides are given, the parallelogram which is rectangle has greatest area.*



Solution Let $ABCD$ be parallelogram in which $AB = a$ and $AD = b$. Let h be its altitude corresponding to base AB, then

$$ar\,(\,||\, gm\ ABCD) = AB \times h = ah$$

$\because$ Sides a and b are given.

$\therefore$ With the same sides a and b we can construct infinitely many parallelogram with different heights.

Now, $\quad ar\,(\,||\, gm\ ABCD) = ah$

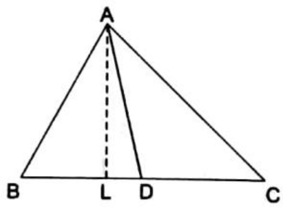
$\Rightarrow \quad ar\,(\,||\, gm\ ABCD)$ is greatest when h is maximum. [$\because$ a is constant]

But the maximum value which h can attains is $AD = b$ and this is possible when $AD \perp AB$,

i.e., $||\, gm\ ABCD$ becomes a rectangle.

Thus $ar\,(\,||\, gm\ ABCD)$ is greatest when $AD \perp AB$ *i.e.*, when $||\, gm\ ABCD$ is rectangle.

Example 3 *Show that a median of a triangle divides it into two triangles of equal area.*



Solution In $\Delta\, ABC$, AD is median. Draw $AL \perp BC$. D is mid point of BC.

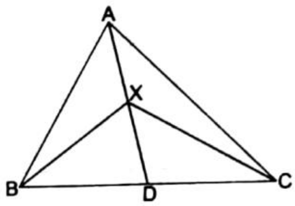
$\Rightarrow \quad BD = DC$

$\Rightarrow \quad BD \times AL = DC \times AL$ (Multiplying by AL)

$\Rightarrow \quad \frac{1}{2}(BD \times AL) = \frac{1}{2}(DC \times AL)$

$\Rightarrow \quad ar\,(\Delta\, ABD) = ar\,(\Delta\, ADC)$

Example 4 *AD is one of the medians of a Δ ABC. X is any point on AD. Show that*



$$ar(\Delta ABX) = ar(\Delta ACX)$$

Solution AD is a median in ΔABC

$\therefore$ $ar(\Delta ABD) = ar(\Delta ACD)$ ...(i)

In ΔXBC, XD is a median

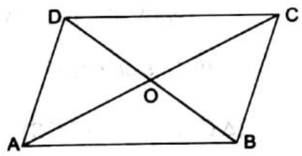
$\therefore$ $ar(\Delta XBD) = ar(\Delta XCD)$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$ar(\Delta ABD) - ar(\Delta XBD) = ar(\Delta ACD) - ar(\Delta XCD)$$

$\Rightarrow$ $ar(\Delta ABX) = ar(\Delta ACX)$

Example 5 *Show that the diagonals of a parallelogram divide it into four triangles of equal area.*



Solution AC and BD intersect at O in $||$ gm $ABCD$.

$\because$ The diagonals of a parallelogram bisect each other at the point of intersection.

$\therefore$ $AO = OC$

and $OB = OD$

Also, the median of a triangle divides it into two equal parts.

In $\Delta ABC, BO$ is the median

$\therefore$ $ar(\Delta OAB) = ar(\Delta OBC)$ ...(i)

In $\Delta BCD, CO$ is the median

$\therefore$ $ar(\Delta OBC) = ar(\Delta OCD)$ ...(ii)

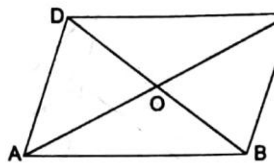
In $\Delta ACD, DO$ is the median.

$\therefore$ $ar(\Delta OCD) = ar(\Delta OAD)$ ...(iii)

from Eqs. (i), (ii) and (iii), we get

$$\begin{aligned} ar(\Delta OAB) &= ar(\Delta OBC) \\ &= ar(\Delta OCD) \\ &= ar(\Delta OAD) \end{aligned}$$

Example 6 *The diagonals of a ABCD, AC and BD intersect in O. Prove that if BO = OD, the Δ ABC and Δ ADC are equal in area.*



Solution In $\Delta\, ABD$, we have

$$BO = OD$$

$\Rightarrow$ O is mid point of BD

$\Rightarrow$ AO is the median.

$$\Rightarrow \quad ar(\Delta\, AOB) = ar(\Delta\, AOD) \quad \ldots(i)$$

[$\because$ Median divides Δ into two triangles of equal area]

In $\Delta\, CBD$, O is the mid point of BD

$\therefore$ CO is a median.

$$\Rightarrow \quad ar(\Delta\, COB) = ar(\Delta\, COD) \quad \ldots(ii)$$

On adding Eqs. (i) and (ii), we have

$$ar(\Delta\, AOB) + ar(\Delta\, COB) = ar(\Delta\, AOD) + ar(\Delta\, COD)$$

$$\Rightarrow \quad ar(\Delta\, ABC) = ar(\Delta\, ADC)$$

Example 7 *ABCD is a parallelogram. O is any point in its interior.*
Prove that

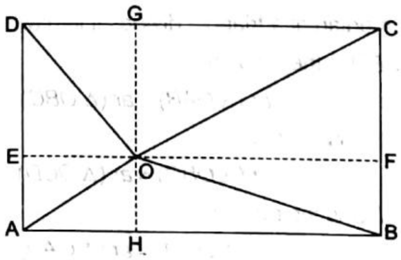
(*a*) $ar(\Delta\, AOB) + ar(\Delta\, COD) = ar(\Delta\, BOC) + ar(\Delta\, AOD)$

(*b*) $ar(\Delta\, AOB) + ar(\Delta\, COD) = \frac{1}{2} ar(\| gm\ ABCD)$

Solution (a) $\because$ $GN \| AD$ and $EF \| DC$

$$\therefore \quad OG \| DE \text{ and } OE \| GD$$

$$\Rightarrow \quad EOGD \text{ is parallelogram}$$



Similarly, $EAHO$, $HBFO$ and $FOGC$ are parallelogram

Now, OD is a diagonal of $\| gm\ EOGD$

$$\Rightarrow \quad ar(\Delta EOD) = ar(\Delta\, DOG) \quad \ldots(i)$$

OA is a diagonal of $\| gm\ EAHO$

$$\Rightarrow \quad ar(\Delta\, EOA) = ar(\Delta\, AOH) \quad \ldots(ii)$$

OB is a diagonal of $||$ gm $HBFO$

$$\Rightarrow \quad ar(\Delta BOF) = ar(\Delta BOH) \quad \ldots\text{(iii)}$$

OC is a diagonal of $||$ gm $FOGC$.

$$\Rightarrow \quad ar(\Delta FOC) = ar(\Delta COG) \quad \ldots\text{(iv)}$$

Adding Eqs. (i),(ii),(iii),(iv), we get

$$ar(\Delta EOD) + ar(\Delta EOA) + ar(\Delta BOF) + ar(\Delta FOC)$$
$$= ar(\Delta DOG) + ar(\Delta AOH) + ar(\Delta BOH) + ar(\Delta COG)$$
$$\Rightarrow \quad ar(\Delta AOD) + ar(\Delta BOC) = ar(\Delta AOB) + ar(\Delta COD)$$

(b) $\because$ ΔAOB and $||$ gm $ABFE$ are on same base AB and between the same parallel lines AB and EF

$$\therefore \quad ar(\Delta AOB) = \frac{1}{2} ar(||\, gm\ ABFE) \quad \ldots\text{(v)}$$

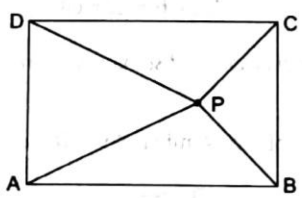
Similarly,

$$ar(\Delta COD) = \frac{1}{2} ar(||\, gm\ DEFC) \quad \ldots\text{(vi)}$$

On adding Eqs. (v) and (vi)

$$ar(\Delta AOB) + ar(\Delta COD)$$
$$= \frac{1}{2} ar(||\, gm\ ABCD)$$

Example 8 *ABCD is a parallelogram. P is any point within it. Show that the sum of the areas of ΔPAB, ΔPCD is equal to half the area of parallelogram.*



Solution If $AB = CD = a$

Let x, y be the lengths of the perpendicular from P on AB and CD respectively.

Then,
$$\Delta PAB + \Delta PCD = \frac{1}{2} ax + \frac{1}{2} ay$$
$$= \frac{1}{2} a(x + y) = \frac{1}{2} ah$$
$$= \frac{1}{2} \times \text{area of } ||\, gm\ ABCD$$

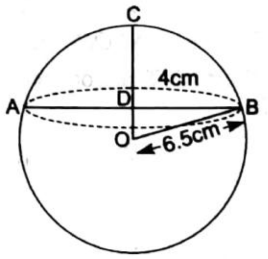
Where h is height of parallelogram.

Additional Solved Examples

Additional Solved Examples

Example 1. *A ball of diameter 13 cm is floating so that top of the ball is 4 cm above the smooth surface of the pond. What is the circumference (in cm) of the circle formed by the contact of water surface with the ball ?*

Solution We should find the circumference of the circle on AB as diameter.



$$CD = 4 \text{ cm}$$

$$OC = OB = \frac{13}{2} = 6.5 \text{ cm}$$

$$OD = 6.5 \text{ cm} - 4 \text{ cm} = 2.5 \text{ cm}$$

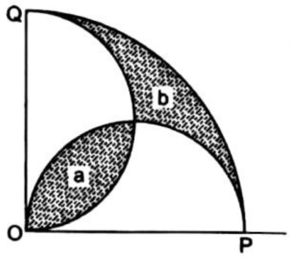
So, $$DB = \sqrt{(6.5)^2 - (2.5)^2} = 6 \text{ cm}$$

So, circumference of circle

$$= 2\pi \times 6 = 12\pi \text{ cm}$$

Example 2. *OPQ is a quadrant of a circle and semicircles are drawn on OP and OQ. Show that shaded areas a and b are equal.*

Solution Area of quadrant = areas of two semi circles + $b - a$.



i.e., $$\frac{1}{4}\pi r^2 = \frac{1}{2}\pi\left(\frac{r}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{r}{2}\right)^2 + b - a$$

$$\Rightarrow \quad \frac{1}{4}\pi r^2 = \frac{1}{4}\pi r^2 + b - a$$

$$\Rightarrow \quad b - a = 0$$

$$\Rightarrow \quad a = b$$

Example 3. *AB is a line segment of length 48 cm, C is its middle point. On AB, AC, CB semicircles are described.Determine the radius of the circle inscribed in the space enclosed by three semicircles.*

Solution Let X, Y be the centres of the semi circles described on AC, CB respectively as diameters O is the centre of the circle inscribed in the space enclosed by the three semicircles and r the radius of this circle

$$AX = XC = 12 \text{ cm}$$
$$OC \perp AB$$
$$OC = CD - OD = 24 - r$$
$$OX = OY = 12 + r$$

ΔXCO is right angled at C

So, $$OX^2 = XC^2 + OC^2$$

$\Rightarrow$ $$(12 + r)^2 = 12^2 + (24 - r)^2$$

$\Rightarrow$ $$72r = 576 \Rightarrow r = 8$$

$\therefore$ Radius of circle is 8 cm.

Example 4. *The radii of two circles in a plane are 13 and 18. Let AB be a diameter of the larger circle and BC a chord of larger circle tangent to smaller circle at D. Find AD.*

Solution $\because$ $$CB = 13 \text{ cm}, CD = 8 \text{cm}$$

$\therefore$ $$BD = \sqrt{13^2 - 8^2} = \sqrt{105} \text{ cm}$$

In ΔADB, C is mid point of AB.

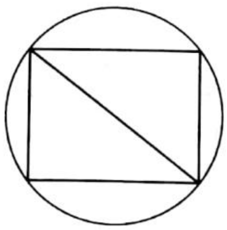
By Appolonius theorem

$$AD^2 + BD^2 = 2(AC^2 + CD^2)$$

$\Rightarrow$ $$AD^2 = 2(13^2 + 8^2) - 105 = 361$$

$\therefore$ $$AD = 19 \text{ cm}$$

Example 5. *Let A be the area of a square inscribed in a circle of radius r. Let B be the area of a hexagon inscribed in same circle. Obtain formula for A and B.*



Compute the ratio B / A.

Solution Each side of a square inscribed in a circle of radius $r = r\sqrt{2}$

$\therefore$ $$A = (r\sqrt{2})^2 = 2r^2$$

Each side of a regular hexagon inscribed in a circle of radius $r = r$

$\therefore$ $B = 6 \times$ area of an equilateral triangle having each side r

$$= 6 \times \frac{\sqrt{3}}{4} r^2 = \frac{3\sqrt{3}r^2}{2}$$

$\therefore$ $$B / A = \left[\frac{\frac{3\sqrt{3}}{2} r^2}{(2r^2)}\right] = \left[\frac{3\sqrt{3}}{4}\right]$$

Example 6. *ΔABC is an isosceles triangle. XY is drawn parallel to the base cutting the sides in X and Y, show that B, C, X, Y lies on a circle.*

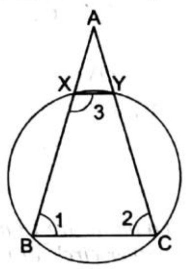
Solution $\because XY \parallel BC$ and AB meets them

$\therefore$ $$\angle BXY + \angle XBC = 2 \text{ right angles} \quad \ldots(i)$$

$\because$ $$AB = AC, \angle B = \angle C \quad \ldots(ii)$$

From Eqs. (i) and (ii)

$$\angle BXY + \angle BCY = 2 \text{ right angles}$$



Since, a pair of opposite angles of the quadrilateral. $BCYX$ is supplementary.

$\therefore$ Quadrilateral $BCYX$ is cyclic.

i.e., the points B, C, X, Y lie on circle.

Example 7. *The angles of a polygon having no reflex angle are in AP. The smallest angle is $2\pi/3$ radians and common difference is 5°. Find the number of sides.*

Solution Let the number of sides be n.

a the first term of AP

d common difference

Then, $a = 120°, d = 5$

$\therefore$ $$\frac{n}{2}\{2a + (n-1)d\}$$
$$= (2n-4) \times 90° = 180°n - 360°$$

We get $n^2 - 25n + 144 = 0$

So that $n = 16, 9$

If $n = 16$,

the greatest angle (in degrees)

$$= a + (n-1)d = 120 + 15.5 = 195$$

Which is not possible since polygon has no reflex angle.

The only other possibility being $n = 9$

It follows that the polygon has 9 sides.

Example 8. *Show that a triangle with sides 3, 4, 5, is the only right angled triangle with integer sides, whose semiperimeter equal its area.*

Solution Let a, b be the legs and c the hypotenuse of a right angle triangle having its area equal to its semiperimeter

Then, $$a^2 + b^2 = c^2 \quad \ldots(i)$$

$$\frac{1}{2}(a + b + c) = \frac{1}{2}ab \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$[ab - (a + b)]^2 = (a + b)^2 - 2ab$$

$$ab - 2(a + b) + 2 = 0$$

$\therefore$ $$(a - 2)(b - 2) = 2$$

$\because$ a, b are +ve integers

We must have either $a - 2 = 2$

$$b - 2 = 1\ ;\ \text{or } a - 2 = 1, b - 2 = 2$$

$\Rightarrow$ either $a = 4, b = 3$ or $a = 3, b = 4$

In either case $c = 5$

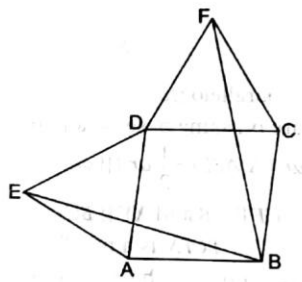
Thus a triangle with sides 3, 4, 5 is the only right angled triangle with integer sides whose semiperimeter equals its area.

Example 9. *ABCD is a square of which no angle is 60°. Equilateral Δ ADE and ΔDCF are drawn outwardly on the sides AD and DC.*

Show that $\Delta\ ABE \cong \Delta\ CFB$

Solution $AE = AD$ [$\because \Delta\ ADE$ is equilateral]

In $\Delta\ ABE$ and $\Delta\ CFB$,



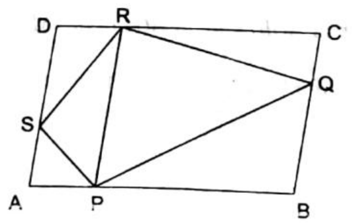
$$AB = CF,\ AE = BC$$

$$\angle\ EAB = \angle\ BCF$$

Therefore, $\Delta\ ABE \cong \Delta\ CFB$

Example 10. *ABCD is a parallelogram P, Q, R and S are points on sides AB, BC, CD, DA respectively, such that AP = DR. If area of || gm ABCD is 16 cm^2. Find area of the quadrilateral PQRS.*

Solution $\because$ $AP = DR$ and $AP \parallel DR$



$\therefore$ $APRD$ is a parallelogram

Consequently $PR \parallel AD$.

$\because$ Δ *PRS* and || *gm PRDA* have the same base *PR* and have the same altitude.

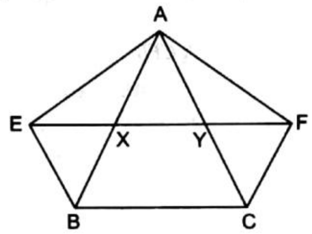
$\therefore$ $$\Delta\ PQR = \frac{1}{2} \times ar\,(||\,gm\ PBCR)$$

$$\begin{aligned} ar\,(\text{quad. } PSRQ) &= ar\,(\Delta\ PSR) + ar\,(\Delta\ PQR) \\ &= \frac{1}{2} \times ar\,(||\,gm\ PRDA) + \frac{1}{2} \times ar\,(||\,gm\ PRCB) \\ &= \frac{1}{2} \times ar\,(||\,gm\ ABCD) = \frac{1}{2} \times 16\,\text{cm}^2 = 8\,\text{cm}^2 \end{aligned}$$

Example 11. *XY is a line parallel to side BC of* Δ *ABC. BE* || *AC and CF* || *AB meet XY in E and F respectively.*

Show that ar (Δ *ABE*) = *ar* (Δ *ACF*)

Solution $\because$ *XY* || *BC* and *BE* || *CY*



$\therefore$ *BCYE* is a parallelogram.

$\because$ Δ *ABE* and || *gm BCYE* are on same base *BE* and between the same parallel lines *BE* and *AC*

$\therefore$ $$ar\,(\Delta\ ABE) = \frac{1}{2}\,ar\,(||\,gm\ BCYE) \quad \text{...(i)}$$

Now, *CF* || *AB* and *XY* || *BC*

$\Rightarrow$ *BCFX* is a parallelogram.

$\because$ Δ *ACF* and || *gm BCFX* are on same base *CF* and between the same parallel lines *AB* and *FC*.

$\therefore$ $$ar\,(\Delta\ ACF) = \frac{1}{2}\,ar\,(||\,gm\ BCFX) \quad \text{...(ii)}$$

But || *gm BCFX* and || *gm BCYE* are on same base *BC* and between the same parallel lines *BC* and *EF*

$\therefore$ $$ar\,(||\,gm\ BCFX) = ar\,(||\,gm\ BCYE) \quad \text{...(iii)}$$

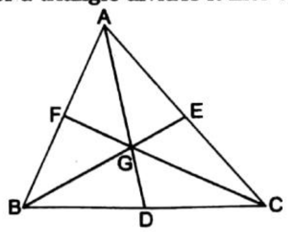
from Eqs. (i), (ii) and (iii), we get

$$ar\,(\Delta\ ABE) = ar\,(\Delta\ ACF)$$

Example 12. *If the medians of a* Δ *ABC intersect at G. Show that*

$$ar\,(\Delta\ AGB) = ar\,(\Delta\ AGC) = ar\,(\Delta BGC) = \frac{1}{3}\,ar\,(\Delta\ ABC)$$

Solution We know that the median of a triangle divides it into two triangles of equal area.



In Δ *ABC*, *AD* is the median

$\Rightarrow \quad ar(\Delta ABD) = ar(\Delta ACD) \quad ...(i)$

In Δ *GBC*, *GD* is the median

$\Rightarrow \quad ar(\Delta GBD) = ar(\Delta GCD) \quad ...(ii)$

Subtracting Eq. (ii) from Eq. (i), we get

$$ar(\Delta ABD) - ar(\Delta GBD) = ar(\Delta ACD) - ar(\Delta GCD)$$

$\Rightarrow \quad ar(\Delta AGB) = ar(\Delta AGC) \quad ...(iii)$

Similarly,

$$ar(\Delta AGB) = ar(\Delta BGC) \quad ...(iv)$$

From Eqs. (iii) and (iv), we get

$$ar(\Delta AGB) = ar(\Delta BGC) = ar(\Delta AGC)$$

But, $\quad ar(\Delta AGB) + ar(\Delta ABGC) + ar(\Delta AGC) = ar(\Delta ABC)$

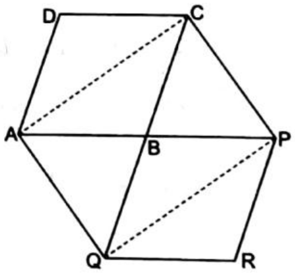
$\therefore \quad 3ar(\Delta AGB) = ar(\Delta ABC)$

$\Rightarrow \quad ar(\Delta AGB) = \frac{1}{3} ar(\Delta ABC)$

Hence, $\quad ar(\Delta AGB) = ar(\Delta AGC) = ar(\Delta BGC) = \frac{1}{3} ar(\Delta ABC)$

Example 13. *The side AB of a parallelogram ABCD is produed to any point P. A line through A parallel to CP meets CB produced in Q and the parallelogram PBQR completed.*
Show that ar (||gm ABCD) = ar (||gm BPRQ)

Solution Join *AC* and *PQ*



$\because \quad ar(\Delta ABC) = \frac{1}{2} ar(||gm\ ABCD) \quad ...(i)$

$$ar(\Delta PBQ) = \frac{1}{2} ar(||gm\ BPRQ) \quad ...(ii)$$

Δ *ACQ* and Δ *AQP* are on the same base *AQ* and between same parallel *AQ* and *CP*.

$\therefore \quad ar(\Delta ACQ) = ar(\Delta AQP)$

$\Rightarrow \quad ar(\Delta ACQ) - ar(\Delta ABQ) = ar(\Delta AQP) - ar(\Delta ABQ)$

$\Rightarrow \quad ar(\Delta ABC) = ar(\Delta BPQ) = \frac{1}{2} ar(||gm\ ABCD) = \frac{1}{2} ar(||gm\ BPRQ)$

[from Eqs. (i) and (ii)]

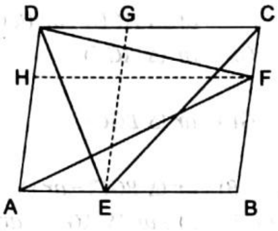
$\Rightarrow \quad ar(||gm\ ABCD) = ar(||gm\ BPRQ)$

Example 14. *In a || gm ABCD, E and F are any two points on sides AB and BC respectively. Show that ar (Δ ADF) = ar (Δ DCE)*

Solution Draw $EG \parallel AD$ and $FH \parallel BA$

$\because$ $FH \parallel AB$

$\therefore$ $ABFH$ is parallelogram



AF is a diagonal of || gm $ABFH$

$$\therefore \quad ar(\Delta AFH) = \frac{1}{2} ar(\parallel gm\ ABFH) \quad \ldots(i)$$

In || gm $DCFH$, DF is a diagonal

$$\therefore \quad ar(\Delta DFH) = \frac{1}{2} ar(\parallel gm\ DCFH) \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$ar(\Delta AFH) + ar(\Delta DFH)$$
$$= \frac{1}{2} ar(\parallel gm\ ABFH) + \frac{1}{2} ar(\parallel gm\ DCFH)$$
$$= \frac{1}{2}[ar(\parallel gm\ ABFH) + ar(\parallel gm\ DCFH)]$$
$$= \frac{1}{2} ar(\parallel gm\ ABCD)$$

$$\Rightarrow \quad ar(\Delta ADF) = \frac{1}{2} ar(\parallel gm\ ABCD) \quad \ldots(iii)$$

In || gm $AEGD$; DE is diagonal

$$\therefore \quad ar(\Delta DEG) = \frac{1}{2} ar(\parallel gm\ AEGD) \quad \ldots(iv)$$

In || gm $CBEG$; CE is diagonal.

$$\therefore \quad ar(\Delta CEG) = \frac{1}{2} ar(\parallel gm\ CBEG) \quad \ldots(v)$$

From Eqs. (iv) and (v), we get

$$ar(\Delta DEG) + ar(\Delta CEG)$$
$$= \frac{1}{2} ar(\parallel gm\ AEGD) + \frac{1}{2} ar(\parallel gm\ CBEG)$$
$$= \frac{1}{2}[ar(\parallel gm\ AEGD) + ar(\parallel gm\ CBEG)]$$
$$= \frac{1}{2}[ar(\parallel gm\ ABCD)]$$

$$\Rightarrow \quad ar(\Delta DCE) = \frac{1}{2} ar(\parallel gm\ ABCD) \quad \ldots(vi)$$

From Eqs. (iii) and (vi), we get

$$ar(\Delta ADF) = ar(\Delta DCE)$$

Example 15. *$BC \parallel XY$, $BX \parallel CA$ and $AB \parallel YC$*

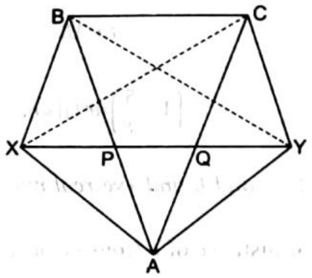
Prove that ar (Δ ABX) = ar (Δ ACY)

Solution Join XC and BY

$\because$ Δ BXC and Δ BCY are on same base BC and between the same parallel lines BC and XY.

$\therefore$ $$ar\ (\Delta\ BXC) = ar\ (\Delta\ BCY) \quad \text{...(i)}$$

Also, Δ BXC and Δ ABX are on the same base BX and between same parallel lines BX and AC



$\therefore$ $$ar\ (\Delta\ BXC) = ar\ (\Delta\ ABX) \quad \text{...(ii)}$$

Δ BCY and Δ ACY are on the same base CY and between the same parallel AB and CY

$\therefore$ $$ar\ (\Delta\ BCY) = ar\ (\Delta\ ACY) \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

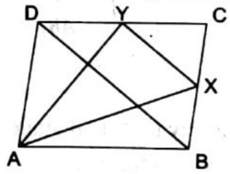
$$ar\ (\Delta\ ABX) = ar\ (\Delta\ ACY)$$

Example 16. *ABCD is a parallelogram. X and Y are mid points of BC and CD respectively. Prove that*

$$ar\ (\Delta\ AXY) = \frac{3}{8} ar\ (\| gm\ ABCD).$$

Solution Join BD

$\because$ X and Y are mid points of side BC and CD respectively in Δ BCD



$\therefore$ $$XY \parallel BD$$

and $$XY = \frac{1}{2} BD$$

$\Rightarrow$ $$ar\ (\Delta\ CYX) = \frac{1}{4}\ ar\ (\Delta\ DBC)$$

$$= \frac{1}{8} ar\ (\| gm\ ABCD) \quad \text{...(i)}$$

$$\left[\because \frac{1}{2} ar\ (\| gm\ ABCD) = ar\ (\Delta\ BCD)\right]$$

$\because$ $\| gm$ $ABCD$ and Δ ABX are between same parallel lines AD and BD and $BX = \frac{1}{2} BC$

$\therefore$ $$ar\ (\Delta\ ABX) = \frac{1}{4} ar\ (\| gm\ ABCD) \quad \text{...(ii)}$$

Similarly,

$$ar(\Delta AYD) = \frac{1}{4} ar(||gm\ ABCD) \quad ...(iii)$$

Now,

$$ar(\Delta AXY)$$
$$= ar(||gm\ ABCD) - [ar(\Delta ABX) + ar(\Delta AYD) + ar(\Delta CYX)]$$
$$= ar(||gm\ ABCD) - \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{8}\right)$$
$$ar(||gm\ ABCD)$$

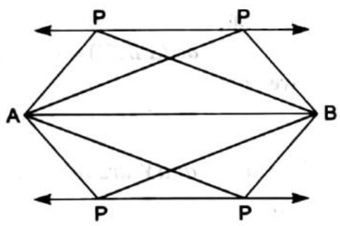
From Eqs. (i), (ii) and (iii)

$$\left(1 - \frac{5}{8}\right) ar(||gm\ ABCD) = \frac{3}{8} ar(||gm\ ABCD)$$

Example 17. *Given two points A and B and +ve real number x. Find the locus of a point P such that ar (Δ PAB) = x*

Solution Let the perpendicular distance of P from AB be h.

Then, $ar(\Delta PAB) = x$



$\Rightarrow$
$$\frac{1}{2} \times AB \times h = x$$

$\Rightarrow$
$$h = \frac{2x}{AB}$$

$\because$ AB and x are given.

$\therefore$ h is a fixed +ve real number.

Thus, the point P moves in such a way that its distance from AB is always same.

i.e., P lies on a line parallel to AB at a distance h from it.

But there are two such lines on either side of AB.

Hence, the locus of P is a pair of lines at a distance $h = \frac{2x}{AB}$ parallel to AB.

Example 18. Let G be the centroid of the ΔABC in which the angle at C is obtuse and let AD and CF be medians from A and C respectively onto the sides BC and AB. If the four points B, D, G and F are concyclic. Show that $\frac{AC}{BC} > \sqrt{2}$. If, further, P is a point on the line BG extended such that $AGCP$ is a parallelogram, show that the ΔABC and ΔGAP are similar.

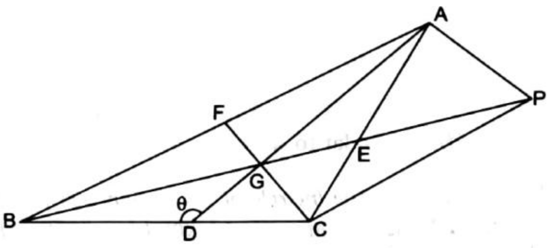
Solution If $\angle ADB = \theta$, then from ΔABD and ΔADC, we get

$$AB^2 = AD^2 + BD^2 - 2AD \cdot BD \cdot \cos\theta$$
$$AC^2 = AD^2 + DC^2 + 2AD \cdot DC \cos\theta$$

Adding,

$$a^2 + b^2 = 2AD^2 + \frac{1}{2}a^2$$

i.e.,
$$AD^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4}$$



Similarly, the other medians are given by

$$BE^2 = \frac{c^2 + a^2}{2} - \frac{b^2}{4},$$

$$CF^2 = \frac{a^2 + b^2}{2} - \frac{c^2}{4}$$

Since B, D, G, F lie on a circle we have

$$AF \cdot AB = AG \cdot AD$$

But
$$AG = \frac{2}{3}AD$$

Hence,
$$\frac{1}{2}c^2 = \frac{2}{3}AD^2 = \frac{1}{3}\left(b^2 + c^2 - \frac{a^2}{2}\right)$$

$\Rightarrow$
$$\frac{3}{2}c^2 = b^2 + c^2 - \frac{1}{2}a^2$$

$\Rightarrow$
$$b^2 = \frac{1}{2}(a^2 + a^2)$$

For the first part we have

$$b^2 = c^2 + a^2 - 2ca\cos B = 2b^2 - 2ca\cos B;$$

i.e.,
$$b^2 = 2ca\cos B,$$

$$a = c\cos B + b\cos C < c\cos B$$

Since $C > 90°$. Hence, $2a^2 < 2c\cos B = b^2; \frac{b}{a} > \sqrt{2}$.

This proves the first part.

For the second part, let the line passing through C and parallel to AG meet BG produced in P. Given that $AGCP$ is a parallelogram. So, AC and GP have the same mid point E.

Hence,
$$GP = 2GE = \frac{2}{3}BE$$

$$AG = \frac{2}{3}AD, AP = CG = \frac{2}{3}CF$$

Since,
$$b^2 = \frac{1}{2}(c^2 + a^2), \text{ we get}$$

$$AD^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2 = \frac{3}{4}c^2$$

$$BE^2 = \frac{1}{2}c^2 + \frac{1}{2}a^2 - \frac{1}{4}b^2 = \frac{3}{4}b^2$$

$$CF^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2 = \frac{3}{4}a^2$$

Hence,
$$AG = \frac{2}{3}\cdot\frac{\sqrt{3}}{2}c = \frac{1}{\sqrt{3}}c,$$

$$GP = \frac{2}{3}\cdot\frac{\sqrt{3}}{2}b = \frac{1}{\sqrt{3}}b,$$

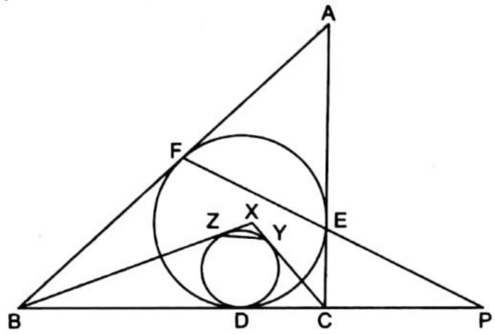
$$PA = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} a = \frac{1}{\sqrt{3}} a$$

Thus,
$$\frac{AG}{BA} = \frac{GP}{AC} = \frac{PA}{CB} = \frac{1}{\sqrt{3}}$$

So, AGP is similar to ABC.

Example 19. *The incircle of ABC touches BC, CA and AB at D, E and F respectively. X is a point inside Δ ABC such that the incircle of Δ XBC touches BC at D also and touches CX and XB at Y and Z respectively. Prove that EFZY is a cyclic quadrilateral.*

Solution Let P be the intersection of EF with BC. Then, by Menelaus' Theorem we have



$$\frac{BP}{PC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \qquad \text{...(i)}$$

Since, $CE = CD$, $EA = AF$ and $FB = BD$, we get

$$\frac{BP}{PC} \cdot \frac{CD}{BD} = 1$$

so that

$$\frac{BP}{PC} = \frac{BD}{CD} \qquad \text{...(ii)}$$

Since, $XZ = XY$, $BZ = BD$ and $CY = CD$, we have from Eq. (ii)

$$\frac{PB}{PC} \cdot \frac{CY}{YX} \cdot \frac{XZ}{ZB} = \frac{BD}{CD} \cdot \frac{CD}{YX} \cdot \frac{XY}{BD} = 1$$

Hence, by Menelaus' Theorem P, Z and Y are collinear.

Since, $PF \cdot PE = PD^2$ and $PZ \cdot PY = PD^2$ we have

$$PF \cdot PE = PZ \cdot PY$$

Hence, $EFZY$ is a cyclic quadrilateral.

Comment If $AB = AC$, then $BD = DC$ and then it can easily be proved that AD is the perpendicular bisector of EF and YZ so that $EFZY$ is an isosceles trapezoid and is a cyclic trapezoid.

Example 20. Let A, B and C be non-collinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$.

Solution *First Solution*:

From the hypothesis we have

$$AX^2 + AB^2 = CX^2 + CB^2 \qquad \text{...(i)}$$

If B_1 is the mid point of BX, applying the first theorem of the median in the triangles ΔABX, ΔCBX, we get

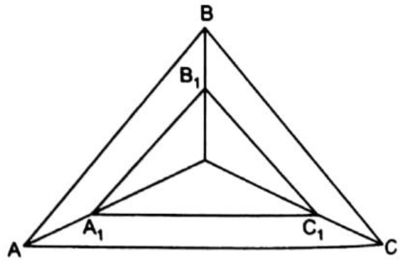
$$2AB_1^2 + 2BB_1^2 = 2CB_1^2 + 2BB_1^2 \text{ or } AB_1 = CB_1 \qquad \text{...(ii)}$$

This indicates that the perpendicular bisector of the side AC passes through the point B_1. Let A_1, C_1 be the mid point of AX and CX, respectively.

Similarly, we obtain that the perpendicular bisectors of BC and AB pass through the mid points A_1 and C_1 respectively. ...(iii)

Furthermore we obtain

$$AB \parallel A_1B_1,\ AC \parallel A_1C_1 \text{ and } BC \parallel B_1C_1. \qquad \text{...(iv)}$$



From Eqs. (iii) and (iv) we get the circumcentre O of ABC is the orthocentre H_1 of $A_1B_1C_1$. ...(v)

Also from Eq. (iv) the $\Delta\ ABC$ and $A_1B_1C_1$ are similar with X the centre of similarity and ratio $\frac{1}{2}$. ...(vi)

So, their orthocentres H and H_1 lie in the same straight line with the point X and $HH_1 = H_1X$ ...(vii)

Combining Eqs. (v) and (vii) we get $HO = OX$; that is the point X is known (constant), because X is symmetric to H with respect to the orthocentre O of ABC.

Example 21. A hexagon is inscribed in a circle with radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation

$$2r^3 - 7r - 3 = 0$$

Solution Equal chords subtend equal angles at the centre of a circle, if each of sides of length i subtends an angle α, $(i = 1, 2, 3)$ at the centre of the given circle, then $2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 360°$,

whence
$$\frac{\alpha_1}{2} + \frac{\alpha_2}{2} = 90° - \frac{\alpha_3}{2},$$

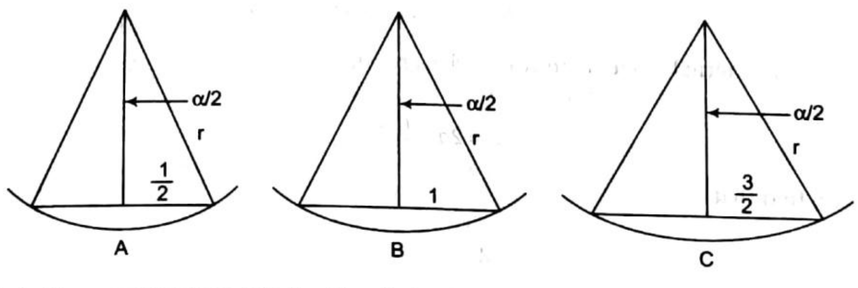
and
$$\cos\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) = \cos\left(90° - \frac{\alpha_3}{2}\right)$$
$$= \sin\frac{\alpha_3}{2}$$

Now, we apply the addition formula for the cosine :

$$\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} = \sin\frac{\alpha_3}{2}, \qquad \text{...(i)}$$

where (see figures)

$$\sin\frac{\alpha_1}{2} = \frac{1/2}{r}, \cos\frac{\alpha_1}{2} = \sqrt{\frac{4r^2 - 1}{2r}},$$
$$\sin\frac{\alpha_2}{2} = \frac{1}{r}, \cos\frac{\alpha_2}{2} = \frac{\sqrt{r^2 - 1}}{r},$$
$$\sin\frac{\alpha_3}{2} = \frac{3/2}{r}$$



We substitute these expressions into Eq. (i) and obtain, after multiplying both sides by $2r^2$,

$$\sqrt{4r^2 - 1} \cdot \sqrt{r^2 - 1} - 1 = 3r$$

Now, write it in the form

$$\sqrt{(4r^2 - 1)(r^2 - 1)} = 3r + 1$$

and square, obtaining

$$(4r^2 - 1)(r^3 - 1) = 9r^2 + 6r + 1$$

which is equivalent to

$$r(2r^3 - 7r - 3) = 0$$

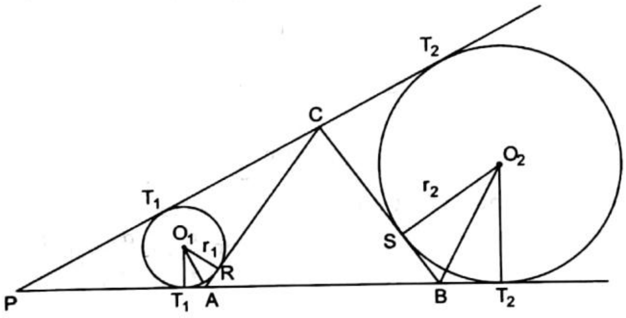
Since $r \neq 0$, we have

$$2r^3 - 7r - 3 = 0$$

which was to be shown.

Example 22. *Let ΔABC be equilateral. On side AB produced, we choose a point P such that A lies between P and B. We now denote a as the length of sides of ΔABC; r_1 as the radius of incircle of ΔPAC; and r_2 as the exradius of ΔPBC with respect to side BC. Determine the sum $r_1 + r_2$ as a function of a alone.*

Solution Looking at the figure, we see that $\angle T_1O_1R = 60°$ since it is the supplement of $\angle T_1AR = 120°$ (as an exterior angle for ΔABC). Hence, $\angle AO_1R = 30°$. Similarly, we obtain $\angle BO_2S = 30°$.



Since, tangents drawn to a circle an external point are equal, we have

$$\begin{aligned} T_1T_2 &= T_1A + AB + BT_2 \\ &= RA + AB + SB \\ &= r_1 \tan 30° + a + r_2 \tan 30° = \frac{r_1 + r_2}{\sqrt{3}} + a \end{aligned}$$

and

$$\begin{aligned} T_1'T_2' &= T_1'C + CT_2' \\ &= CR + CS = (a - RA) + (a - SB) = 2a - \frac{r_1 + r_2}{\sqrt{3}} \end{aligned}$$

Since, common external tangents to two circles are equal

$$T_1T_2 = T_1'T_2'$$

Hence,

$$\frac{r_1 + r_2}{\sqrt{3}} + a = 2a - \frac{r_1 + r_2}{\sqrt{3}},$$

whence we find that

$$r_1 + r_2 = \frac{a\sqrt{3}}{2}$$

Example 23. Let ABC be a triangle and a circle Γ' be drawn lying inside the triangle, touching its incircle Γ externally and also touching the two sides AB and AC. Show that the ratio of the radii of the circles Γ' and Γ is equal to $\tan^2\left(\frac{\pi - A}{4}\right)$.

Solution Let r, r' be the radii and I, I' be the centres of Γ, Γ' respectively.

Then,

$$\frac{r}{AI} = \sin\frac{A}{2} = \frac{r'}{AI'}$$

Hence,

$$\sin\frac{A}{2} = \frac{r - r'}{AI - AI'}$$

$$= \frac{r - r'}{II'};$$

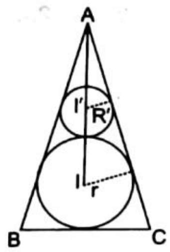
$$\frac{\sin A/2}{1} = \frac{r - r'}{r + r'},$$

$$\frac{1 - \sin A/2}{1 + \sin A/2} = \frac{(r + r') - (r - r')}{(r + r') + (r - r')} = \frac{2r'}{2r}$$

$$\frac{r'}{r} = \frac{1 - \cos(\pi/2 - A/2)}{1 + \cos(\pi/2 - A/2)}$$

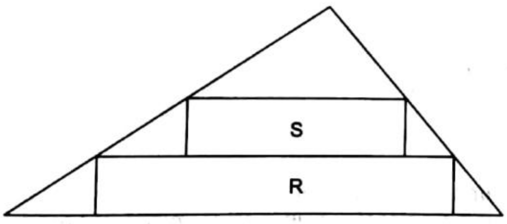
$$= \frac{2\sin^2\left(\frac{\pi - A}{4}\right)}{2\cos^2\left(\frac{\pi - A}{4}\right)}$$

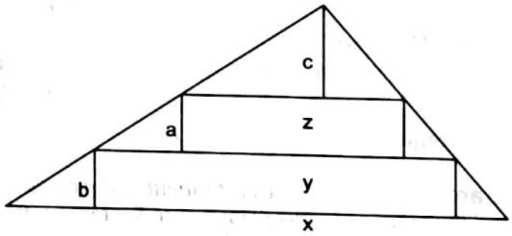
$$= \tan^2\left(\frac{\pi - A}{4}\right)$$



This proves the result.

Example 24. *Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown : Let A(X) denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\frac{A(R) + A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.*





Solution As in the figure

$$\frac{A(R) + A(S)}{A(T)} = \frac{ay + bz}{hx/2},$$

where $h = a + b + c$, the altitude of T. By similar triangles

$$\frac{x}{h} = \frac{y}{b + c} = \frac{z}{c},$$

so

$$\frac{A(R) + A(S)}{A(T)} = \frac{a\frac{(b + c)x}{h} + b\frac{cx}{h}}{hx/2} = \frac{2}{h^2}(ab + ac + bc)$$

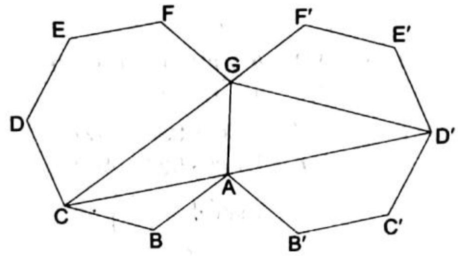
We need to maximize $ab + ac + bc$ subject to $a + b + c = h$. One way to do this is first to fix a, so $b + c = h - a$. Then,

$$ab + ac + bc = a(h - a) + bc,$$

and bc is maximized when $b = c$. We now wish to maximize $2ab + b^2$ subject to $a + 2b = h$. This is a straight-forward calculus problem giving $a = b = c = h/3$. Hence, the maximum ratio is 2/3 (independent of T).

Example 25. *Given is a regular 7-gon ABCDEFG. The sides have length 1. Prove for the diagonals AC and AD.*

$$\frac{1}{AC}+\frac{1}{AD}=1.$$



Solution Reflect the 7-gon with AG as an axis to obtain another 7-gon $AB'C'D'E'F'G'$.

Since, $$\angle GAC + \angle GAD' = \frac{4\pi}{7}+\frac{3\pi}{7}=\pi,$$

C, A and D' are collinear, besides

$$\angle GCA = \angle GD'A = \frac{\pi}{7}$$

$$= \angle CAB = \angle ACB$$

Therefore, $\Delta GCD \sim \Delta BAC$ giving

$$\frac{AC}{AB}=\frac{CD'}{CG}=\frac{AC+AD'}{AD}=\frac{AC+AD}{AD}.$$

But $AB = 1$ so

$$\frac{1}{AC}+\frac{1}{AD}=\frac{AC+AD}{AC\cdot AD}=\frac{1}{AB}=1,$$

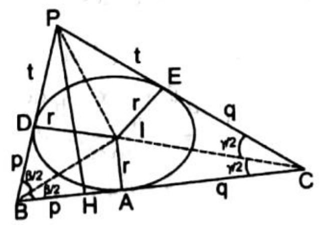
as required.

Example 26. *The line l is tangent to the circle S at the point A; B and C are points on l on opposite sides of A and the other tangents from B, C to S intersect at a point P. If B, C vary along l in such a way that the product $|AB|\cdot|AC|$ is constant, find the locus of P.*

Solution Let S be the incircle $S(1, 2)$ of ΔBCP. We denote $\angle PBA = \beta$, $\angle PCA = \gamma$.

$$\overline{AB} = p\overline{AC} = q \text{ with } pq = k^2\text{, a constant.}$$

Let S touch BP and CP and D and E respectively. For ΔPEI



we have $$\angle EIP = \frac{1}{2}(\beta+\gamma)$$

Thus, $$t = r\tan\frac{1}{2}(\beta+\gamma)=\frac{(p+q)r^2}{pq-r^2}$$

The semiperimeter of ΔBCP is

$$p + q + t = p + q + \frac{(p+q)r^2}{pq - r^2} = \frac{pq(p+q)}{pq - r^2}$$

The area F, of ΔBCP is

$$r\,\frac{pq(p+q)}{pq - r^2} = \frac{1}{2}(p+q)\,PH,$$

where PH is the altitude to BC. It follows immediately that

$$PH = \frac{2pqr}{pq - r^2} = \frac{2k^2 r}{k^2 - r}$$

So, the locus of P is a line parallel to BC.

Example 27. *In a Δ ABC, $\angle A$ is twice $\angle B$. Show that $a^2 = b(b + c)$.*

Solution $A = 2B \Rightarrow \angle C = \pi - 3B$, Since

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

we have

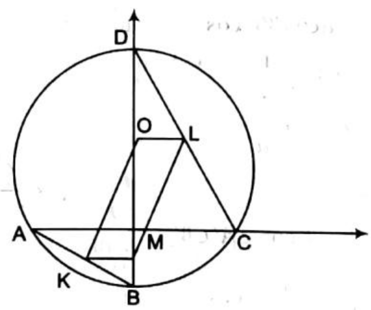
$$a^2 = b(b+c) \Leftrightarrow \sin^2 A = \sin B(\sin B + \sin C) \Leftrightarrow \sin^2 2B = \sin B(\sin B + \sin 3B)$$

$$\Leftrightarrow \quad \sin^2 2B = \sin B \cdot 2\sin 2B \cdot \cos B$$

$$\Leftrightarrow \quad \sin^2 2B = 2\sin B\cos B \cdot \sin 2B$$

$$\Leftrightarrow \quad \sin^2 2B = \sin^2 2B$$

Example 28. *Let AC and BD be two chords of a circle with centre Q such that they intersect at right-angles inside the circle at the point M. Suppose K and L are the mid points of the chords AB and CD respectively. Prove that OKML is a parallelogram.*



Solution Choose M as the origin, AC as the x-axis and BD as the y-axis. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. The x-coordinates of A and C given by

$$x^2 + 2gx + c = 0$$

If A is $(-g - \sqrt{g^2 - c}, 0)$, then C is $(-g + \sqrt{g^2 - c}, 0)$. The y-coordinates of B and D are given by

$$y^2 + 2fy + c = 0$$

If is $(0, -f - \sqrt{f^2 - x})$, then D is $(0, -f + \sqrt{f^2 - c})$.

Thus, K is $\left(\frac{-g-\sqrt{g^2-c}}{2}, \frac{-f-\sqrt{f^2-c}}{2}\right)$,

$$L \text{ is } \left(\frac{-g+\sqrt{g^2-c}}{2}, \frac{-f+\sqrt{f^2-c}}{2}\right)$$

Hence, the mid point of KL is $\left(\frac{-g}{2}, \frac{-f}{2}\right)$. Since, O is $(-g, -f)$ and M is $(0, 0)$ the mid point of OM also is $\left(\frac{-g}{2}, \frac{-f}{2}\right)$. Hence, OM and KL have the same mid point and so $OKML$ is a parallelogram.

Aliter Let the radius of the circle be R, $AB = a$, $CD = b$,
$\angle ADB = \theta$ and $\angle CAD = \phi$. Then, $\theta + \phi = 90°$ from the right angled triangle AMD.

Thus,
$$a = AB = 2R\sin\theta,$$
$$b = CD = 2R\sin\phi = 2R\cos\theta,$$
$$a^2 + b^2 = 4R^2.$$

Hence, $OK^2 = R^2 - \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2$. But L is the circumcentre of the right angled $\Delta\, CMD$.

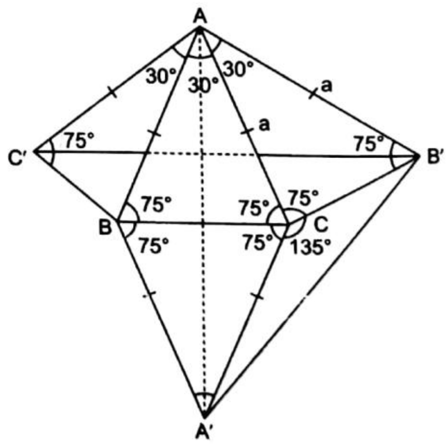
Hence, $\frac{1}{2}b = LC = LM$. Thus, $OK = ML$.

Similarly $OL = KM$. This proves the result. The arguments show that M may be outside the circle also.

Example 29. *In ΔABC, $AB = AC$ and $\angle BAC = 30°$, A', B', C' are reflections of A, B, C respectively on BC, CA, AB, show that $\Delta A'B'C'$ is equilateral.*

Solution $\Delta A'BC$, $\Delta B'AC$, $\Delta C'AB$ are reflections of ΔABC on BC, CA, AB respectively, $\angle BAC = 90°$, If $AB = AC = a$, $B'C' = \sqrt{2}\,a$ since $\Delta B'AC'$ is a right angled isosceles triangle.

$$BC = 2a\cos 75° = 2a\cos(30° + 45°)$$
$$= 2a(\cos 30°\cos 45° - \sin 30°\sin 45°)$$
$$= 2a\left(\frac{\sqrt{3}}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{2}}\right)$$
$$= \frac{(\sqrt{3}-1)}{\sqrt{2}}a$$



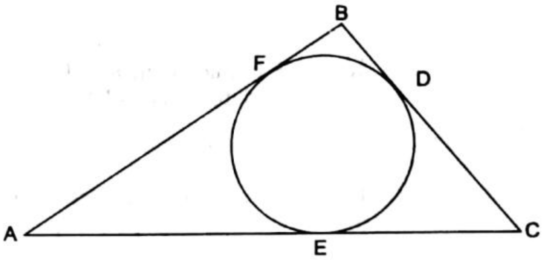
By the cosine formula (in $\Delta A'CB'$).

$$(A'B')^2 = A'C^2 + CB'^2 - 2A'C' \cdot CB'\cos\angle A'CB'$$
$$= a^2 + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 - 2a\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)a\cos 135°$$
(since $CB' = BC$)
$$= a^2\left[1 + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 + 2\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\right]$$
$$= a^2(1 + 2 - \sqrt{3} + \sqrt{3} - 1)$$
$$= 2a^2 \quad (\text{since } \cos 135° = \cos(90 + 45°) = -\sin 45°)$$

Hence, $A'B' = \sqrt{2}a = B'C'$. We can similarly show that $A'C' = \sqrt{2}a$ so that $\Delta A'B'C'$ is equilateral.

Example 30. *In a Δ ABC, the incircle touches the sides BC, CA and AB respectively at D, E and F. If the radius of the incircle is 4 units and if BD, CE and AF are consecutive integers, find the sides of the Δ ABC.*

Solution Suppose that $BD = x$,



$$CE = x + 1$$

and $AF = x + 2.$

Then, $CD = CE = x + 1,$

$$AE = AF = x + 2,$$

$$BF = BD = x,$$

Hence, $a = BC = x + x + 1 = 2x + 1,$

$$b = CA = x + 1 + x + 2 = 2x + 3,$$

$$c = AB = x + 2 + x = 2x + 2;$$

$$s = \frac{a+b+c}{2} = 3x + 3;$$

$$s - a = x + 2, s - b = x, s - c = x + 1;$$

Since, $r = \frac{\Delta}{s} = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)},$

we get $4 = \sqrt{\frac{(x+2)x(x+1)}{3x+3}}$

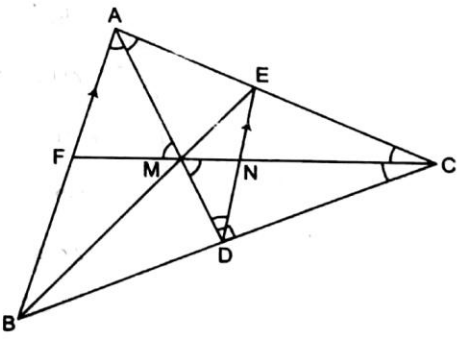
i.e., $16 = \frac{x(x+2)}{3}; x^2 + 2x = 48$

i.e., $(x + 8)(x - 6) = 0; x = 6 \text{ or } -8$

Since, x cannot be negative $x = 6$ and $a = 13, b = 15, c = 14.$

Example 31. *Let ABC be an acute angled triangle in which D, E, F are points on BC, CA, AB respectively such that AD $\perp$ BC; AE = EC and CF bisects $\angle C$ internally. Suppose CF meets AD and DE in M and N respectively. If FM = 2, MN = 1, NC = 3, find the perimeter of the Δ ABC.*

Solution $FN = 3 = NC$. So, N is mid point of FC and E the mid point of AC. Hence $ND \parallel AB$ and so D is the mid point of BC. Thus AD is altitude as well as median, so that $AB = AC$. AD is also angle bisector of $\angle A$. ΔAFM and ΔDNM are similar since



$$\angle FAM = \angle MDN,$$

$\angle AMF = \angle DMN.$

So, $\frac{AM}{MD} = \frac{FM}{MN} = \frac{2}{1}$

Thus, M is the centroid of ΔABC. Since, CF passes through M, CF is also a median, *i.e.*, angle bisector CR is also median. Hence, $CA = CB$; Consequently ΔABC is equilateral $CF = 6 =$ altitude of an equilateral triangle. Side of the equilateral triangle $= \frac{12}{\sqrt{3}}$. Perimeter $= \frac{12}{\sqrt{3}}$.

Aliter Having proved $AB = AC$, consider M which is the incentre and MD the inradius. D is the point where the incircle touches BC. Now BM is the angle bisector of ΔFBC.

$$\frac{FM}{MC} = \frac{FB}{BC} = \frac{2}{4} = \frac{1}{2}$$

So, $FB = \frac{BC}{2} = BD \cdot F$ is the point where the incircle touches AB. But angle bisector CF meets AB at F. $CF \perp AB$. So, CF is also median, altitude and angle bisector. Hence, the triangle is equilateral.

Aliter Consider ΔAFC and the fact that AM is the angle bisector of $\angle A$

$$\frac{AF}{AC} = \frac{FM}{MC} = \frac{1}{2},$$

$$AF = \frac{AC}{2} = \frac{AB}{2}$$

Since, $AB = AC$. So, F is mid point of AB. Hence, CF is the median, angle bisector and altitude, so $AB = CA = CB$. Hence, triangle is equilateral.

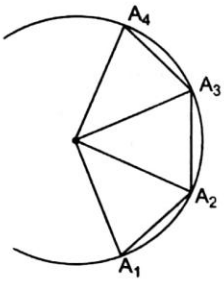
Note Can also be solved using coordinate geometry with DA as origin BC as x-axis and AD as y-axis.

Example 32. Let $A_1A_2 \ldots A_n$ be an n-sided regular polygon such that

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}.$$

Determine n, the number of sides of the polygon.

Solution Let O be the centre of the polygon and $OA_1 = r$. Let θ be the angle subtended by any side at the centre.



Then, $$\theta = \frac{2\pi}{n}.$$

Also, $$A_1A_2 = 2r\sin\frac{\theta}{2}, A_1A_3 = 2r\sin\theta, A_1A_4 = 2r\sin\frac{3\theta}{2}$$

$\therefore$ $$\frac{1}{2r\sin\frac{\theta}{2}} = \frac{1}{2r\sin\theta} + \frac{1}{2r\sin\frac{3\theta}{2}}$$

$\Rightarrow$ $$\frac{1}{\sin\frac{\theta}{2}} = \frac{1}{\sin\theta} + \frac{1}{\sin\frac{3\theta}{2}}$$

$\Rightarrow$ $$\frac{1}{\sin\frac{\theta}{2}} = \frac{\sin\theta + \sin\frac{3\theta}{2}}{\sin\theta\cdot\sin\frac{3\theta}{2}}$$

$\Rightarrow$ $$2\sin\frac{3\theta}{2}\cos\frac{\theta}{2} = \sin\theta + \sin\frac{3\theta}{2}$$

$\Rightarrow$ $$\sin 2\theta + \sin\theta = \sin\theta + \sin\frac{3\theta}{2}$$

$\Rightarrow$ $$\sin 2\theta - \sin\frac{3\theta}{2} = 0$$

$\Rightarrow$ $$2\cos\frac{7\theta}{4}\cdot\sin\frac{\theta}{4} = 0$$

$\therefore$ $$\frac{7\theta}{4} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$\Rightarrow$ $$\theta = \frac{2\pi}{7}, \frac{6\pi}{7}, \ldots$$

Since, $\theta = \frac{2\pi}{0}$, we must have $\theta = \frac{2\pi}{7}$ and $n = 7$.

Let us Practice

Let us Practice

Level 1

1. Let $ABCD$ be a square and M, N points on sides AB, BC respectively, such that $\angle MDN = 45°$. If R is the mid point of MN, show that $RP = RQ$ where P, Q are the points of intersection of AC with the lines MD, ND.

2. Prove that the inradius of a right-angled triangle with integer sides is an integer.

3. The cyclic octagon $A\ B\ C\ D\ E\ F\ G\ H$ has sides, a, a, a, a, b, b, b, b respectively. Find the radius of the circle that circumscribes $A\ B\ C\ D\ E\ F\ G\ H$.

4. $ABCD$ is quadrilateral and P, Q are mid points of CD, AB, AP, DQ meet at X, and BP, CQ meet at Y. Prove that area of ΔADX + area of ΔBCY = area of quadrilateral $PXQY$.

5. $ABCD$ is a cyclic quadrilateral; x, y, z are the distances of A from the lines BD, BC, CD respectively. Prove that
$$\frac{BD}{x} = \frac{BC}{y} + \frac{CD}{z}$$

6. $ABCD$ is a cyclic quadrilateral with $AC \perp BD$ and AC meets BD at E. Prove that
$$EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$$
where R is the radius of the circumscribed circle.

7. Let Γ and Γ' be concentric circles. Let ABC, $A'B'C'$ be any 2 equilateral triangles inscribed in Γ and Γ' respectively. If P and P' are any two points on Γ and Γ' respectively, show that
$$P'A^2 + P'B^2 + P'C^2 = A'P^2 + B'P^2 + C'P^2$$

8. The internal bisector of $\angle A$ in a ΔABC with $AC > AB$, meets the circumcircle Γ of the triangle in D. Join D to the centre O of the circle Γ and suppose DO meets AC in E, possibly when extended. Given that BE is perpendicular to AD, show that AO is parallel to BD.

9. Let AC be a line segment in the plane and B a point between A and C. Construct isosceles ΔPAB and ΔQBC on one side of the segment AC such that $\angle APB = \angle BQC = 120°$ and an isosceles ΔRAC on the other side of AC such that $\angle ARC = 120°$ Show that PQR is an equilateral triangle.

10. In a ΔABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72°$.

11. Let BE and CF be the altitudes of an acute ΔABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

12. The circumference of a circle is divided into eight arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose $p + r = q + s$. Prove that $ABCD$ is a cyclic quadrilateral.

13. In an acute ΔABC, points D, E, F are located on the sides BC, CA, AB respectively such that
$$\frac{CD}{CE} = \frac{CA}{CB}, \frac{AE}{AF} = \frac{AB}{AC}, \frac{BF}{BD} = \frac{BC}{BA}$$
Prove that AD, BE, CF are the altitudes of ABC.

14. Suppose P is an interior point of a ΔABC such that the ratios
$$\frac{d(A, BC)}{d(P, BC)}, \frac{d(B, CA)}{d(P, CA)}, \frac{d(C, AB)}{d(P, AB)}$$
are all equal. Find the common value of these ratios. [Here $d(X, YZ)$ denotes the perpendicular distance from a point X to the line YZ.]

15. Let ABC be a triangle in which $AB = AC$ and $\angle CAB = 90°$. Suppose M and N are points on

the hypotenuse BC such that $BM^2 + CN^2 = MN^2$. Prove that $\angle MAN = 45°$.

16. Let $ABCD$ be a quadrilateral; X and Y be the mid points of AC and BD respectively and the lines through X and Y respectively parallel to BD, AC meet in O. Let P, Q, R, S be the mid points of AB, BC, CD, DA respectively. Prove that

(a) quadrilaterals $APOS$ and $APXS$ have the same area;

(b) the areas of the quadrilateral $APOS, BQOP, CROQ, DSOR$ are all equal.

17. Consider in the plane a circle Γ with centre O and a line l not intersecting circle Γ. Prove that there is a unique point Q on the perpendicular drawn from O to the line l, such that for any point P on the line l, PQ represents the length of the tangent from P to the circle Γ.

18. In Δ ABC, let D be th mid point of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, Determine $\angle BAD$.

19. Let $ABCD$ be a convex quadrilateral; P, Q, R, S be the mid points of AB, BC, CD, DA respectively such that Δ AQR and Δ CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.

20. Let $ABCD$ be a quadrilateral in which AB is parallel to CD and perpendicular to AD; $AB = 3CD$; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.

21. A 6×6 square is dissected in to 9 rectangles by lines parallel to its sides such that all these rectangles have integer sides. Prove that there are always two congruent rectangles.

22. Let ABC be an acute-angled triangle and let D, E, F be the feet of perpendiculars from A,B,C respectively to BC,CA,AB. Let the perpendiculars from F to CB, CA, AD, BE meet them in P, Q, M, N respectively. Prove that P, Q, M, N are collinear.

23. Let ABC be an acute-angled triangle; AD be bisector of $\angle BAC$ with D on BC; and BE be the altitude from B on AC. Show that $\angle CED > 45°$.

24. A trapezium $ABCD$, in which AB is parallel to CD, is inscribed in a circle with centre O. Suppose the diagonals AC and BD of the trapezium intersect at M and $OM = 2$.

(a) If $\angle AMB$ is 60°, determine, with proof the difference between the lengths of the parallel sides.

(b) If $\angle AMD$ is 60°, find the difference between the lengths of the parallel sides.

25. Let ABC be an acute-angled triangle; let D, F be the mid-points of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular at B to BC meet in N. Prove that ND is equal to the circumradius of ABC.

26. Let ABC be a triangle in which $AB = AC$ and let I be its incentre. Suppose $BC = AB + AI$. Find $\angle BAC$.

27. A convex polygon Γ is such that the distance between any two vertices of Γ does not exceed 1.

(i) Prove that the distance between any two points on the boundary of Γ does not exceed 1.

(ii) If X and Y are two distinct points inside Γ, prove that there exists a point Z on the boundary of Γ such that $XZ + YZ \le 1$.

28. Let $ABCDEF$ be a convex hexagon in which the diagonals AD, BE, CF are concurrent at O. Suppose the area of Δ OAF is the geometric mean of those of OAB and OEF, and the area of Δ OBC is the geometric mean of those of OAB and OCD. Prove that the area of Δ OED is the geometric mean of those of OCD and OEF.

29. Let ABC be a triangle in which $\angle A = 60°$ Let BE and CF be the bisectors of the angles $\angle B$ and $\angle C$ with E on AC and F on AB. Let M be the reflection of A in the line EF. Prove that M lies on BC.

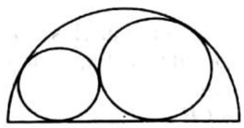
Level 2

1. A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units find the area of the rectangle $ABCD$.

2. In an acute-angled $\Delta\ ABC$, $\angle A = 30°$, H is the orthocentre, and M is the mid point of BC. On the line HM, take a point T such that $HM = MT$. Show that $AT = 2BC$.

3. The inscribed circumference in the $\Delta\ ABC$ is tangent to BC, CA and AB at D, E and F, respectively. Suppose that this circumference meets AD again at its mid-point X; that is, $AX = XD$. The lines XB and XC meet the inscribed circumference again at Y and Z, respectively.
Show that $EY = FZ$.

4. Two externally tangent circles of radii R_1 and R_2 are internally tangent to a semicircle of radius 1, as in the figure.



Prove that $R_1 + R_2 \le 2(\sqrt{2} - 1)$
with equality holds if and only if $R_1 = R_2$.

5. T_1 is an isosceles triangle with circumcircle K. Let T_2 be another isosceles triangle inscribed in K whose base is one of the equal sides of T_1 and which overlaps the interior of T_1. Similarly create isosceles triangles T_3 from T_2, T_4 from T_3 and so on. Do the triangles T_n approach an equilateral triangle as $n \to \infty$?

6. The incircle of ΔABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of $\Delta\ ABC$.

7. In a convex quadrilateral $PQRS$, $PQ = RS$, $(\sqrt{3} + 1)QR = SP$ and $\angle RSP - \angle SPQ = 30°$. Prove that $\angle PQR - \angle QRS = 90°$

8. Let ABC be a triangle in which no angle is 90°. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements;
(a) If P is the incentre or an excentre of ABC, then P is the circumcentre of $A_1B_1C_1$;
(b) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$;
(c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.

9. Let ABC be a triangle and D be the mid point of side BC. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15°$. Show that $\angle ADC$ is obtuse. Further, if O is the circumcentre of ADC, prove that $\Delta\ AOD$ is equilateral.

10. For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following six statements :
(a_1) AB is parallel to DE ;
(a_2) $AE = BD$;
(b_1) BC is parallel to EF;
(b_2) $BF = CE$;
(c_1) CD is parallel to FA;
(c_2) $CA = DF$.
(a) Show that, if all the six statements are true; then the hexagon is cyclic (*i.e.*, it can be inscribed in a circle).
(b) Prove that, in fact, any five of these six statements also imply that the hexagon is cyclic.

11. Let ABC be a triangle with sides a, b, c. Consider a $\Delta\ A_1B_1C_1$ with sides equal to $a + \dfrac{b}{2}$, $b + \dfrac{c}{2}$, $c + \dfrac{a}{2}$.
Show that $[A_1B_1C_1] \ge \dfrac{9}{4}[ABC]$,
where $[XYZ]$ denotes the area of the $\Delta\ XYZ$.

12. Consider an acute $\Delta\ ABC$ and let P be an interior point of ABC. Suppose the lines BP and CP, when produced, meet AC and AB in E and F respectively. Let D be the point where AP intersects the line segment EF and K be the foot of perpendicular from D on to BC. Show that DK bisects $\angle EKF$.

13. Let R denote the circumradius of a Δ ABC; a, b, c its sides BC, CA, AB; and r_a, r_b, r_c its exradii opposite A, B, C. If $2R \leq r_a$, prove that

(i) $a > b$ and $a > c$;

(ii) $2R > r_b$ and $2R > r_c$.

14. Consider a convex quadrilateral $ABCD$, in which K, L, M, N are the mid points of the sides AB, BC, CD, DA respectively. Suppose

(a) BD bisects KM at Q;

(b) $QA = QB = QC = QD$; and

(c) $LK/LM = CD/CB$.

Prove that $ABCD$ is a square.

15. Let M be the mid point of side BC of a Δ ABC. Let the median AM intersect the incircle of ABC at K and L, K being nearer to A than L. If $AK = KL = LM$, prove that the sides of Δ ABC are in the ratio $5:10:13$ in some order.

16. In a non-equilateral Δ ABC, the sides a, b, c form an arithmetic progression. Let I and O denote the incentre and circumcentre of the triangle respectively.

(i) Prove that IO is perpendicular to BI.

(ii) Suppose BI extended meets AC in K and D, E are the mid points of BC, BA respectively. Prove that I is the circumcentre of Δ DKE.

17. In a cyclic quadrilateral $ABCD$, $AB = a, BC = b, CD = c$, $\angle ABC = 120°$, and $\angle ABD = 30°$. Prove that

(i) $c \geq a + b$;

(ii) $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$.

18. Let ABC be a triangle in which $AB = AC$. Let D be the mid point of BC and P be a point on AD. Suppose E is the foot of perpendicular from P on AC. If $\dfrac{AP}{PD} = \dfrac{BP}{PE} = \lambda$, $\dfrac{BD}{AD} = m$ and $z = m^2(1+\lambda)$, prove that

$$z^2 - (\lambda^3 - \lambda^2 - 2)z + 1 = 0.$$

Hence, show that $\lambda \geq 2$ and $\lambda = 2$, if and only if and only if ABC is equilateral.

19. In a Δ ABC right angled at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that

$$\frac{5}{2} < \frac{AB}{BC} < 3.$$

20. Let ABC be a triangle, I its incentre; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circumcircle of $\Delta A_1B_1C_1$ passes through A, Prove that B_1, C_1, I, I_1 are concyclic, where I_1 is the incentre of Δ $A_1B_1C_1$.

21. Let ABC be a triangle; $\Gamma_A, \Gamma_B, \Gamma_C$ be the three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB and BC; and Γ_C touches BC and CA. Let Γ be a circle touching circles $\Gamma_A, \Gamma_B, \Gamma_C$ externally. Prove that the line joining the circumcentre O and the incentre I of ΔABC passes through the centre of Γ.

22. Let ABC be a triangle and let P be interior point such that $\angle BPC = 90°$, $\angle BAP = \angle BCP$. Let M, N be the mid points of AC, BC respectively. Suppose, $BP = 2PM$. Prove that A, P, N are collinear.

23. Let ABC be an acute angles triangle and let H be its orthocentre. Let h_{max} denote the largest altitude of the ΔABC. Prove that $AH + BH + CH \leq 2h_{max}$.

24. Let ABC be an acute-angled triangle with altitude AK. Let H be its orthocentre and O be its circumcentre. Suppose KOH is an acute-angled triangle and P its circumcentre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid points of AB and AC.

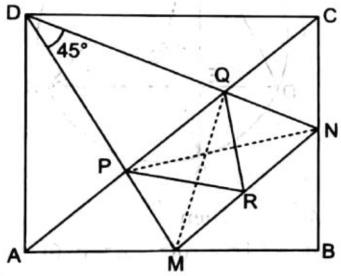
25. Let ABC be a triangle with circumcircle Γ. Let M be a point in the interior of Δ ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB; and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

Solutions

Solutions

Level 1

1. As construction join PN and QM. In the figure AC is a diagonal of the quadrilateral $ABCD$ with $\angle ACB = 45°$ and $\angle MDN = 45°$ (by hypothesis). So, $\angle PDN = \angle PCN$ each angle being equal to 45°. Now, as $\angle PDN$ and $\angle PCN$ are equal angles subtended by the same line segment PN at points D and C, the quadrilateral $PNCD$ is cyclic. So $\angle DCN + \angle DPN = 180°$. But $\angle NCD = 90°$ which implies that $\angle NPD = 90°$. So $\angle NPM = 90°$, as $\angle NPD$ and $\angle NPM$ form a linear pair. Now, as $\angle NPM = 90°$, MN is the diameter of the circumcircle of the right ΔNPM with R as its centre as R is the mid point of MN by hypothesis.



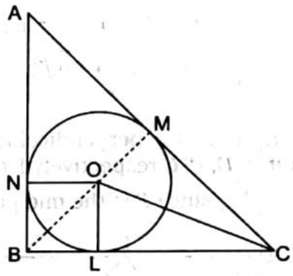
Similarly by showing that quadrilateral $QMAD$ is a cyclic one and arguing as before, we can conclude that MN is the diameter of the circumcircle of the right ΔNQM with $\angle NQM = 90°$ and R as its centre.

The two preceding paragraphs imply that $MNQM$ is a cyclic quadrilateral with R as its centre. Hence $RP = RQ$.

2. Let ABC be a right triangle with $\angle B = 90°$.

Let O be its incentre and L, M, N the points of contact of the incircle with the a, b, c respectively.

Suppose that the inradius is r. Now as $\angle ABC = 90°$, the quadrilateral $NBLO$ is a square. So $NB = BL = r$. Also, as the two tangents drawn from an external point to a circle are of equal length, we have



$$AM = AN = AB - NB = c - r \text{ and}$$
$$CM = CL = BC - CL = a - r.$$

So, $b = AC = AM + CM = c - r + a - r$

$$= c + a - 2r \Rightarrow r = \frac{b - (c + a)}{2}$$

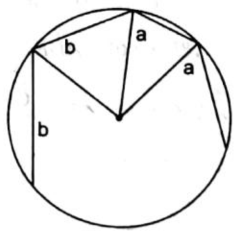
As $\angle B = 90° \Rightarrow b^2 = c^2 + a^2$, we have

(i) if c and a are both odd or both even, $c^2 + a^2$ is even $\Rightarrow b^2$ is given $\Rightarrow b$ is even $\Rightarrow b - (c + a)$ is even.

(ii) if one of c and a is even and the other odd, $c^2 + a^2$ is odd $\Rightarrow b^2$ is odd $\Rightarrow b$ is odd $\Rightarrow$ $b - (c + a) =$ an even number.

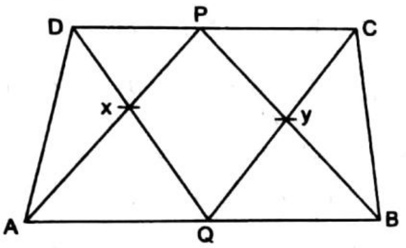
So, in any case, if a, b, c are integers, we have $r = \dfrac{b - (c + a)}{2} =$ an integer.

3. Let r be the radius of the circumcircle and θ, φ angle subtended by the sides of lengths a and b respectively at the centre of the circle. Then, $4\theta + 4\varphi = 2\pi$ and hence $\varphi = \dfrac{\pi}{2} - \theta$. Using cosine rule, we have



$$a^2 = 2r^2 - 2r^2\cos\theta = 2r^2(1-\cos\theta)$$
$$b^2 = 2r^2 - 2r^2\sin\theta = 2r^2(1-\sin\theta)$$
$$\left(\frac{a}{b}\right)^2 = \frac{1-\cos\theta}{1-\sin\theta} = \frac{2t^2}{(1-t)^2}, \text{ where } t = \tan\frac{\theta}{2}$$
$$\therefore \quad \frac{a}{b} = \frac{\sqrt{2}\,t}{1-t} \Rightarrow \frac{a}{\sqrt{2}b} = \frac{t}{1-t} \Rightarrow t = \frac{a}{a+\sqrt{2}b}$$
$$\therefore \quad 1-\cos\theta = \frac{2b^2}{1+t^2} = \frac{a^2}{a^2+b^2+ab\sqrt{2}}$$
$$\therefore \qquad r = \sqrt{\frac{a^2+b^2+ab\sqrt{2}}{2}}$$

4. Let d_1, d_2, d_3 be the perpendicular distances of the points D, P, C respectively from AB. Then, $d_2 = \frac{d_1+d_3}{2}$, since P is the mid point of CD.



Area of quad. $PXQY$

$= \text{area of } \Delta DQC - \text{area } \Delta DXP - \text{area of } \Delta PYC$

$= \frac{1}{2}\cdot CD\cdot d_2 - \text{area of } \Delta DXP - \text{area of } \Delta PYC$

$= \frac{1}{2}\cdot CD\cdot d_2 - (\text{area of } \Delta DAP - \text{area of } \Delta DAX)$

$\qquad - (\text{area of } \Delta PBC - \text{area of } \Delta BYC)$

$= \frac{1}{2}CD\cdot d_2 - \frac{1}{2}\cdot PD\cdot d_1 - \frac{1}{2}\cdot PC\cdot d_3$

$\qquad + \text{area of } \Delta DAX + \text{area of } \Delta BYC$

$= \frac{1}{2}CD\cdot d_2 - \frac{1}{2}\cdot\frac{1}{2}CD\cdot d_1 - \frac{1}{2}\cdot\frac{1}{2}CD\cdot d_3$

$\qquad + \text{area of } \Delta ADX + \text{area of } \Delta BCY$

$= \frac{1}{2}CD\cdot\left(d_2 - \frac{d_1+d_3}{2}\right) + \text{area of } \Delta ADX + \text{area}$

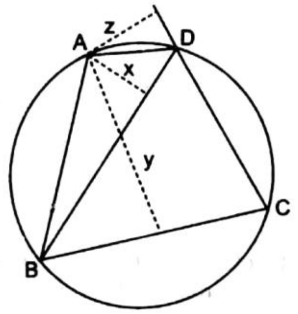
of ΔBCY = area of ΔADX + area of $\Delta\ BCY$

5. $\frac{BD}{x} = \cot\angle ABD + \cot\angle ADB\,\frac{CB}{y}$

$\qquad = \cot\angle ABC + \cot\angle ACB\,\frac{CD}{z}$

$\qquad = \cot\angle ACD + \cot\angle ADC\cdot\angle ACB = \angle ADB$

and $\qquad \angle ACD = \angle ABD$

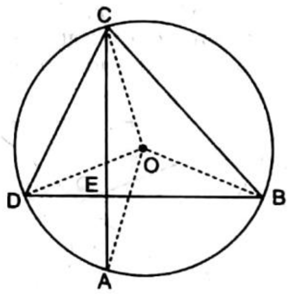


$$\therefore \frac{BC}{y} + \frac{CD}{z} - \frac{BD}{x} = \cot\angle ABC + \cot\angle ADC = 0$$

Since, $\qquad \angle ABC + \angle ADC = 180°$

6. Let O be the centre of the circle.

$$\angle AOB + \angle COD = 2(\angle ACD + \angle CBD)$$
$$= 2\times 90° = 180°$$



So $\angle AOB = \theta$, Then

$$AB^2 + CD^2 = 2(R^2 - R^2\cos\theta)$$
$$+ 2(R^2 - R^2\cos(\pi-\theta)) = 4R^2$$

Similarly,

$$BC^2 + AD^2 = 4R^2 EA^2 + EB^2 + EC^2 + ED^2$$
$$= \frac{1}{2}\angle(AB^2 + BC^2 + CD^2 + DA^2) = 4R^2$$

7. Let O be the centre and let r be the radius of the inner circle and R the radius of the outer circle. Let $\angle POA' = \theta$. Then

$$(PA')^2 = (OP)^2 + (OA')^2 - 2\cdot OA'\cdot OP\cos\theta$$
$$= r^2 + R^2 - 2r\,R\cos\theta$$
$$(PB')^2 = r^2 + R^2 - 2r\,R\cos(\theta - 120°)$$
$$(PC')^2 = r^2 + R^2 - 2r\,R\cos(\theta + 120°)$$
$$(PA')^2 + (PB')^2 + (PC')^2$$
$$= 3r^2 + 3R^2 - 2r\,R(\cos\theta + \cos(\theta - 120°)$$
$$+ \cos(\theta + 120°))$$

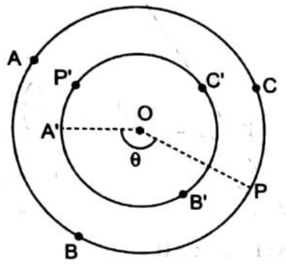
$= 3r^2 + 3R^2 - 2rR(\cos\theta + 2 \cdot \cos\theta \cos 120°)$

$= 3r^2 + 3R^2 - 2rR(\cos\theta - 2 \cdot \frac{1}{2} \cos\theta)$

$= 3r^2 + 3R^2$

Similarly, $P'A^2 + P'B^2 + P'C^2 = 3r^2 + 3R^2$.

Thus, they are equal.



Aliter O is the centroid of both the ΔABC and $\Delta A'B'C'$. In a ΔXYZ, if G is the centroid an P is any point, then

$PX^2 + PY^2 + PZ^2 = 3PG^2 + QX^2 + GY^2 + GZ^2$

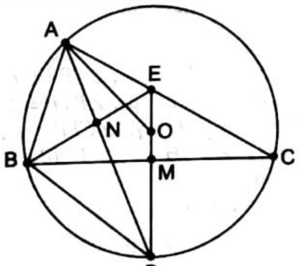
In ΔABC using P' as point and O as the centroid.

$PA^2 + P'B^2 + P'C^2$

$= 3P'O^2 + OA^2 + OB^2 + OC^2 = 3R^2 + 3r^2$

Similarly, the other expression too is equal to $3R^2 + 3r^2$. Hence the result.

8. We consider here the case when ABC is an acute-angled triangle; the cases when $\angle A$ is obtuse or one of the $\angle B$ and $\angle C$ is obtuse may be handled similarly.



Let M be the point of intersection of DE and BC; let AD intersect BE in N. Since ME is the perpendicular bisector of BC, we have $BE = CE$. Since AN is the internal bisector of $\angle A$ and is perpendicular to BE, it must bisect BE; *i.e.*, $BN = NE$. This in turn implies that DN bisects $\angle BDE$. But $\angle BDA = \angle BCA = \angle C$. Thus $\angle ODA = \angle C$. Since $OD = OA$, we get $\angle OAD = \angle C$. It follows that $\angle BDA = \angle C = \angle OAD$. This implies that OA is parallel to BD.

9. We give here two different cases.

Case I Drop perpendiculars from P and Q to AC and extend them to meet AR, RC in K, L respectively. Join KB, PB, QB, LB, KL (Fig.1)

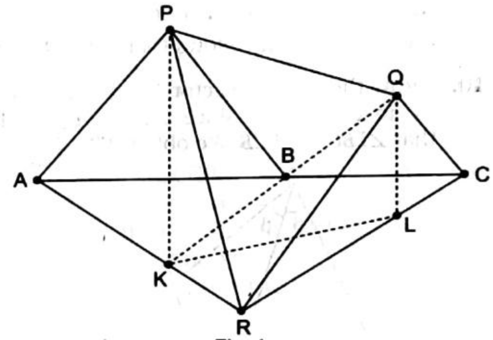


Fig. 1

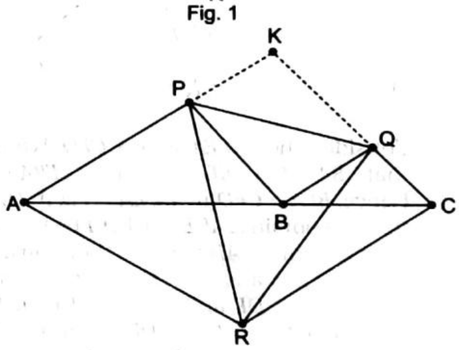


Fig. 2

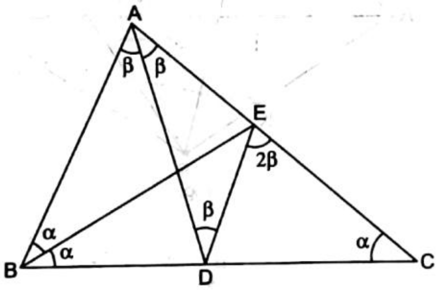
Observe that K, B, Q are collinear and so are P, B, L (This is because $\angle QBC = \angle PBA = \angle KBA$ and similarly $\angle PBA = \angle CBL$.) by symmetry we see that $\angle KPQ = \angle PKL$ and $\angle KPB = \angle PKB$. If follows that $\angle LPQ = \angle LKQ$ and hence K, L, Q, P are concyclic. We also note that $\angle KPL + \angle KRL = 60° + 120° = 180°$. This implies that P, K, R, L are concyclic. We conclude that P, K, R, L, Q are concyclic. This gives

$\angle PRQ = \angle PKQ = 60°$,

$\angle RPQ = \angle RKQ = \angle RAP = 60°$

Case II Produce AP and CQ to meet at K. Observe that $AKCR$ is a rhombus and $BQKP$ is a parallelogram. (See Fig.2.) Put $AP = x, CQ = y$. Then $PK = BQ = y$, $KQ = PB = x$ and $AR = RC = CK = KA = x + y$. Using cosine rule in ΔPKQ, we get $PQ^2 = x^2 + y^2 - 2xy\cos 120° = x^2 + y + xy$.
Similarly cosine rule in ΔQCR gives $QR^2 = y^2 + (x+y)^2 - 2xy\cos 60°$
$= x^2 + y^2 + xy$ and cosine rule in ΔPAR gives $RP^2 = x^2 + (x+y)^2 - 2xy\cos 60°$
$= x^2 + y^2 + xy$. If follows that $PQ = QR = RP$.

10. Draw the angle bisector BE of $\angle ABC$ to meet AC in E. Join ED. Since, $\angle B = 2\angle C$, it follows that $\angle EBC = \angle ECB$. We obtain $EB = EC$.



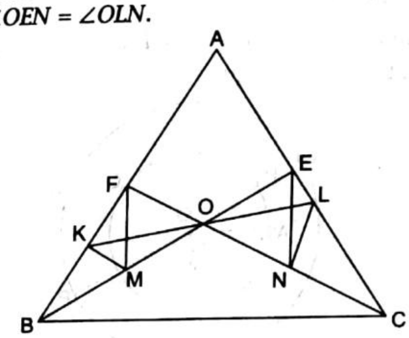
Consider the Δ BEA and Δ CED. We observe that $BA = CD$, $BE = CE$ and $\angle EBA = \angle ECD$. Hence, $BEA = CED$ giving $EA = ED$. If $\angle DAC = \beta$, then we obtain $\angle ADE = \beta$, Let I be the point of intersection of AD and BE. Now consider the triangles AIB and DIE. They are similar since $\angle BAI = \beta = \angle IDE$ and $\angle AIB = \angle DIE$. It follows that $\angle DEI = \angle ABI = \angle DBI$. Thus BDE is isosceles and $DB = DE = EA$. We also observe that $\angle CED = \angle EAD + \angle EDA = 2\beta = \angle A$. This implies that ED is parallel to AB. Since $BD = AE$, we conclude that $BC = AC$. In particular $\angle A = 2\angle C$. Thus the total angle of ABC is $5\angle C$ giving $\angle C = 36°$. We obtain $\angle A = 72°$.

Aliter We make use of the charactarisation, in a $\Delta ABC, \angle B = 2\angle C$ if and only if $b^2 = c(c + a)$. Note that $CD = c$ and $BD = a - c$. Since AD is the angle bisector, we also have

$$\frac{a-c}{c} = \frac{c}{b}.$$

This gives $c^2 = ab - bc$ and hence $b^2 = ca + ab - bc$. It follows that $b(b + c) = a(b + c)$ so that $a = b$. Hence, $\angle A = 2\angle C$ as well and we get $\angle C = 36°$. In turn $\angle A = 72°$.

11. Observe that $KMOF$ and $ONLE$ are cyclic quadrilaterals. Hence, $\angle FMO = \angle FKO$ and $\angle OEN = \angle OLN$.

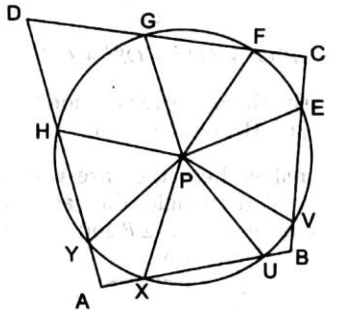


However we see that

$$\angle OLN = \frac{\pi}{2} - \angle NOL = \frac{\pi}{2} - \angle KOF = \angle OKF.$$

If follows that $\angle FMO = \angle OEN$.
This forces that FM is parallel to EN.

12. Let the lengths of the arcs XY, UV, EF, GH be respectively p, q, r, s. We also the following notations : (See figure)



$\angle XAY = \alpha_1$, $\angle AYP = \alpha_2$, $\angle YPX = \alpha_3$, $\angle PXA = \alpha_4$, $\angle UBY = \beta_1$, $\angle BVP = \beta_2$, $\angle VPU = \beta_3$, $\angle PUB = \beta_4$, $\angle ECF = \gamma_1$, $\angle CFP = \gamma_2$, $\angle FPE = \gamma_3$, $\angle PEC = \gamma_4$, $\angle GDH = \delta_1$, $\angle DHP = \delta_2$, $\angle HPG = \delta_3$, $\angle PGD = \delta_4$.
We observe that

$$\Sigma\alpha_j = \Sigma\beta_j = \Sigma\gamma_j = \Sigma\delta_j = 2\pi$$

It follows that

$$\Sigma(\alpha_j + \gamma_j) = \Sigma(\beta_j + \delta_j)$$

On the other hand, we also have $\alpha_2 = \beta_4$ since $PY = PU$. Similarly we have other relations; $\beta_2 = \gamma_4$, $\gamma_2 = \delta_4$ and $\delta_2 = \alpha_4$. It follows that

$$\alpha_1 + \alpha_3 + \gamma_1 + \gamma_3 = \beta_1 + \beta_3 + \delta_1 + \delta_3$$

But $p + r = q + s$ implies that $\alpha_3 + \gamma_3 = \beta_3 + \delta_3$. We thus obtain

$$\alpha_1 + \gamma_1 = \beta_1 + \delta_1$$

Since, $\alpha_1 + \gamma_1 + \beta_1 + \delta_1 = 360°$, it follows that $ABCD$ is a cyclic quadrilateral.

13. Put $CD = x$. Then, with usual notations, we get

$$CE = \frac{CD \cdot CB}{CA} = \frac{ax}{b}$$

Since, $AE = AC - CE = b - CE$, we obtain

$$AE = \frac{b^2 - ax}{b},$$

$$AF = \frac{AE \cdot AC}{AB} = \frac{b^2 - ax}{c}$$

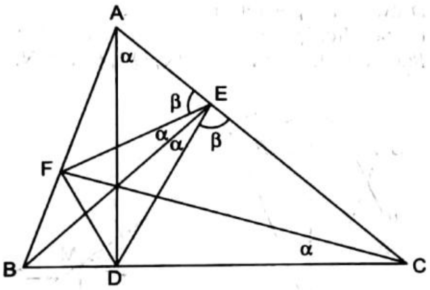


Fig. 1

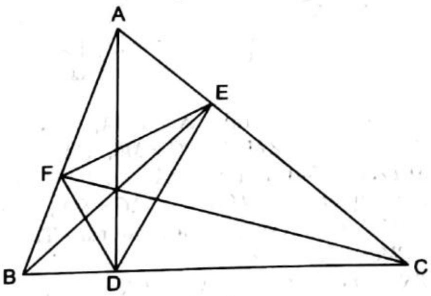


Fig. 2

This in turn gives

$$BF = AB - AF = \frac{c^2 - b^2 + ax}{c}$$

Finally, we obtain

$$BD = \frac{c^2 - b^2 + ax}{a}$$

Using $BD = a - x$, we get

$$x = \frac{a^2 - c^2 + b^2}{2a}$$

However, if L is the foot of perpendicular from A on to BC, then using Pythagoras theorem in $\Delta\ ALB$ and $\Delta\ ALC$, we get

$$b^2 - LC^2 = c^2 - (a - LC)^2$$

Which reduces to $LC = (a^2 - c^2 + b^2)/2a$. We conclude that $LC = DC$ proving $L = D$. Or we can also infer that $x = b \cos C$ from cosine rule in $\Delta\ ABC$. This implies that $CD = CL$, since $CL = b \cos C$ from right $\Delta\ ALC$. Thus, AD is altitude on to BC. Similar proof works for the remaining altitudes.

Aliter We see that $CD \cdot CB = CE \cdot CA$, so that $ABDE$ is a cyclic quadrilateral. Similarly we infer that $BCEF$ and $CAFD$ are also cyclic quadrilaterals. (See Fig. 2.) Thus $\angle AEF = \angle B = \angle CED$. Moreover $\angle BED = \angle DAF = \angle DCF = \angle BCF = \angle BEF$. It follows that $\angle BEA = \angle BEC$ and hence each is a right angle thus proving that BE is an altitude. Similarly we prove that CF and AD are altitudes. (Note that the concurrence of the lines AD, BE, CF are not required.)

14. Let AP, BP, CP when extended, meet the sides BC, CA, AB in D, E, F respectively. Draw AK, PL perpendicular to BC with K, L on BC. (See Fig. 3)

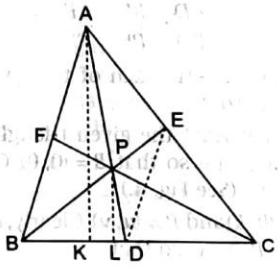


Fig. 3

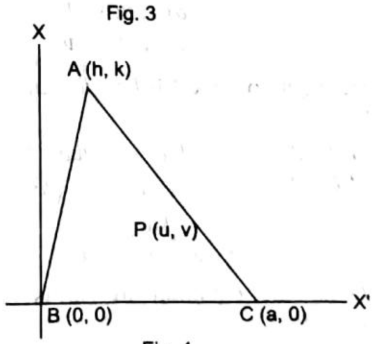


Fig. 4

Now, $$\frac{d(A,BC)}{d(P,BC)} = \frac{AK}{PL} = \frac{AD}{PD}$$

Similarly, $$\frac{d(B,CA)}{d(P,CA)} = \frac{BE}{PE}$$

and $$\frac{d(C,AB)}{d(P,AB)} = \frac{CF}{PF}$$

So, we obtain

$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF} \text{ and hence } \frac{AP}{PD} = \frac{BP}{PE} = \frac{CP}{PF}.$$

From $\frac{AP}{PD} = \frac{BP}{PE}$ and $\angle APB = \angle DPE$,

it follows that triangles APB and DPE are similar. So, $\angle ABP = \angle DEP$ and hence AB is parallel to DE.

Similarly, BC is parallel to EF and AC is parallel to DF. Using these we obtain

$$\frac{BD}{DC} = \frac{AE}{EC} = \frac{AF}{FB} = \frac{DC}{BD}$$

whence $BD^2 = CD^2$ or which is same as $BD = CD$. Thus D is the mid point of BC. Similarly E, F are the mid-points of CA and AB respectively.

We infer that AD, BE, CF are indeed the medians of the $\Delta\ ABC$ and hence P is the centroid of the triangle. So

$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF} = 3$$

and consequently each of the given ratios is also equal to 3.

Aliter Let ABC, the given triangle be placed in the xy-plane so that $B = (0, 0)$, $C = (a, 0)$ (on the x-axis). (See Fig. 4.)

Let $A = (h, k)$ and $P = (u, v)$. Clearly, $d(A, BC) = k$ and $d(P, BC) = v$, so that

$$\frac{d(A, BC)}{d(P, BC)} = \frac{k}{u}$$

The equation to CA is $kx - (h - a)y - ka = 0$.

So,

$$\frac{d(B, CA)}{d(P, CA)} = \frac{-ka}{\sqrt{k^2 + (h-a)^2}} \Big/ \frac{(ku - (h-a)v - ka)}{\sqrt{k^2 + (h-a)^2}}$$

$$= \frac{-ka}{ku - (h-a)v - ka}$$

Again the equation to AB is $kx - hy = 0$. Therefore,

$$\frac{d(C, AB)}{d(P, AB)} = \frac{ka}{\sqrt{h^2 + k^2}} \Big/ \frac{(ku - hv)}{\sqrt{h^2 + k^2}} = \frac{ka}{ku - hv}.$$

From the equality of these ratios, we get

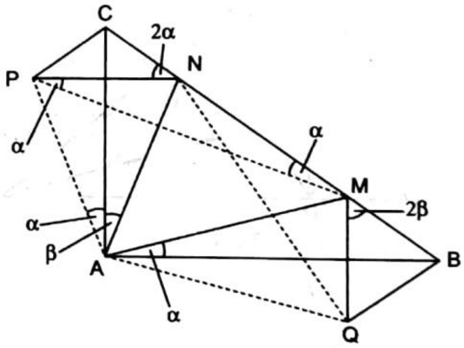
$$\frac{k}{v} = \frac{-ka}{ku - (h-a)v - ka} = \frac{ka}{ku - hv}.$$

The equality of the first and third ratios gives $ku - (h + a)v = 0$. Similarly the equalilty of second and third ratios gives $2ku - (2h - a)v = ka$. Solving for u and v, we get

$$u = \frac{h+a}{3}, v = \frac{k}{3}.$$

Thus, P is the centroid of the triangle and each of the ratios is equal to $\frac{k}{v} = 3$

15. Draw CP perpendicular to CB and BQ perpendicular to CB such that $CP = BM$, $BQ = CN$. Join PA, PM, PN, QA, QM, QN.



In $\Delta\ CPA$ and $\Delta\ BMA$, we have $\angle PCA = 45° = \angle MBA$; $PC = MB$, $CA = BA$. So, $\Delta CPA \equiv \Delta BMA$. Hence $\angle PAC = \angle BAM = \alpha$, say. Consequently, $\angle MAP = \angle BAC = 90°$, whence $PAMC$ is a cyclic quadrilateral. Therefore $\angle PMC = \angle PAC = \alpha$, Again $PN^2 = PC^2 + CN^2 = BM^2 + CN^2 = MN^2$. So, $PN = MN$, giving $\angle NPM = \angle NMP = \alpha$, in ΔPMN. Hence $\angle PNC = 2\alpha$. Likewise $\angle QMB = 2\beta$, where $\beta = \angle CAN$. Also $\Delta NCP \equiv \Delta QBM$, as $CP = BM$, $NC = BQ$ and $\angle NCP = 90° = \angle QBM$. Therefore, $\angle CPN = \angle BMQ = 2\beta$, wherce $2\alpha + 2\beta = 90°$; $\alpha + \beta = 45°$; finally $\angle MAN = 90° - (\alpha + \beta) = 45°$.

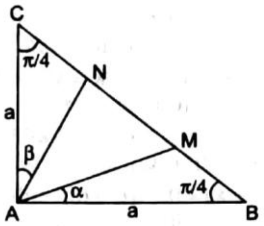
Aliter Let $AB = AC = a$, so that $BC = \sqrt{2}a$; and $\angle MAB = \alpha$, $\angle CAN = \beta$.

By the Sine Law, we have from ΔABM that

$$\frac{BM}{\sin\alpha} = \frac{AB}{\sin(\alpha + 45°)}$$

So $$BM = \frac{a\sqrt{2}\sin\alpha}{\cos\alpha + \sin\alpha} = \frac{a\sqrt{2}u}{1+u},$$

where $u = \tan\alpha$.



Similarly, $CN = \dfrac{a\sqrt{2}v}{1+v}$, where $v = \tan\beta$.

But $$BM^2 + CN^2 = MN^2 = (BC - MB - NC)^2$$
$$= BC^2 + BM^2 + CN^2$$
$$- 2BC \cdot MB - 2BC \cdot NC + MB \cdot NC$$

So, $BC^2 - 2BC \cdot MB - 2BC \cdot NC + 2MB \cdot NC = 0$

This reduces to

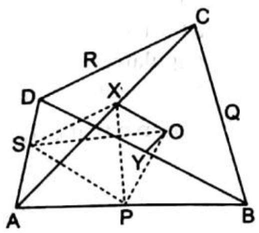
$$2a^2 - 2\sqrt{2}a\,\frac{a\sqrt{2}u}{1+u} - 2\sqrt{2}a\,\frac{a\sqrt{2}v}{1+v} + \frac{4a^2uv}{(1+u)(1+v)} = 0$$

Multiplying by $(1+u)(1+v)/2a^2$, we obtain $(1+u)(1+v) - 2u(1+v) - 2v(1+u) + 2uv = 0.$ Simplification gives $1 - u - v - uv = 0$. So

$$\tan(\alpha + \beta) = \frac{u+v}{1-uv} = 1$$

This gives $\alpha + \beta = 45°$, whence $\angle MAN = 45°$, as well.

16. We use the facts : (i) the line joining the mid points of the sides of a triangle is parallel to the third side; (ii) and any median of a triangle bisects its area; (iii) two triangles having equal bases and bounded by same parallel lines have equal area.

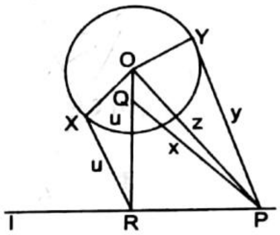


(a) Now BD is parallel to PS as well as OX is parallel to PS. Hence $[PXS] = [POS]$. Adding $[PAS]$ to both sides we get $[APXS] = [APOS]$. This proves part (a).

(b) Now, $$[APXS] = [APX] + [ASX]$$
$$= \frac{1}{2}[ABX] + \frac{1}{2}[ADX]$$
$$= \frac{1}{4}[ABC] + \frac{1}{4}[ADC]$$
$$= \frac{1}{4}[ABCD]$$

Hence by (a), $[APOS] = \frac{1}{4}[ABCD]$. Similarly, by symmetry each of the areas $[AQOP], [CROQ]$ and $[DSOR]$ is equal to $\frac{1}{4}[ABCD]$. Thus, the four given areas are equal. This proves part. (b). [Note $[\cdot]$ denotes area].

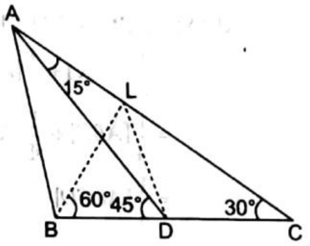
17. Let R be the foot of the perpendicular from O to the line l and u be the length of the tangent RX from R to circle Γ. On OR take a point Q such that $QR = u$.



We show that Q is the desired point. To this end, take any point P on line l and let y be the length of the tangent PY from P to Γ.

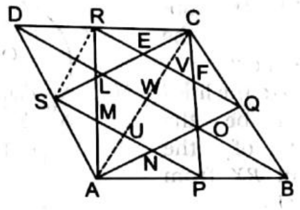
Further let r be the radius of the circle Γ and let y be the length of the tangent PY from P to Γ. Join OP, QP. Let $QP = x$, $OP = z$, $RP = t$. From right angled triangles POY, OXP, ORP, PQR we have respectively $z^2 = r^2 + y^2$, $OR^2 = r^2 + u^2$, $z^2 = OR^2 + t^2 = r^2 + u^2 + t^2$, $x^2 = u^2 + t^2$. So we obtain $y^2 = z^2 - r^2 = r^2 + u^2 + t^2 - r^2 = u^2 + t^2 = x^2$. Hence $y = x$. This gives $PY = PX$ which is what we needed to show.

18. Draw BL perpendicular to AC and join L to D. Since $\angle BCL = 30°$, we get $\angle CBL = 60°$. Since BLC is a right triangle with $\angle BCL = 30°$, we have $BL = BC/2 = BD$. Thus in Δ BLD, we observe that $BL = BD$ and $\angle DBL = 60°$. This implies that BLD is an equilateral triangle and hence $LB = LD$. Using $\angle LDB = 60°$ and $\angle ADB = 45°$, we get $\angle ADL = 15°$. But $\angle DAL = 15°$. Thus $LD = LA$. We hence have $LD = LA = LB$. This implies that L is the circumcentre of the Δ . Thus,



$$\angle BAD = \frac{1}{2}\angle BLD = \frac{1}{2} \times 60^\circ = 30^\circ$$

19. We have $QR = BD/2 = PS$. Since AQR and CSP are both equilateral and $QR = PS$, they must be congruent triangles. This implies that $AQ = QR = RA = CS = SP = PC$.

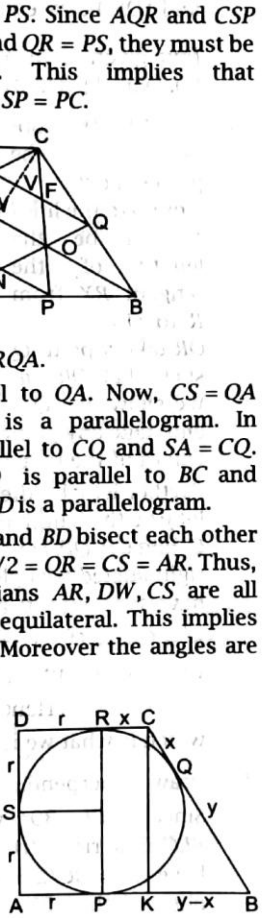


Also $\angle CEF = 60^\circ = \angle RQA$.

Hence, CS is parallel to QA. Now, $CS = QA$ implies that $CSQA$ is a parallelogram. In particular SA is parallel to CQ and $SA = CQ$. This shows that AD is parallel to BC and $AD = BC$. Hence, $ABCD$ is a parallelogram.

Let the diagonal AC and BD bisect each other at W. Then, $DW = DB/2 = QR = CS = AR$. Thus, in $\Delta\ ADC$, the medians AR, DW, CS are all equal. Thus, ADC is equilateral. This implies $ABCD$ is a rhombus. Moreover the angles are 60° and 120°.

20. Let P, Q, R, S be the points of contact of incircle with the sides AB, BC, CD, DA respectively. Since AD is perpendicular to AB and AB is parallel to DC, we see that $AP = AS = SD = DR = r$, the radius of the inscribed circle. Let $BP = BQ = y$ and $CQ = CR = x$. Using $AB = 3CD$, we get $r + y = 3(r + x)$.



Since the area of $ABCD$ is 4, we also get

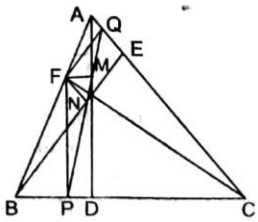
$$4 = \frac{1}{2} AD\,(AB + CD)$$

$$= \frac{1}{2}(2r)\,(4(r + x)).$$

Thus we obtain $r(r + x) = 1$. Using Pythagoras theorem, we obtain $BC^2 = BK^2 + CK^2$. However $BC = y + x$, $BK = y - x$ and $CK = 2r$. Substituting these and simplifying, we get $xy = r^2$. But $r + y = 3(r + x)$ gives $y = 2r + 3x$. Thus $r^2 = x(2r + 3x)$ and this simplifies to $(r - 3x)(r + x) = 0$. We conclude that $r = 3x$. Now the relation $r(r + x) = 1$ implies that $4r^2 = 3$, giving $r = \sqrt{3}/2$.

21. Consider the dissection of the given 6×6 square in to non-congruent rectangles with least possible areas. The only rectangle with area 1 is an 1×1 rectangle. Similarly, we get 1×2, 1×3 rectangles for areas 2, 3, units. In the case of 4 units we may have either a 1×4 rectangle or a 2×2 square. Similarly, there can be a 1×5 rectangle for area 5 units and 1×6 or 2×3 rectangle for 6 units. Any rectangle with area 7 units must be 1×7 rectangle, which is not possible since the largest side could be 6 units. And any rectangle with area 8 units must be a 2×4 rectangle. If there is any dissection of the given 6×6 square in to 9 non-congruent rectangles with areas $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8 \le a_9$, then we observe that $a_1 \ge 1$, $a_2 \ge 2$, $a_3 \ge 3$, $a_4 \ge 4$, $a_5 \ge 4$, $a_6 \ge 5$, $a_7 \ge 6$, $a_8 \ge 6$, $a_9 \ge 8$, and hence the total area of all the rectangles is $a_1 + a_2 + \ldots + a_9$ $\ge 1 + 2 + 3 + 4 + 4 + 5 + 6 + 6 + 8 = 39 > 36$, Which is the area of the given square. Hence if a 6×6 square is dissected in to 9 rectangles as stipulated in the problem, there must be two congruent rectangles.

22. Observe that C, Q, F, P are concyclic.



Hence, $\angle CQP = \angle CEP = 90^\circ - \angle FCP = \angle B$

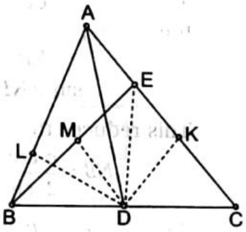
Similarly the concyclicity of F, M, Q, A given

$$\angle AQN = 90° + \angle FQM = 90° + \angle FAM$$
$$= 90° + 90° - \angle B = 180° - \angle B$$

Thus we obtain $\angle CQP + \angle AQN = 180°$. It follows that Q, N, P lie on the same line.

We can similarly prove that $\angle CPQ + \angle BPM = 180°$. This implies that P, M, Q are collinear. Thus M, N both lie on the line joining P and Q.

23. Draw DL perpendicular to AB; DK perpendicular to AC; and DM perpendicular to BE. Then $EM = DK$. Since AD bisects $\angle A$, we observe

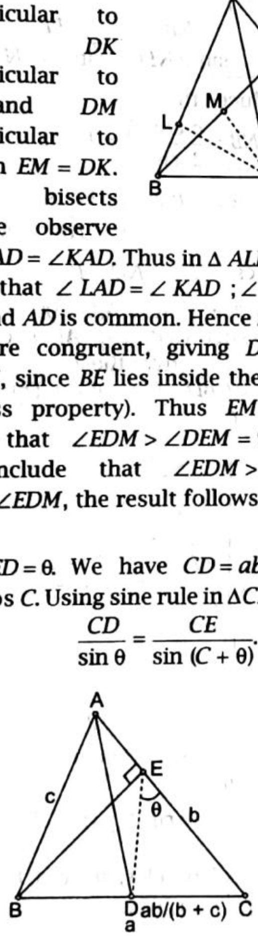


that $\angle BAD = \angle KAD$. Thus in $\Delta\, ALD$ and $\Delta\, AKD$, we see that $\angle LAD = \angle KAD$; $\angle AKD = 90° = \angle ALD$ and AD is common. Hence $\Delta\, ALD$ and $\Delta\, AKD$ are congruent, giving $DL = DK$. But $DL > DM$, since BE lies inside the triangle (by acuteness property). Thus $EM > DM$. This implies that $\angle EDM > \angle DEM = 90° - \angle EDM$. We conclude that $\angle EDM > 45°$. Since $\angle CED = \angle EDM$, the result follows.

Aliter

Let $\angle CED = \theta$. We have $CD = ab/(b + c)$ and $CE = a\cos C$. Using sine rule in ΔCED, we have

$$\frac{CD}{\sin\theta} = \frac{CE}{\sin(C+\theta)}.$$



This reduces to

$$(b + c)\sin\theta\cos C = b\sin C\cos\theta + b\cos C\sin\theta.$$

Simplification gives

$$c\sin\theta\cos C = b\sin C\cos\theta$$

so that $\quad \tan\theta = \dfrac{b\sin C}{c\cos C} = \dfrac{\sin B}{\cos C}$

$$= \frac{\sin B}{\sin(\pi/2 - C)}$$

Since, ABC is acute-angled, we have $A < \pi/2$. Hence, $B + C > \pi/2$ or $B > (\pi/2) - C$. Therefore $\sin B > \sin(\pi/2 - C)$. This implies that $\tan\theta > 1$ and hence $\theta > \pi/4$.

24. Suppose $\angle AMB = 60°$. Then, AMB and CMD are equilateral triangle. Draw OK perpendicular to BD. (see Fig. 1). Note that OM bisects $\angle AMB$ so that $\angle OMK = 30°$. Hence, $OK = OM/2 = 1$. It follows that $KM = \sqrt{OM^2 - OK^2} = \sqrt{3}$. We also observe that

$$AB - CD = BM - MD$$
$$= BK + KM - (DK - KM) + 2KM,$$

since K is the mid-point of BD. Hence $AB - CD = 2\sqrt{3}$.

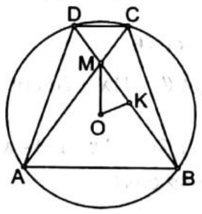


Fig. 1

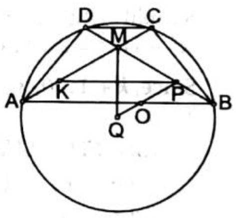


Fig. 2

Suppose $\angle AMD = 60°$ so that $\angle AMB = 120°$. Draw PQ through O parallel to AC (with Q on AB and P on BD). (see Fig. 2). Again OM bisects $\angle AMB$ so that $\angle OPM = \angle OMP = 60°$. Thus OMP is an equilateral triangle. Hence diameter perpendicular to BD also bisects MP. This gives $DM = PB$. In the triangles DMC and BPQ, we have $BP = DM$, $\angle DMC = 120° = \angle BPQ$ and $\angle DCM = \angle PBQ$ (property of cyclic quadrilateral). Hence DMC and BPQ are congruent so that $DC = BQ$. Thus $AB - DC = AQ$. Note that $AQ = KP$ since $KAQP$ is a parallelogram. But KP is twice the altitude of $\Delta\, OPM$. Since $OM = 2$, the altitude of OPM is $2 \times \sqrt{3}/2 = \sqrt{3}$. This gives $AQ = 2\sqrt{3}$.

Aliter

Using some trigonometry, we can get solutions for both the parts simultaneously. Let K, L be the mid points of AB and CD respectively. Then L, M, O, K are collinear (see Fig. 3 and Fig. 4). Let $\angle AMK = \theta (= \angle DML)$, and

$OM = d$. Since AMB and CMD are similar triangles, if $MD = MC = x$, then $MA = MB = kx$ for some positive constant k.

Now $MK = kx\cos\theta$, $ML = x\cos\theta$, so that $OK = [kx\cos\theta - d]$ and $OL = x\cos\theta + d$. Also $AK = kx\sin\theta$ and $DL = x\sin\theta$. Using

$$AK^2 + OK^2 = AO^2 = DO^2 = DL^2 + OL^2,$$

we get $k^2x^2\sin^2\theta + (kx\cos\theta - d)^2$

$$= x^2\sin^2\theta + (x\cos\theta + d)^2.$$

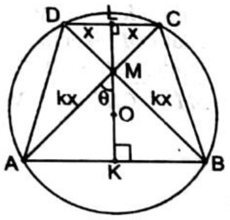


Fig. 3

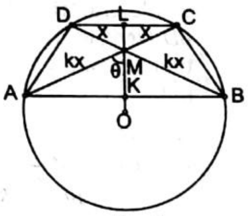


Fig. 4

Simplification gives

$(k^2 - 1)x^2 = 2xd(k+1)\cos\theta$

Since, $k + 1 > 0$, we get $(k-1)x = 2d\cos\theta$. Thus

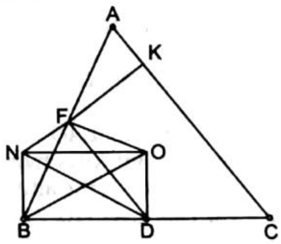
$$AB - CD = 2(AK - LD) = 2(kx\sin\theta - x\sin\theta)$$

$$= 2(k-1)x\sin\theta = 4d\cos\theta\sin\theta$$

$$= 2d\sin 2\theta$$

If $\angle AMB = 60°$, then $2\theta = 60°$. If $\angle AMD = 60°$, then $2\theta = 120°$. In either case $\sin 2\theta = \sqrt{3}/2$. If $d = 2$, then $AB - CD = 2\sqrt{3}$, in both the cases.

25. Let O be the circumcentre of ABC. Join OD, ON and OF. We show that $BDON$ is a rectangle. It follows that $DN = BO = R$, the circumradius of ABC.



Observe that $\angle NBC = \angle NKC = 90°$. Hence $BCKN$ is a cyclic quadrilateral. Thus $\angle KNB = 180° - \angle BCA$. But $\angle BOA = 2\angle BCA$ and OF bisects $\angle BOA$. Hence, $\angle BOF = \angle BCA$. We thus obtain

$$\angle FNB + \angle BOF = \angle KNB + \angle BCK = 180°$$

This implies that B, O, F, N are concyclic. Hence, $\angle BFO = \angle BNO$. But observe that $\angle BFO = 90°$ since OF is perpendicular to AB. Thus $\angle BNO = 90°$. Since NB and OD are perpendicular to BC, it follows that $BDON$ is a rectangle.

Aliter We can also get the conclusion using trigonometry. Observe that $\angle NFB = \angle AFK = 90° - \angle A$ and $\angle BNF = 180° - \angle B$ since $BCKN$ is a cyclic quadrilateral. Using the sine rule in the $\Delta\, BFN$,

$$\frac{NB}{\sin\angle NFB} = \frac{BF}{\sin\angle BFN}$$

This reduces to

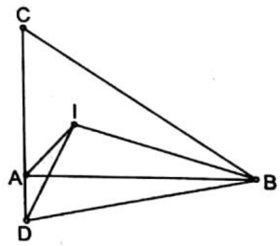
$$NB = \frac{c}{2}\frac{\cos A}{\sin C} = R\cos A$$

But $BD = a/2 = R\sin A$. Thus

$$ND^2 = NB^2 + BD^2 = R^2$$

This gives $\qquad ND = R$

26. We observe that $\angle AIB = 90° + (C/2)$. Extend CA to D such that $AD = AI$. Then, $CD = CB$ by the hypothesis. Hence, $\angle CDB = \angle CBD = 90° - (C/2)$. Thus



$$\angle AIB + \angle ADB = 90° + \frac{C}{2} + 90° - \frac{C}{2} = 180°$$

Hence, $ADBI$ is a cyclic quadrilateral. This implies that

$$\angle ADI = \angle ABI = \frac{B}{2}$$

But ADI is isosceles, since $AD = AI$. This gives

$$\angle DAI = 180° - 2(\angle ADI) = 180° - B$$

Thus, $\angle CAI = B$ and this gives $A = 2B$. Since $C = B$, we obtain $4B = 180°$ and hence $B = 45°$. We thus get $A = 2B = 90°$.

27. (i) Let S and T be two points on the boundary of Γ, with S lying on the side AB and T lying on

the side PQ of Γ. (see Fig. 1). Join TA, TB, TS. Now ST lies between TA and TB in $\Delta\ TAB$. One of $\angle AST$ and $\angle BST$ is at least 90°, say $\angle AST \geq 90^\circ$. Hence, $AT \geq TS$. But AT lies inside $\Delta\ APQ$ and one of $\angle ATP$ and $\angle ATQ$ is at least 90°, say $\angle ATP \geq 90^\circ$. Then $AP \geq AT$. Thus, we get $TS \leq AT \leq AP \leq 1$.

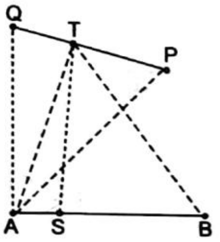


Fig. 1

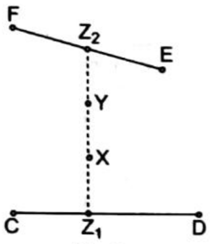


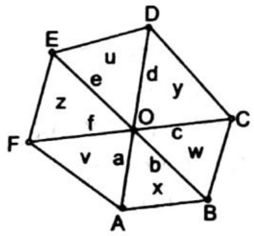
Fig. 2

(ii) Let X and Y be points in the interior Γ. Join XY and produce them on either side to meet the sides CD and EF of Γ at Z_1 and Z_2 respectively. We have

$$(XZ_1 + YZ_1) + (XZ_2 + YZ_2)$$
$$= (XZ_1 + XZ_2) + (YZ_1 + YZ_2) = 2Z_1Z_2 < 2.$$

by the first part. Therefore, one of the same $XZ_1 + YZ_1$ and $XZ_2 + YZ_2$ is at most 1. We may choose Z accordingly as Z_1 or Z_2.

28. Let $OA = a, OB = b, OC = c, OD = d, OE = e, OF = f$, $[OAB] = x$, $[OCD] = y$, $[OEF] = z$, $[ODE] = u$, $[OFA] = v$ and $[OBC] = w$. We are given that $v^2 = zx$, $w^2 = xy$ and we have to prove that $u^2 = yz$.



Since, $\angle AOB = \angle DOE$, we have

$$\frac{u}{x} = \frac{\frac{1}{2} de \sin \angle DOE}{\frac{1}{2} ab \sin \angle AOB} = \frac{de}{ab}$$

Similarly, we obtain

$$\frac{v}{y} = \frac{fa}{cd}, \frac{w}{z} = \frac{bc}{ef}$$

Multiplying, these three equalities, we get $uvw = xyz$.

Hence, $\quad x^2y^2z^2 = u^2v^2w^2 = u^2(zx)(xy)$.

This gives $u^2 = yz$, as desired.

29. Draw $AL \perp EF$ and extend it to meet AB in M. We show that $AL = LM$. First we show that A, F, I, E are concyclic. We have

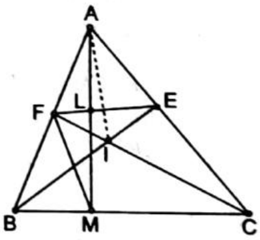
$$\angle BIC = 90^\circ + \frac{\angle A}{2} = 90^\circ + 30^\circ = 120^\circ.$$

Hence, $\angle FIE = \angle BIC = 120^\circ$. Since $\angle A = 60^\circ$, it follows that A, F, I, E are concyclic.

Hence,

$$\angle BEF = \angle IEF = \angle IAF = \angle A/2.$$ This gives

$$\angle AFE = \angle ABE + \angle BEF = \frac{\angle B}{2} + \frac{\angle A}{2}$$



Since $\angle ALF = 90^\circ$, we see that

$$\angle FAM = 90^\circ - \angle AFE = 90^\circ - \frac{\angle B}{2} - \frac{\angle A}{2}$$
$$= \frac{\angle C}{2} = \angle FCM.$$

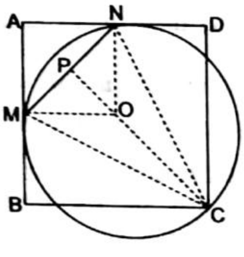
This implies that F, M, C, A are concyclic. It follows that

$$\angle FMA = \angle FCA$$
$$= \frac{\angle C}{2} = \angle FAM$$

Hence, FMA is an isosceles triangle. But $FL \perp AM$. Hence, L is the mid point of AM or $AL = LM$.

Level 2

1. Let O be the centre of the circle and P be the foot of perpendicular from C to MN.



Then, OM is perpendicular to AB, ON is perpendicular to AD and $OM = ON =$ the radius of the circle. So, $AMON$ is a square.

$$\angle MCN = \frac{1}{2}\angle MON = 45^\circ$$

$$\angle CMP + \angle CNP = 135^\circ = \angle CMP + \angle CMB$$
$$= \angle CNP + \angle CND.$$

Hence, $\angle CNP = \angle CMB$ and $\angle CMP = \angle CND$. Thus we see that the right ΔCNP and $\Delta\ CMB$ are similar, and $\angle CMP$ and $\angle CND$ are similar. So

$$\frac{CN}{CM} = \frac{CP}{CB}, \frac{CM}{CN} = \frac{CP}{CD}$$

Multiplying, $1 = \dfrac{CP^2}{CB \cdot CD}$. Hence, the area of the rectangle is

$$CB \cdot CD = CP^2 = 5^2 = 25$$

Aliter Let $AB = a$, $BC = b$, $BM = x$, $DN = y$ and $r =$ radius of the circle. Producing MO to meet the opposite side we can see that

$$x^2 + y^2 = OC^2 = r^2.$$

Thus

$$CM^2 = b^2 + x^2 = (r+y)^2 + x^2 = 2r^2 + 2ry = 2br$$
$$CN^2 = a^2 + y^2 = (r+x)^2 + y^2 = 2r^2 + 2rx = 2ar$$

Also $\quad \dfrac{CP}{CN} = \sin \angle CNP;$

$$CM = 2r \sin \angle CNM.$$

Hence, $\quad CM \cdot CN = 2r \cdot CP = 10r.$

Thus $\quad (2ar)(2br) = (10r)^2 \Rightarrow ab = 25$

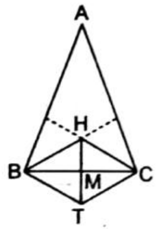
2. The diagonals BC and TH of the quadrilateral $BTCH$ are bisected at M. Hence, $BTCH$ is a parallelogram. Since $CH \perp AB$ and $CH \parallel TB$, we have $TB \perp AB$.

Similarly $TC \perp AC$ since $BH \perp AC$. Thus the circle on AT as diameter passes through B and C. This is the circumcircle of $\Delta\ ABC$. If R is its radius, then

$$AT = 2R, BC = 2R \sin A = 2R \sin 30^\circ = R$$

Hence $AT = 2BC$.

Aliter We can assume that the circumcentre of ΔABC is at the origin. If R is the circumradius, $BC = 2R \sin A = R$. Also if Z_1, Z_2, Z_3 are complex numbers representing A, B, C respectively than $Z_1 + Z_2 + Z_3$ represents H and M is $\dfrac{Z_2 + Z_3}{2}$. If t represents T, then



$$\frac{t + Z_1 + Z_2 + Z_3}{2} = \frac{Z_2 + Z_3}{2}$$

$$\Rightarrow \qquad t = -Z_1 \Rightarrow AT = 2T = 2BC$$

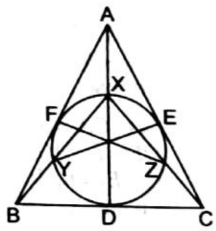
3. Since $\angle BFY = \angle BXF$ and $\angle FBY = \angle XBF$ we have ΔBFY and ΔBXF are similar, so that

$FY : FX = BF : BX \quad \ldots$(i)

Similarly, we get

$DY : DX = BD : BX \quad \ldots$(ii)

As $BF = BD$, we have from Eqs. (i) and (ii) that



$$FY : FX = DY : DX$$

Since $AX = DX$, we get

$$FY : FX = DY : AX \quad \ldots\text{(iii)}$$

Since, X, F, Y, D are concyclic we have

$$\angle FYD = \angle AXF \quad \ldots\text{(iv)}$$

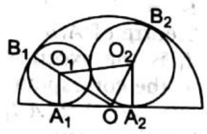
Thus, we get from Eqs. (iii) and (iv) that ΔFYD is similar to ΔFXA.

Hence, $\angle YFD = \angle XFA = \angle XDF$ so that $FY \parallel XD$. Similarly we have $EZ \parallel XD$. Thus, $FY \parallel EZ$.

Therefore, $FYZE$ is an isosceles trapezoid and then $EY = FZ$.

4. Let O_1, O_2 and O denote the centres of the circles and let A_1, A_2, B_1 and B_2 denote the points of tangency of these circles with the

semicircle, as shown in the diagram. Then, $O_1O_2 = R_1 + R_2$, $O_1A_1 = R_1$ and $O_2A_2 = R_2$, so



$$A_1A_2 = \sqrt{(O_1O_2)^2 - (O_1A_1 - O_2A_2)^2}$$
$$= \sqrt{(R_1 + R_2)^2 - (R_1 - R_2)^2} = 2\sqrt{R_1R_2}$$

Also $OB_1 = 1, OB_2 = 1, O_1B_1 = R_1$ and $O_2B_2 = R_2$ so that $OO_1 = 1 - R_1$ and $OO_2 = 1 - R_2$.

Therefore, $A_1A_2 = OA_1 + OA_2$

$$= \sqrt{(OO_1)^2 - (O_1A_1)^2} + \sqrt{(OO_2)^2 - (O_2A_2)^2}$$
$$= \sqrt{(1 - R_1)^2 - R_1^2} + \sqrt{(1 - R_2)^2 - R_2^2}$$
$$= \sqrt{1 - 2R_1} + \sqrt{1 - 2R_2}$$

Thus, $\sqrt{1 - 2R_1} + \sqrt{1 - 2R_2} = 2\sqrt{R_1R_2}$

Squaring, then dividing each term by 2 and rearranging the terms, we get

$$\sqrt{(1 - 2R_1)(1 - 2R_2)} = 2R_1R_2 + R_1 + R_2 - 1.$$

Square both sides and simplify.

$$8R_1R_2 = (2R_1R_2 + R_1 + R_2)^2 \quad \ldots(i)$$

So, $2\sqrt{2R_1R_2} = 2R_1R_2 + R_1 + R_2$

Thus, $R_1 + R_2 = 2\sqrt{R_1R_2}(\sqrt{2} - \sqrt{R_1R_2}) \quad \ldots(ii)$

$$\le (R_1 + R_2)(\sqrt{2} - \sqrt{R_1R_2})$$

and therefore $\sqrt{R_1R_2} \le \sqrt{2} - 1$

Now consider the function $f(x) = 2x(\sqrt{2} - x)$. $f(x)$ is increasing on the interval $(0, 1/\sqrt{2})$ since $f'(x) = 2\sqrt{2} - 4x > 0$ for x in the interval. Since $0 < \sqrt{R_1R_2} \le \sqrt{2} - 1 < 1/\sqrt{2}$ and $R_1 + R_2 = f(\sqrt{R_1R_2})$ from Eq. (ii), $R_1 + R_2$ attains its maximum when $\sqrt{R_1R_2} = \sqrt{2} - 1$. Hence

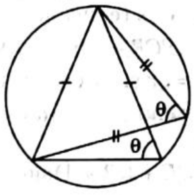
$$R_1 + R_2 \le 2(\sqrt{2} - 1)[\sqrt{2} - (\sqrt{2} - 1)] = 2(\sqrt{2} - 1).$$

Equally holds when $R_1 = R_2$

5. Note that the base angle of T_n is equal to the angle opposite the base of T_{n+1} (as the figure indicates). Therefore if θ is the base angle for T_n, then the base angle for the next triangle (T_{n+1}) is

$$\frac{180° - \theta}{2} = 90° - \frac{\theta}{2}.$$

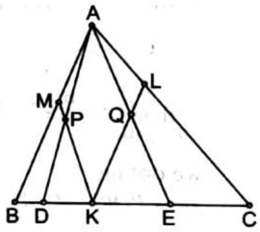
Suppose now that θ is the base angle for T_1. Then, the base angle for T_n is



$$90° - \frac{90°}{2} + \frac{90°}{4} - \frac{90°}{8} + \ldots + (-1)^{n-2}\frac{90°}{2^{n-2}} + (-1)^{n-1}\frac{\theta}{2^{n-1}}.$$

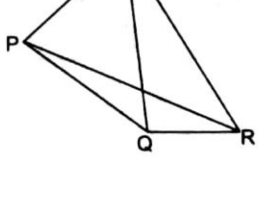
Note that the limit as $n \to \infty$ of the above is $\dfrac{90°}{1 + 1/2} = 60°$ by the formula for the sum of an infinite geometric series. Since each T_n is isosceles, the angles of T_n do approach 60° as $n \to \infty$.

6. Let AP, AQ produced meet BC in D, E respectively. Since MK is parallel to AE, we have $\angle AEK = \angle MKB$. Since $BK = BM$, both being tangents to the circle from B, $\angle MKB = \angle BMK$. This with the fact that MK is parallel to AE gives us $\angle AEK = \angle MAE$. This shows that $MAEK$ is an isosceles trapezoid. We conclude that $MA = KE$. Similarly, we can prove that $AL = DK$. But $AM = AL$. We get that $DK = KE$. Since KP is parallel to AE, we get $DP = PA$ and similarly $EQ = QA$. This implies that PQ is parallel to DE and hence bisects AB, AC when produced.



[The same argument holds even if one or both of P and Q lie outside ΔABC].

7. Let denote the area of figure. We have



$$[PQRS] = [PQR] + [RSP] = [QRS] + [SPQ].$$

Let us write $PQ = p, QR = q, RS = r, SP = s$. The above relations reduce to

$$pq \sin \angle PQR + rs \sin \angle RSP$$
$$= qr \sin \angle QRS + sp \sin \angle SPQ$$

Using $p = r$ and $(\sqrt{3} + 1)q = s$ and dividing by pq, we get

$$\sin \angle PQR + (\sqrt{3} + 1)\sin \angle RSP$$
$$= \sin \angle QRS + (\sqrt{3} + 1)\sin \angle SPQ.$$

Therefore, $\sin \angle PQR - \sin \angle QRS$

$$= (\sqrt{3} + 1)(\sin \angle SPQ - \sin \angle RSP).$$

This can be written in the form

$$2 \sin \frac{\angle PQR - \angle QRS}{2} \cos \frac{\angle PQR + \angle QRS}{2}$$
$$= (\sqrt{3} + 1)2 \sin \frac{\angle SPQ - \angle RSP}{2}$$
$$\cos \frac{\angle SPQ + \angle RSP}{2}$$

Using the relations

$$\cos \frac{\angle PQR + \angle QRS}{2} = -\cos \frac{\angle SPQ + \angle RSP}{2}$$

and $\sin \dfrac{\angle SPQ - \angle RSP}{2} = -\sin 15° = -\dfrac{(\sqrt{3} - 1)}{2\sqrt{2}}$,

we obtain

$$\sin \frac{\angle PQR - \angle QRS}{2} = (\sqrt{3} + 1)\left[-\frac{(\sqrt{3} - 1)}{2\sqrt{2}}\right] = \frac{1}{\sqrt{2}}$$

This shows that

$$\frac{\angle PQR - \angle QRS}{2} = \frac{\pi}{4}$$

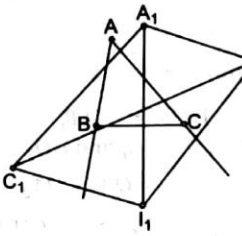
or $$\frac{3\pi}{4}$$

Using the convexity of $PQRS$, we can rule out the latter alternative. We obtain

$$\angle PQR - \angle QRS = \frac{\pi}{2}.$$

8. (a)

If $P = I$ is the incentre of Δ ABC, and r its inradius, then it is clear that $A_1I = B_1I = C_1I = 2r$. It follows that I is the circumcentre of $A_1B_1C_1$. On the otherhand if $P = I_1$ is the excentre of ABC opposite A and r_1 the corresponding exradius, then again we see that $A_1I_1 = B_1I_1 = C_1I_1 = 2r_1$. Thus I_1 is the circumcentre of $A_1B_1C_1$.



(b) Let $P = O$ be the circumcentre of ABC. By definition, it follows that OA_1 bisects and is bisected by BC and so on. Let D, E, F be the mid points of BC, CA, AB respectively. Then, FE is parallel to BC. ButE, F are also mid points of OB_1, OC_1 and hence FE is parallel to B_1C_1 as well. We conclude that BC is parallel to B_1C_1. Since, OA_1 is perpendicular to BC, it follows that OA_1 is perpendicular to B_1C_1. Similarly OB_1 is perpendicular to C_1A_1 and OC_1 is perpendicular to A_1B_1. These imply that O is the orthocentre of $A_1B_1C_1$. (This applies whether O is inside or outside ABC).

(c) Let $P = H$, the orthocentre of ABC. We consider two possibilities; H falls inside ABC and H falls outside ABC.

Suppose H is inside ABC; this happens if ABC is an acute triangle. It is known that A_1, B_1, C_1 lie on the circumcircle of ABC. Thus $\angle C_1A_1A = \angle C_1CA = 90° - A$. Similarly $\angle B_1A_1A = \angle B_1BA = 90° - A$. These show that $\angle C_1A_1A = \angle B_1A_1A$. Thus A_1A is an internal bisector of $\angle C_1A_1B_1$. Similarly we can show that B_1 bisects $\angle A_1B_1C_1$ and C_1C bisects $\angle B_1C_1A_1$. Since A_1A, B_1B, C_1C concur at H, we conclude that H is the incentre of $A_1B_1C_1$.

OR

If D, E, F are the feet of perpendiculars of A, B, C to the sides BC, CA, AB respectively, then we see that EF, FD, DE are respectively parallel to B_1C_1, C_1A_1, A_1B_1. This implies that $\angle C_1A_1H = \angle FDH = \angle ABE = 90° - A$, as $BDHF$ is a cyclic quadrilateral. Similarly, we can show that $\angle B_1A_1H = 90° - A$. It follows that A_1H is

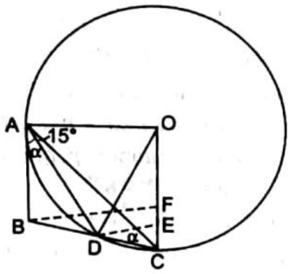
the internal bisector of $\angle C_1A_1B_1$. We can proceed as in the earlier case.

If H is outside ABC, the same proofs go through again, except that two of A_1H, B_1H, C_1H are external angle bisectors and one of these is an internal angle bisector. Thus, H becomes an excentre of $\Delta A_1B_1C_1$.

9. Let α denote the equal angles $\angle BAD = \angle DCA$. Using sine rule in

ΔDAB and ΔDAC, we get

$$\frac{AD}{\sin B} = \frac{BD}{\sin \alpha}, \frac{CD}{\sin 15^\circ} = \frac{AD}{\sin \alpha}.$$



Eliminating α (using $BD = DC$ and $2\alpha + B + 15^\circ = \pi$), we obtain $1 + \cos(B + 15^\circ) = 2\sin B \sin 15^\circ$. But we know that

$2\sin B \sin 15^\circ = \cos(B - 15^\circ) - \cos(B + 15^\circ)$. Putting $\beta = B - 15^\circ$, we get a relation $1 + 2\cos(\beta + 30) = \cos\beta$. We write this in the form

$$(1 - \sqrt{3})\cos\beta + \sin\beta = 1.$$

Since $\sin\beta \le 1$, it follows that $(1 - \sqrt{3})\cos\beta \ge 0$. We conclude that $\cos\beta \le 0$ and hence that β is obtuse. So is angle B and hence $\angle ADC$.

We have the relation $(1 - \sqrt{3})\cos\beta + \sin\beta = 1$. If we set $x = \tan(\beta/2)$, then we get, using $\cos\beta = (1 - x^2)/(1 + x^2), \sin\beta = 2x/(1 + x^2)$.

$$(\sqrt{3} - 2)x^2 + 2x - \sqrt{3} = 0.$$

Solving for x, we obtain $x = 1$ or $x = \sqrt{3}(2 + \sqrt{3})$. If $x = \sqrt{3}(2 + \sqrt{3})$, then $\tan(\beta/2) > 2 + \sqrt{3} = \tan 75^\circ$ giving us $\beta > 150^\circ$. This forces that $B > 165^\circ$ and hence $B + A > 165^\circ + 15^\circ = 180^\circ$, a contradiction, thus $x = 1$ giving us $\beta = \pi/2$. This gives $B = 105^\circ$ and hence $\alpha = 30^\circ$. Thus $\angle DAO = 60^\circ$. Since $OA = OD$, the result follows.

OR

Let m_a denote the median AD. Then, we can compute

$$\cos\alpha = \frac{c^2 + m_a^2 - (a^2/4)}{2cm_a}, \sin\alpha = \frac{2\Delta}{cm_a},$$

where Δ denotes the area of ΔABC. These two expressions give

$$\cot\alpha = \frac{c^2 + m_a^2 - (a^2/4)}{4\Delta}$$

Similarly, we obtain

$$\cos\angle CAD = \frac{b^2 + m_a^2 - (a^2/4)}{4\Delta}$$

Thus, we get

$$\cot\alpha - \cot 15^\circ = \frac{c^2 - a^2}{4\Delta}$$

Similarly, we can also obtain

$$\cot B - \cot\alpha = \frac{c^2 - a^2}{4\Delta},$$

giving us the relation

$$\cot B = 2\cot\alpha - \cot 15^\circ$$

If B is acute, then

$$2\cot\alpha > \cot 15^\circ = 2 + \sqrt{3} > 2\sqrt{3}.$$

It follows that $\cot\alpha > \sqrt{3}$. This implies that $\alpha < 30^\circ$ and hence

$$B = 180^\circ - 2\alpha - 15^\circ > 105^\circ$$

This contradiction forces that $\angle B$ is obtuse and consequently $\angle ADC$ is obtuse.

Since $\angle BAD = \alpha = \angle ACD$, the line AB is tangent to the circumcircle Γ of ADC at A. Hence, OA is perpendicular to AB. Draw DE and BF perpendicular to AC and join OD. Since $\angle DAC = 15^\circ$, we see that $\angle DOC = 30^\circ$ and hence $DE = OD/2$. But DE is parallel to BF and $BD = DC$ shows that $BF = 2DE$. We conclude that $BF = DO$. But $DO = AO$, both being radii of Γ. Thus $BF = AO$. Using right ΔBFO and ΔBAO, we infer that $AB = OF$. We conclude that $ABFO$ is a rectangle. In particular $\angle AOF = 90^\circ$. It follows that

$$\angle AOD = 90^\circ - \angle DOC = 90^\circ - 30^\circ = 60^\circ$$

Since, $OA = OD$, we conclude that AOD is equilateral.

OR

Note that ΔABD and ΔCBA are similar. Thus we have the ratios

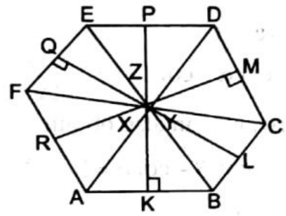
$$\frac{AB}{BD} = \frac{CB}{BA}$$

This reduces to $a^2 = 2c^2$ giving us $a = \sqrt{2}c$.
This is equivalent to $\sin^2(\alpha + 15°) = 2\sin^2\alpha$.
We write this in the form

$$\cos 15° + \cot\alpha \sin 15° = \sqrt{2}.$$

Solving for $\cot\alpha$, we get $\cot\alpha = \sqrt{3}$. We conclude that $\alpha = 30°$ and the result follows.

10. (a) Suppose all the six statements are true. Then $ABDE, BCEF, CDFA$ are isosceles trapeziums; if K, L, M, P, Q, R are the mid points of AB, BC, CD, DE, EF, FA respectively, then we see that $KP \perp AB, ED$; $LQ \perp BC, EF$ and $MR \perp CD, FA$.



If AD, BE, CF themselves concur at a point O, then $OA = OB = OC = OD = OE = OF$. ($O$ is on the perpendicular bisector of each of the sides). Hence, A, B, C, D, E, F are concyclic and lie on a circle with centre O. Otherwise these lines AD, BE, CF form a triangle, say XYZ. (See Fig.) Then, KX, MY, QZ, when extended, become the internal angle bisectors of the ΔXYZ and hence concur at the incentre O' of XYZ. As earlier O' lies on the perpendicular bisector of each of the sides. Hence, $O'A = O'B = O'C = O'D = O'E = O'F$, giving the concyclicity of A, B, C, D, E, F.

(b) Suppose $(a_1), (a_2), (b_1), (b_2)$ are true. Then, we see that $AD = BE = CF$. Assume that (c_1) is true. Then, CD is parallel to AF. It follows that ΔYCD and ΔYFA are similar. This gives

$$\frac{FY}{AY} = \frac{YC}{YD} = \frac{FY + YC}{AY + YD} = \frac{FC}{AD} = 1$$

We obtain $FY = AY$ and $YC = YD$. This forces that ΔCYA and ΔDYE are congruent. In particular $AC = DF$ so that (c_2) is true. The conclusion follows from (a). Now assume that (c_2) is true; *i.e.*, $AC = FD$. We have seen that $AD = BE = CF$. It follows that ΔFDC and ΔACD are congruent. In particular $\angle ADC = \angle FCD$. Similarly, we can show that $\angle CFA = \angle DAF$. We conclude that CD is parallel to AF giving (c_1).

11. It is easy to observe that there is a triangle with sides $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$. Using Heron's formula, we get

$$16[ABC]^2 = (a + b + c)(a + b - c)(b + c - a)(c + a - b)$$

and $$16[A_1B_1C_1]^2 = \frac{3}{16}(a + b + c)(-a + b + 3c)(-b + c + 3a)(-c + a + 3b).$$

Since a, b, c are the sides of a triangle, there are positive real numbers p, q, r such that $a = q + r$, $b = r + p$, $c = p + q$. Using these relations we obtain

$$\frac{[ABC]^2}{[A_1B_1C_1]^2} = \frac{16pqr}{3(2p + q)(2q + r)(2r + p)}.$$

Thus it is sufficient to prove that

$$(2p + q)(2q + r)(2r + p) \geq 27pqr,$$

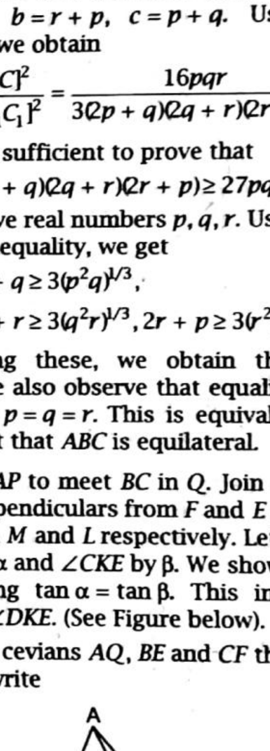
for positive real numbers p, q, r. Using AM-GM inequality, we get

$$2p + q \geq 3(p^2q)^{1/3},$$
$$2q + r \geq 3(q^2r)^{1/3}, 2r + p \geq 3(r^2p)^{1/3}.$$

Multiplying these, we obtain the desired result. We also observe that equality holds if and only $p = q = r$. This is equivalent to the statement that ABC is equilateral.

12. Produce AP to meet BC in Q. Join KE and KF. Draw perpendiculars from F and E on to BC to meet it in M and L respectively. Let us denote $\angle BKF$ by α and $\angle CKE$ by β. We show that $\alpha = \beta$ by proving $\tan\alpha = \tan\beta$. This implies that $\angle DKF = \angle DKE$. (See Figure below).

Since the cevians AQ, BE and CF the concur, we may write



$$\frac{BQ}{QC} = \frac{z}{y}, \frac{CE}{EA} = \frac{x}{z}, \frac{AF}{FB} = \frac{y}{x},$$

We observe that

$$\frac{FD}{DE} = \frac{[AFD]}{[AED]} = \frac{[PFD]}{[PED]} = \frac{[AFP]}{[AEP]}.$$

However standard computations involving bases give

$$[AFP] = \frac{y}{y+x}[ABP],$$

$$[AEP] = \frac{z}{z+x}[ACP]$$

and $$[ABP] = \frac{z}{x+y+z}[ABC],$$

$$[ACP] = \frac{y}{x+y+z}[ABC].$$

Thus, we obtain

$$\frac{FD}{DE} = \frac{x+z}{x+y}.$$

On the other hand

$$\tan\alpha = \frac{FM}{KM} = \frac{FB\sin B}{KM},\ \tan\beta = \frac{EL}{KL} = \frac{EC\sin C}{KL}.$$

Using $FB = \left(\frac{x}{x+y}\right)AB,\ EC = \left(\frac{x}{x+z}\right)AC$ and $AB\sin B = AC\sin C$, we obtain

$$\frac{\tan\alpha}{\tan\beta} = \left(\frac{x+z}{x+y}\right)\left(\frac{KL}{KM}\right) = \left(\frac{x+z}{x+y}\right)\left(\frac{DE}{FD}\right)$$

$$= \left(\frac{x+z}{x+y}\right)\left(\frac{x+y}{x+z}\right) = 1$$

We conclude that $\alpha = \beta$.

13. We know that, $2R = \frac{abc}{2\Delta}$ and $r_a = \frac{\Delta}{s-a}$, where a, b, c are the sides of the $\Delta\ ABC$, $s = \frac{a+b+c}{2}$ and Δ is the area of ABC. Thus, the given condition $2R \le r_a$ translates to

$$abc \le \frac{2\Delta^2}{s-a}$$

Putting $s-a=p,\ s-b=q,\ s-c=r$, we get $a=q+r, b=r+p, c=p+q$ and the condition now is

$$p(p+q)(q+r)(r+p) \le 2\Delta^2$$

But Heron's formula gives, $\Delta^2 = s(s-a)(s-b)(s-c) = pqr(p+q+r)$. We obtain $(p+q)(q+r)(r+p) \le 2qr(p+q+r)$. Expanding and effecting some cancellations, we get

$$p^2(q+r) + p(q^2+r^2) \le qr(q+r) \qquad (*)$$

Suppose $a \le b$. This implies that $q+r \le r+p$ and hence $q \le p$. This implies that $q^2r \le p^2r$ and $qr^2 \le pr^2$ giving

$$qr(q+r) \le p^2r + pr^2 < p^2r + pr^2 + p^2q + pq^2$$

$$= p^2(q+r) + p(q^2+r^2)$$

which contradicts (*). Similary, $a \le c$ is also not possible. This proves (i). Suppose $2R \le r_b$. As above this takes the form

$$q^2(r+p) + q(r^2+p^2) \le pr(p+r) \qquad (**)$$

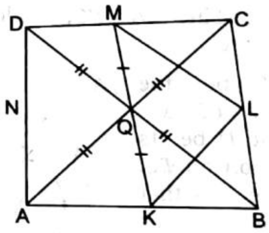
Since, $a > b$ and $a > c$, we have $q > p$, $r > p$. Thus $q^2r > p^2r$ and $qr^2 > pr^2$. Hence,

$$q^2(r+p) + q(r^2+p^2) > q^2r + qr^2$$

$$> p^2r + pr^2 = pr(p+r)$$

which contradicts (**). Hence, $2R > r_b$. Similarly, we can prove that $2R > r_c$. This proves (ii).

14. Observe that $KLMN$ is a paralellogram, Q is the mid point of MK and hence NL also passes through Q. Let T be the point of intersection of AC and BD, and let S be the point of intersection of BD and MN.



Consider the $\Delta\ MNK$. Note that SQ is parallel to NK and Q is the mid point of MK. Hence S is the mid point of MN. Since MN is parallel to AC, it follows that T is the mid point of AC. Now Q is the circumcentre of ΔABC and the median BT passes through Q. Here, there are two possibilities.

(i) ABC is a right triangle with $\angle ABC = 90°$ and $T = Q$; and

(ii) $T \ne Q$ in which case BT is perpendicular to AC.

Suppose $\angle ABC = 90°$ and $T = Q$. Observe that Q is the circumcentre of the ΔDCB and hence $\angle DCB = 90°$. Similarly $DAB = 90°$. It follows that $\angle ADC = 90°$ and $ABCD$ is a rectangle. This implies that $KLMN$ is a rhombus. Hence, $LK/LM = 1$ and this gives $CD = CB$. Thus, $ABCD$ is a square.

In the second case, observe that BD is perpendicular to AC, KL is parallel to AC and LM is parallel to BD. Hence, it follows

that ML is perpendicular to LK. Similar reasoning shows that $KLMN$ is a rectangle.

Using $LM/LM = CD/CB$, we get that CBD is similar to LMK. In particular, $\angle LMK = \angle CBD = \alpha$ (say). Since LM is parallel to DB, , we also get $\angle BQK = \alpha$. Since $KLMN$ is a cyclic quadrilateral we also get $\angle LNK = \angle LMK = \alpha$. Using the fact that BD is parallel to NK, we get $\angle LQB = \angle LNK = \alpha$. Since BD bisects $\angle CBA$, we also have $\angle KBQ = \alpha$. Thus

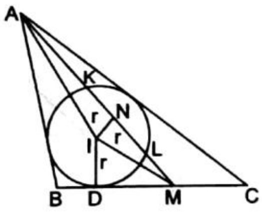
$$QK = KB = BL = LQ$$

and BL is parallel to QK. This gives QM is parallel to LC and

$$QM = QL = BL = LC$$

It follows that $QLCM$ is a parallelogram. But $\angle LCM = 90°$. Hence, $\angle MQL = 90°$. This implies that $KLMN$ is a square. Also observe that $\angle LQK = 90°$ and hence $\angle CBA = \angle LQK = 90°$. This gives $\angle CDA = 90°$ and hence $ABCD$ is a rectangle. Since $BA = BC$, it follows that $ABCD$ is a square.

15. Let I be the incentre of ΔABC and D be its projection on BC. Observe that $AB \neq AC$ as $AB = AC$ implies that $D = L = M$. So, assume that $AC > AB$. Let N be the projection of I on KL. Then, the perpendicular IN from I to KL is a bisector of KL and as $AK = LM$, it is a bisector of AM also. Hence, $AI = IM$.



But $$AI = \frac{r}{\sin (A/2)} = r \operatorname{cosec} (A/2)$$

and $$IM^2 = ID^2 + DM^2 = r + (BM - BD)^2$$
$$= r^2 + \left(\frac{a}{2} - (s-b)\right)^2.$$

Hence, $r^2 \operatorname{cosec}^2 (A/2) = r^2 + ((a/2) - (s-b)^2)^2$ giving $r^2 \cot^2 (A/2) = ((b-c)/2)^2$. Since $b > c$, we obtain $r \cot (A/2) = ((b-c)/2)$. So $(s-a) = ((b-c)/2)$. This gives $a = 2c$.

As $KN = NL$ and $AK = KL = LM$, we have $NL = AM/6$. We also have $AN = NM$.

Now, $r^2 = IL^2 = IN^2 + NL^2 = AI^2 - AN^2 + NL^2$

$$= AI^2 - \frac{1}{4} m_a^2 + \frac{1}{36} m_a^2$$
$$= r^2 \operatorname{cosec}^2 (A/2) - \frac{2}{9} m_a^2$$

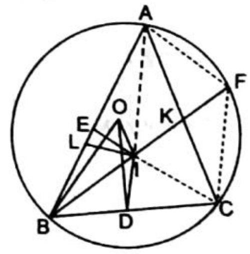
Hence, $r^2 \cot^2 (A/2) = \frac{2}{9} m_a^2$. From the above, we get

$$\left(\frac{b-c}{2}\right)^2 = \frac{2}{9} \cdot \frac{1}{4} (2b^2 + 2c^2 - a^2)$$

Simplification gives $5b^2 + 13c^2 - 18bc = 0$. This can be written as $(b-c)(5b-13c) = 0$. As $b \neq c$, we get $5b - 13c = 0$. To conclude, $a = 2c, 5b = 13c$ yield

$$\frac{a}{10} = \frac{b}{13} = \frac{c}{5}$$

16. (i) Extend BI to meet the circumcircle in F. Then, we know that $FA = FI = FC$ (See Figure). Let $BI : IF = \lambda : \mu$. Applying Stewart's theorem to ΔBAF, we get



$$\lambda AF^2 + \mu AB^2 = (\lambda + \mu)(AI^2 + BI \cdot IF).$$

Similarly, Stewart's theorem to ΔBCF gives

$$\lambda CF^2 + \mu BC^2 = (\lambda + \mu)(CI^2 + BI \cdot IF).$$

Since, $CF = AF$, subtraction gives

$$\mu(AB^2 - BC^2) = (\lambda + \mu)(AI^2 - CI^2)$$

Using the standard notations $AB = c$, $BC = a$, $CA = b$ and $s = (a+b+c)/2$, we get $AI^2 = r^2 + (s-a)^2$ and $CI^2 = r^2 + (s-c)^2$ where r is the inradius of ΔABC.

Thus,

$$\mu(c^2 - a^2) = (\lambda + \mu)((s-a)^2 - (s-c)^2)$$
$$= (\lambda + \mu)(c-a)b$$

It follows that either $c = a$ or $\mu(c+a) = (\lambda + \mu)b$. But $c = a$ implies that $a = b = c$ since a, b, c are in arithmetic progression. However, we have taken a non-equilateral ΔABC. Thus $c \neq a$ and we have $\mu(c+a) = (\lambda + \mu)b$. But $c + a = 2b$ and we obtain $2b\mu = (\lambda + \mu)b$. We conclude that

$\lambda = \mu$. This in turn tells that I is the mid point of BF. Since $OF = OB$, we conclude that OI is perpendicular to BF.

Aliter

Applying Ptolemy's theorem to the cyclic quadrilateral $ABCF$, we get

$$AB \cdot CF + AF \cdot BC = BF \cdot CA$$

Since $CF = AF$, we get $CF(c + a) = BF \cdot b = BF(c + a)/2$. This gives $BF = 2CF = 2IF$. Hence, I is the mid point of BF and as earlier we conclude that OI is perpendicular to BF.

Aliter

Join BO. We have to prove that $\angle BIO = 90°$, which is equivalent to $BI^2 + IO^2 = BO^2$. Draw IL perpendicular to AB. Let R denote the circumradius of ABC and let Δ denote its area. Observe that $BO = R, IO^2 = R^2 - 2Rr$,

$$BI = \frac{BL}{\cos (B/2)} = (s - b)\sqrt{\frac{ca}{s(s - b)}}$$

Thus, we obtain

$$BI^2 = ac(s - b)/s = \frac{ac}{3}$$

Since a, b, c are in arithmetic progression. Thus, we need to prove that

$$\frac{ac}{3} + R^2 - 2Rr = R^2$$

This reduces to proving $2Rr = ac/3$.

But $2Rr = 2 \cdot \dfrac{abc}{4\Delta} \cdot \dfrac{\Delta}{s} = \dfrac{abc}{2s} = \dfrac{abc}{a + b + c} = \dfrac{ac}{3}$,

using $a + c = 2b$. This proves the claim.

(ii) Join ID. Note that $\angle BIO = \angle BDO = 90°$. Hence B, D, I, O are concyclic and hence $\angle BID = \angle BOD = A$. Since, $\angle DBI = \angle KBA = B/2$, it follows that $\Delta\, BAK$ and ΔBID are similar. This gives

$$\frac{BA}{BI} = \frac{BK}{BD} = \frac{AK}{ID}.$$

However, we have seen earlier that $BI = ac/3$. Moreover $AK = bc/(a + c)$. Thus we obtain

$$BK = \frac{BA \cdot BD}{BI} = \frac{1}{2}\sqrt{3ac},$$

$$ID = \frac{AK \cdot BI}{BA} = \frac{1}{2}\sqrt{\frac{ac}{3}}$$

By symmetry, we must have $IE = \dfrac{1}{2}\sqrt{\dfrac{ac}{3}}$. Finally

$$IK = \frac{b}{a + b + c} \cdot BK = \frac{1}{3} BK = \frac{1}{2}\sqrt{\frac{ac}{3}}$$

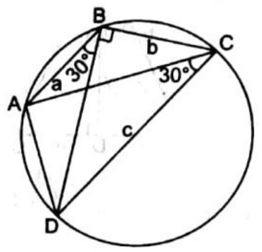
Thus, $ID = IE = IK$ and I is the circumcentre of DKE.

Aliter

Observe that $AK = bc/(a + c) = c/2 = AE$. Since, AI bisects $\angle A$, we see that AIE is congruent to AIK. This gives $IE = IK$. Similarly CID is congruent to CIK giving $ID = IK$. We conclude that

$$ID = IK = IE$$

17. Applying cosine rule to $\Delta\, ABC$, we get



$$AC^2 = a^2 + b^2 - 2ab \cos 120° = a^2 + b^2 + ab$$

Observe that

$$\angle DAC = \angle DBC = 120° - 30° = 90°.$$

Thus, we get

$$c^2 = \frac{AC^2}{\cos^2 30°} = \frac{4}{3}(a^2 + b^2 + ab)$$

So, $c^2 - (a + b)^2 = \dfrac{4}{3}(a^2 + b^2 + ab)$

$$- (a^2 + b^2 + 2ab)$$

$$= \frac{(a - b)^2}{3} \geq 0$$

This proves $c \geq a + b$ and thus (i) is true.

For proving (ii), consider the product

$$Q = (\alpha + \beta + \gamma)(\alpha - \beta - \gamma)(\alpha + \beta - \gamma)(\alpha - \beta + \gamma),$$

where $\alpha = \sqrt{c + a}, \beta = \sqrt{c + b}$ and $\gamma = \sqrt{c - a - b}$. Expanding the product, we get

$$Q = (c + a)^2 + (c + b)^2 + (c - a - b)^2$$

$$-2(c + a)(c + b) - 2(a + a)(c - a - b)$$

$$- 2(c + b)(c - a - b)$$

$$= -3c^2 + 4a^2 + 4b^2 + 4ab = 0$$

Thus at least one of the factors must be equal to 0. Since, $\alpha + \beta + \gamma > 0$ and $\alpha + \beta - \gamma > 0$, it follows that the product of the remaining two factors is 0. This gives

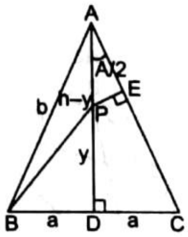
$$\sqrt{c+a} - \sqrt{c+b} = \sqrt{c-a-b}$$

or $\sqrt{c+a} - \sqrt{c+b} = -\sqrt{c-a-b}$

We conclude that

$$|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$$

18. Let $AD = h$, $PD = y$ and $BD = DC = a$. We observe that $BP^2 = a^2 + y^2$. Moreover,



$$PE = PA \sin \angle DAC = (h-y)\frac{DC}{AC} = \frac{a(h-y)}{b},$$

where $b = AC = AB$. Using $AP/PD = (h-y)/y$, we obtain $y = h/(1+\lambda)$.

Thus,

$$\lambda^2 = \frac{BP^2}{PE^2} = \frac{(a^2+y^2)b^2}{(h-y)^2a^2}$$

But $(h-y) = \lambda y = \lambda h/(1+\lambda)$ and $b^2 = a^2 + h^2$. Thus, we obtain

$$\lambda^4 = \frac{(a^2(1+\lambda)^2 + h^2)(a^2+h^2)}{a^2h^2}.$$

Using $m = \dfrac{a}{h}$ and $z = m^2(1+\lambda)$, this simplifies to

$$z^2 - z(\lambda^3 - \lambda^2 - 2) + 1 = 0.$$

Dividing by z, this gives

$$z + \frac{1}{z} = \lambda^3 - \lambda^2 - 2$$

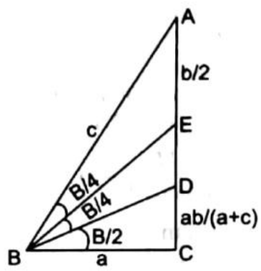
However $z + (1/z) \geq 2$ for any positive real number z. Thus, $\lambda^3 - \lambda^2 - 4 \geq 0$. This may be written in the form $(\lambda - 2)(\lambda^2 + \lambda + 2) \geq 0$. But $\lambda^2 + \lambda + 2 > 0$. (For example, one may check that its discriminant is negative). Hence, $\lambda \geq 2$. If $\lambda = 2$, then $z + (1/z) = 2$ and hence $z = 1$. This gives $m^2 = 1/3$ or $\tan(A/2) = m = 1/\sqrt{3}$. Thus $A = 60°$ and hence $\Delta\, ABC$ is equilateral.

Conversely, if $\Delta\, ABC$ is equilateral, then $m = \tan(A/2) = 1/\sqrt{3}$ and hence $z = (1+\lambda)/3$. Substituting this in the equation satisfied by z, we obtain

$$(1+\lambda)^2 - 3(1+\lambda)(\lambda^3 - \lambda^2 - 2) + 9 = 0.$$

This may be written in the form $(\lambda - 2)(3\lambda^3 + 6\lambda + 8\lambda + 8) = 0$. Here, the second factor is positive because $\lambda > 0$. We conclude that $\lambda = 2$.

19. (1) Since E is the mid point of AC, we have $AE = EC = b/2$. Since, BD bisects $\angle ABC$, we also know that $CD = ab/(a+c)$. Since, BE bisects $\angle ABD$, we also have



$$\frac{BD^2}{BA^2} = \frac{DE^2}{EA^2}$$

However, $\quad BD^2 = BC^2 + CD^2 = a^2 + \dfrac{a^2b^2}{(a+c)^2},$

$$DE^2 = \left(\frac{b}{2} - \frac{ab}{a+c}\right)^2$$

Using these in the above expression and simplifying, we get

$$a^2\{(a+c)^2 + b^2\} = c^2(c-a)^2$$

Using $c^2 = a^2 + b^2$ and eliminating b, we obtain

$$c^3 - 2ac^2 - a^2c - 2a^3 = 0$$

Introducing $t = c/a$, this reduces to a cubic equation;

$$t^3 - 2t^2 - t - 2 = 0.$$

Consider the function $f(t) = t^3 - 2t^2 - t - 2$ for $t > 0$ (as c/a is positive). For $0 < t \leq 2$, we see that $f(t) = t^2(t-2) - t - 2 < 0$. We also observe that $f(t) = (t-2)(t^2-1) - 4$ is strictly increasing on $(2, \infty)$. It is easy to compute

$$f(5/2) = -\frac{11}{8} < 0 \text{ and } f(3) = 4 > 0.$$

Hence, there is a unique value of t in the interval (5/2, 3) such that $f(t) = 0$. We conclude that

$$\frac{5}{2} < \frac{c}{a} < 3$$

(2) Let us take $\angle B/4 = \theta$. Then, $\angle EBC = \angle DBE = \theta$ and $\angle CBD = 2\theta$. Using sine rule in $\Delta\ BEA$ and $\Delta\ BEC$, we get

$$\frac{BE}{\sin A} = \frac{AE}{\sin\theta}, \frac{BE}{\sin 90^\circ} = \frac{CE}{\sin 3\theta}$$

Since, $AE = CE$, we obtain $\sin 3\theta \sin A = \sin\theta$. However $A = 90^\circ - 4\theta$. Thus, we get $\sin 3\theta \cos 4\theta = \sin\theta$. Note that

$$\frac{c}{a} = \frac{1}{\cos 4\theta} = \frac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta$$

This shows that $c/a < 3$. Using $ca = 3 - 4\sin^2\theta$, it is easy to compute $\cos 3\theta = ((c/a) - 1)/2$.

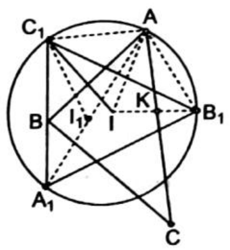
Hence, $$\frac{a}{c} = \cos 4\theta = \frac{1}{2}\left(\frac{c}{a} - 1\right)^2 - 1.$$

Suppose $c/a \le 5/2$. Then, $((c/a) - 1)^2 \le 9/4$ and $a/c \ge 2/5$. Thus

$$\frac{2}{5} \le \frac{a}{c} = \frac{1}{2}\left(\frac{c}{a} - 1\right)^2 - 1 \le \frac{9}{8} - 1 = \frac{1}{8},$$

which is absurd. We conclude that $c/a > 5/2$.

20. Here, $IA_1 = IB_1 = IC_1 = 2r$ where r is inradius of ΔABC.



$\therefore I$ is the circumcentre of $\Delta A_1B_1C_1$.

Let K be the point of intersection of IB_1 and AC.

Then, $IK = r$

$IA = 2r$ and $\angle IKA = 90^\circ$

$\therefore \quad \angle IAK = 30^\circ$

and $\quad \angle IAB_1 = 60^\circ$

Hence, AIB_1 is an equilateral triangle.

Similarly, ΔAIC_1 is an equilateral,

$\therefore$ We get,

$$AB_1 = AC_1 = AI = IB_1 = IC_1 = 2r$$

Hence, $\quad \angle B_1IC_1 = 120^\circ$

$\therefore IB_1AC_1$ is a rhombus and $\angle BA_1C_1 = 120^\circ$

$\angle A_1 = 60^\circ$ (by concyclicity)

A is mid point of arc B_1AC_1 $\quad (\because AB_1 = AC_1)$

Hence, it follows that A_1A bisects $\angle A_1$ and I_1 lies on line A_1A.

$$\Rightarrow \quad \angle B_1I_1C_1 = 90^\circ + \frac{\angle A_1}{2} = 90^\circ + 30^\circ = 120^\circ$$

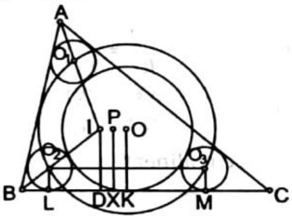
$\therefore \quad B_1, I, I_1, C_1$ are concyclic.

$[\because B_1IC_1 = 120^\circ$ and A is centre$]$

21. Let O_1, O_2 and O_3 be the centre of the circles Γ_A, Γ_B and Γ_C respectively.

Let P be the circumcentre of the $\Delta O_1O_2O_3$.

Let x denote the common radius of three circles $\Gamma_A, \Gamma_B, \Gamma_C$.



P is also the centre of circle Γ.

$[\because O_1P, O_2P$ and O_3P each exceed the radius of Γ by $x]$

Let D, X, K, L and M be respectively the projections of I, P, O, O_1, O_2 on BC.

Now, $$BL = \frac{x(s-b)}{r} \quad \left(\text{form } \frac{BL}{BD} = \frac{LO_2}{DI}\right)$$

(where $ID = r$, $BD = s - b$)

Similarly, $$CM = \frac{x(s-c)}{r}$$

$$\therefore \quad LM = a - \frac{x}{r}(s - b + s - c) = \frac{a}{r}(r - x)$$

PX is perpendicular bisector of LM.

$(\because O_2LMO_3$ is a rectangle and PX is perpendicular bisector of $O_2O_3)$

Thus, $$LX = \frac{1}{2}LM = \frac{a}{2r}(r - x)$$

$$BX = BL + LX$$

$$= \frac{x}{r}(s - b) + \frac{a}{2r}(r - x)$$

$$= \frac{a}{2} - \frac{x(b - c)}{2r}$$

$$DK = BK - BD = \frac{a}{2} - (s - b) = \frac{b - c}{2}$$

$$XK = BK - BX = \frac{a}{2} - \frac{a}{2} + \frac{x(b-c)}{2r} = \frac{x(b-c)}{2r}$$

Hence, we get $\frac{XK}{DK} = \frac{x}{r}$

$\therefore$ Sides of $\Delta O_1O_2O_3$ are

$$O_2O_3 = LM = \frac{a}{r}(r-x)$$

$$O_3O_1 = \frac{b}{r}(r-x)$$

and $$O_1O_2 = \frac{c}{r}(r-x)$$

$\because$ Sides of $O_1O_2O_3 \parallel ABC$.

And I is incentre of $O_1O_2O_3$,

$\therefore$ $$\frac{IP}{IO} = \frac{r-x}{r}$$

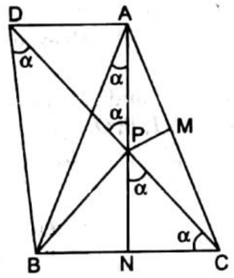
Hence, $$\frac{PO}{IO} = \frac{x}{r}$$

We get, $$\frac{XK}{DK} = \frac{PO}{IO}$$

$\therefore I, P, O$ are collinear.

22. Extend CP to D.

Now, $CP = PD$



Let $\angle BCP = \angle BAP = \alpha$,

$BC = BD$

($\because$ BP is perpendicular bisector of CD)

$\therefore BCD$ is an isosceles triangle.

Thus, $\angle BDP = \alpha$

Then, $\angle BDP = \angle BAP = \alpha$

Hence, B, P, A, D all lie on a circle.

$\angle DAB = \angle DPB = 90°$

($\because$ P is mid point of CD and M is mid point of CA)

$\therefore$ $PM \parallel DA$

where $DA = 2PM = BP$

Thus, $DBPA$ is an isosceles trapezium

and $DB \parallel PA$

Hence, we get

$$\angle DPA = \angle BAP = \angle BCP = \angle NPC$$

$\because$ $\angle BPC = 90°$

and N is mid point of CB.

$\therefore NP = NC = NB$ for right angled ΔBPC.

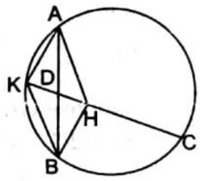
Hence, A, P and N are collinear.

23. Let $\angle C$ be the smallest angle.

So that $CA \geq AB$ and $CB \geq AB$

Here, the altitude through C is the longest one. Let the altitude through C meets AB in D. Let H be the orthocentre of ΔABC.

Let CD extended meet the circumcircle of ABC in K.



We have

$$CD = h_{\max}$$

Using,

$$CD = CH + HD$$

Which reduced to

$$AH + BH \leq CD + HD$$

But $AH = AK$

$BH = BK$ ($\because \Delta DEK \equiv \Delta DBH$)

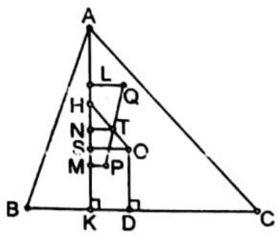
$DH = DK$

Apply Ptolemy's theorem to the cyclic quadrilateral $BCAK$.

We get, $$AB \cdot CK = AC \cdot BK + BC \cdot AK$$
$$\geq AB \cdot BK + AB \cdot AK$$

$\Rightarrow$ $$CK \geq AK + BK$$

24. Let D be the mid point of BC; M that of HK; and T that of OH. Then, PM is perpendicular to HK and PT is perpendicular to OH. Since, Q is the reflection of P in HO, we observe that P, T, Q are collinear and $PT = TQ$. Let QL, TN and OS be the perpendiculars drawn respectively from Q, T and O on to the altitude AK, (See the figure).



We have $LN = NM$, since T is the mid point of QP: $HN = NS$, since T is the mid point of OH; and $HM = MK$, as P is the circumcentre of KHO, we obtain

$$LH + HN = LN = NM = NS + SM$$

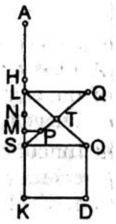
which gives $LH = SM$. We know that,

$$AH = 2OD$$

Thus, $AL = AH - LH = 2OD - LH = 2SK - SM$
$= SK + (SK - SM) = SK + MK$
$= SK + HM = SK + HS + SM$
$= SK + HS + LH = SK + LS = LK$

This shows that L is the mid point of AK and hence lies on the line joining the mid points of AB and AC. We observe that the line joining the mid points of AB and AC is also perpendicular to AK . Since QL is perpendicular to AK, we conclude that Q also lies on the line joining the mid points of AB and AC.

Remark



It may happen that H is above L as in the adjoining figure but the result remains true here as well. We have $HN = NS$, $LN = NM$ and $HM = MK$ as earlier. Thus, $HN = HL + LN$ and $NS = SM + NM$ give $HL = SM$

Now, $AL = AH + HL = 2OD + SM$
$= 2SK + SM$
$= SK + (SK + SM) = SK + MK$
$= SK + HM = SK + HL + LM$
$= SK + SM + LM = LK$

The conclusion that Q lies on the line joining the mid points of AB and AC follows as earlier.

25. Let $A = 2\alpha$. Then $\angle A_1AC = \angle BAA_1 = \alpha$. Thus

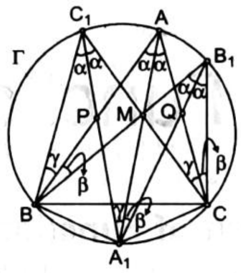
$$\angle A_1B_1C = \alpha = \angle BB_1A_1 = \angle A_1C_1C = \angle BC_1A_1$$

We also have $\angle B_1CQ = \angle AA_1B_1 = \beta$, say. If follows that $\Delta\ MA_1B_1$ and $\Delta\ QCB_1$ are similar and hence

$$\frac{QC}{MA_1} = \frac{B_1C}{B_1A_1}$$

Similarly, $\Delta\ ACM$ and $\Delta\ C_1A_1M$ are similar and we get

$$\frac{AC}{AM} = \frac{C_1A_1}{C_1M}$$



Using the point P, we get similar ratios

$$\frac{PB}{MA_1} = \frac{C_1B}{A_1C_1}, \frac{AB}{AM} = \frac{A_1B_1}{MB_1}$$

Thus, $$\frac{QC}{PB} = \frac{A_1C_1 \cdot B_1C}{C_1B \cdot B_1A_1}$$

and $$\frac{AC}{AB} = \frac{MB_1 \cdot C_1A_1}{A_1B_1 \cdot C_1M}$$

$$= \frac{MB_1}{C_1M}\frac{C_1A_1}{A_1B_1}$$

$$= \frac{MA_1C_1B \cdot QC}{C_1M\ PB \cdot B_1C}$$

However, ΔC_1BM and ΔB_1CM are similar, which gives

$$\frac{B_1C}{C_1B} = \frac{MB}{MC_1}$$

Putting this in the last expression, we get

$$\frac{AC}{AB} = \frac{QC}{PB}$$

We conclude that PQ is parallel to BC.

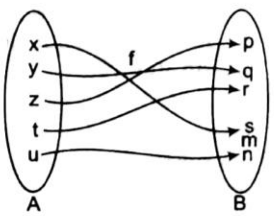
Unit 6 Functions

Unit 6

Functions

Definition of Function

Let A and B be two given non-empty sets, then a function (or mapping or map or transformation) f from A to B written as $f: A \to B$ or $A \xrightarrow{f} B$ (to be read as f is a function from the set A to the set B or A is mapped to B under function f) is a rule which associates **every element** of A to a **unique element** of B.



Here $f(x) = s$, $f(y) = q$, $f(z) = p$, $f(t) = r$, $f(u) = n$. The elements $f(x)$, $f(y)$, $f(z)$, $f(t)$, $f(u)$ of B are called **images** of x, y, z, t and u of A respectively and the elements x, y, z, t and u are called the **pre-images** of the elements s, q, p, r and n of B respectively. The element $m \in B$ has no pre-image in A.

If $f: A \to B$, then the set A is called the **domain** of the function f and is generally denoted by $D(f)$ or D_f and the set B is called the codomain of the function f, generally denoted by C_f or $C(f)$. The set of images of the elements of set A is called the **range** of function, which is generally denoted by R_f or $R(f)$.

Hence, $\quad$ Domain of $f = D_f = \{x : x \in A, (x, f(x)) \in A\}$,

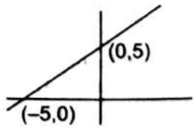
Range of $f = R_f = (f(x) : x \in A, f(x) \in B\} \subseteq B$

A function f is said to be integral, rational, irrational, non-negative, real or complex valued function according as $f(x) \in Z, Q, R - Q, R^+ \cup \{0\}, R$ or C. We shall discuss the nature of real valued functions only.

On the basis of images, a function is classified into two groups - **one-one** and **many-one**.

A function f from A to B (*i.e.*, $f: A \to B$) is said to be one-one (injective) iff different elements in A have different images in B.

i.e., $\quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \forall x_1, x_2 \in A$



For example let us consider a function f defined as $y = f(x) = x + 5$.

Here, $\quad f(x_1) = f(x_2) \Rightarrow x_1 + 5 = x_2 + 5$

$\Rightarrow \quad x_1 = x_2$

i.e., images under function f are equal iff the pre-images are equal. In other words, different elements in the domain of the function have different images. Hence f is one-one or injective. Here we observe that **any straight line parallel to y-axis intersects the graph of the function in at most one point**. Also a straight line parallel to x-axis intersects the graph in at most one point, hence the function is one-one or equivalently, a straight line parallel to x-axis intersects the graph in not more than one point.

A function $f: A \to B$ is said to be **many-one** iff different element in A have same image in B.

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$, for at least two elements x_1 and x_2 of A.

For example let a function f defined as $f(x) = x^2$

We observe that

$$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow \quad x_1^2 - x_2^2 = 0$$

$$\Rightarrow \quad (x_1 + x_2)(x - x_2) = 0$$

$$\Rightarrow \quad x_1 \neq x_2 \quad \text{always}$$

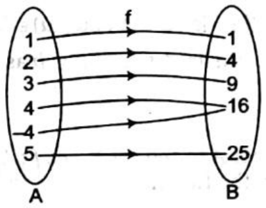
Which shows that images are equal even, if pre-images are not equal. Hence, the function under consideration is many one.

On the basis of existence of pre-images, a function is also classified into two groups-**onto** and **into.**

Let $f : A \to B$. Then, f is said to be onto (surjective) if every element of B has a pre-image in A.

Mathematically, $y \in B \Rightarrow \exists\, x \in A$ such that $y = f(x)$, $\forall\, y \in B$, then f is called onto function or surjective function.

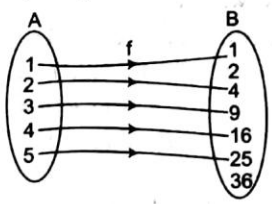
For example, Let $A = \{1, 2, 3, 4, 5, -4\}$, $B = \{1, 4, 9, 16, 25\}$ and $f : A \to B$ defined by $f(x) = x^2$, then, we observe that every element of B has its pre-image in A, which is clear from the arrow diagram given below



Hence, f is an onto function.

A function $f : A \to B$ is said to be into iff there exists at least one element in B having no pre-image in A. *i.e.*, $\exists$ at least one $y \in B$ which has no pre-image x in A or equivalently for at least one elements $y \in B$, there does not exist $x \in A$ such that $y = f(x)$ holds good.

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 1, 9, 16, 25, 36\}$ and $f : A \to B$ defined by $f(x) = x^2$.



Here we observe that $2 \in B$, $36 \in B$ have no pre-image in A. Hence, f is into.

Note

1. The number of functions from a finite set A containing m elements to a finite set B containing n elements is equal to $n^m = (n(B))^{n(A)}$
2. The number of one-one (injective) functions from A to $B = \begin{cases} {}^nP_m; & n \geq m \\ 0; & n < m \end{cases}$
3. The number of surjective functions (onto functions) $= \begin{cases} \sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r r^m; & \text{if } 1 \leq n \leq m \\ 0; & \text{if } n > m \end{cases}$
4. Number of bijective (one-one and onto) functions $= n!$ if $m = n$. If. $m \neq n$ the number of bijective, function is 0. The above results can be proved with the help of permutation and combination).

Real Valued Function of a Real Variable

A function whose domain an range are subsets of the set R of real numbers is called a real valued function.

Value of a Function

If $y = f(x)$ be any function of x and $x = a$ is admissible value of x, then the value of the function is obtained simply by replacing x by a in $y = f(x)$ and $f(a)$.

Equality of Functions

Two functions f and g are said to be equal, *i.e.*, $f = g$, if (i) $D_f = D_g$, (ii) $f(x) = g(x)$ for all $x \in D_f$ (or D_g).

Inverse Image of an Element

Let f be a function defined from the set A to the set B, then the inverse image of an element $b \in B$ under f is denoted by $f^{-1}(b)$ to be read as 'f inverse b'.

$$\therefore \qquad f^{-1}(b) = \{a : a \in A \text{ and } f(a) \in B\}$$

i.e., $f^{-1}(b)$ is the set of all those elements in A which has b as their pre-image.

Inverse Function

Let f be a function defined from the set A to the set B, *i.e.*, $f : A \to B$ and g be a function defined from the set B to the set A *i.e.*, $g : B \to A$; then the function g is said to be inverse of f, if
$g\{f(x)\} = x, \forall x \in A$ and the function g is denoted by f^{-1}.

Example 1 *If $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$ and $f(0) = 1$, for all $x, y \in R$, then find $f(x)$*

Solution We have, $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$, $\forall x, y \in R$, and $f(0) = 1$.

Putting $x = y = 0$, we get

$$f(1) = (\sqrt{f(0)} + \sqrt{f(0)})^2 = (\sqrt{1} + \sqrt{1})^2 = 4 = (1 + 1)^2$$

For $x = 1, y = 0$, $\quad f(1 + 0 + 1) = (\sqrt{f(1)} + \sqrt{f(0)})^2$

$$= (2 + 1)^2 = 3^2$$

$$\Rightarrow \qquad f(2) = (2 + 1)^2$$

and for $x = 1, y = 1$

$$f(1 + 1 + 1) = (\sqrt{f(1)} + \sqrt{f(1)})^2 = (2 + 2)^2 = 4^2$$

$$\Rightarrow \qquad f(3) = (3 + 1)^2$$

Here we observe that

$$f(0) = 1 = (0 + 1)^2$$
$$f(1) = 2^2 = (1 + 1)^2$$
$$f(2) = 3^2 = (2 + 1)^2$$
$$f(3) = 4^2 = (3 + 1)^2$$

The above observation suggests that $f(x) = (x + 1)^2$

Let us verify whether $f(x) = (x + 1)^2$ satisfies the given condition $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$ or not.

We have $\quad \sqrt{f(x)} + \sqrt{f(y)} = x + 1 + y + 1 = x + y + 1 + 1$

$$\Rightarrow \qquad (\sqrt{f(x)} + \sqrt{f(y)})^2 = (x + y + 1 + 1)^2 = f(x + y + 1)$$

Hence, $\quad f(x) = (x + 1)^2$

Example 2 *If* $mf(x) + nf\left(\frac{1}{x}\right) = x - 5, m \neq n$ *and* $x \neq 0$, *find* $f(x)$

Solution We have, $mf(x) + nf\left(\frac{1}{x}\right) = x - 5, m \neq n$ and $x \neq 0, x \in R$ ...(i)

$$\Rightarrow \quad nf(x) + mf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii) by cross-multiplication, we get

$$\frac{f(x)}{-n\left(\frac{1}{x} - 5\right) + m(x-5)} = \frac{f\left(\frac{1}{x}\right)}{-n(x-5) + m\left(\frac{1}{x} - 5\right)} = \frac{1}{m^2 - n^2}$$

$$\Rightarrow \quad f(x) = \frac{mx - \frac{n}{x} + 5(n-m)}{m^2 - n^2}$$

$$\Rightarrow \quad f(x) = \frac{mx - \frac{n}{x}}{m^2 - n^2} - \frac{5}{m+n}$$

Example 3 *If* $f(x) + 2f(1-x) = x^2, \forall x \in R$, *find* $f(x)$

Solution We have, $f(x) + 2f(1-x) = x^2, \forall x \in R$ ...(i)

Putting $1 - x$, for x, we get

$$f(1-x) + 2f(x) = (1-x)^2 \quad \text{...(ii)}$$

Thus, given equation reduces to

$$f(x) + 2(1-x)^2 - 4f(x) = x^2$$

$$\Rightarrow \quad 3f(x) = 2 + 2x^2 - 4x - x^2$$

$$\Rightarrow \quad 3f(x) = x^2 - 4x + 2$$

$$\Rightarrow \quad f(x) = \frac{1}{3}[x^2 - 4x + 2]$$

Algebra of Real Functions

(i) **Addition of two real functions** Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$. Then, we define $(f + g): X \to R$ by

$$(f + g)(x) = f(x) + g(x), \text{ for all } x \in X$$

(ii) **Subtraction of a real function from another** Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$. Then, we define $(f - g); X \to R$ by $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.

(iii) **Multiplication by a scalar** Let $f : X \to R$ be a real valued function and α be a scalar. Here, by scalar, we mean a real number. Then the product αf is a function from X to R defined by $(\alpha f)(x) = \alpha f(x), x \in X$.

(iv) **Multiplication of two real functions** The product (or multiplication) of two real functions $f : X \to R$ and $g : X \to R$ is a function $fg : X \to R$ defined by $(fg)(x) = f(x)g(x), \forall x \in X$. This is also called *pointwise multiplication.*

(v) **Quotient of two real functions** Let f and g be two real functions defined from $X \to R$ where $X \subset R$. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined by, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0, x \in X$

Example 1 *Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions. Find*

$$(f+g)(x), (f-g)(x), (fg)(x), \left(\frac{f}{g}\right)(x).$$

Solution We have,

$$(f+g)(x) = x^2 + 2x + 1,$$
$$(f-g)(x) = x^2 - 2x - 1$$
$$(fg)(x) = x^2(2x+1) = 2x^3 + x^2,$$
$$\left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1}, x \neq -\frac{1}{2}$$

Example 2 *Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f+g)(x), (f-g)(x), (fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.*

Solution We have,

$$(f+g)(x) = \sqrt{x} + x, (f-g)(x) = \sqrt{x} - x,$$
$$(fg)\,x = \sqrt{x}(x) = x^{\frac{3}{2}} \text{ and } \left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}}, x \neq 0$$

Types of Functions

(i) **Identity function** The function f defined by $f(x) = x$ for $\forall x \in R$ is called an identity function of R. For identity function $D_f = R$, $R_f = R$.

(ii) **Constant function** The function f defined by $f(x) = c$ for all $x \in R$, where c is some real number is called a constant function, $D_f = R$, $R_f = \{c\}$.

(iii) **Reciprocal function** The function f defined by $f(x) = \frac{1}{x}$ is called reciprocal function.

$$D_f = R - \{0\}, D_f = R - \{0\}.$$

(iv) **Modulus function** *(or absolute value function)* The function f defined by

$$f(x) = |x| = \begin{cases} x \text{ where } & x \geq 0 \\ -x \text{ where } & x < 0 \end{cases}$$

is called the modulus or absolute value function

$$D_f = R; R_f = [0, \infty)$$

Properties of Modulus Function

(a) $|x| \leq a \Rightarrow -a \leq x \leq a; (a \geq 0)$

(b) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a; (a \geq 0)$

(c) $|x+y| = |x| + |y| \Leftrightarrow x \geq 0$ and $y \geq 0$ or $x \leq 0$ and $y \leq 0$

(d) $|x-y| = |x| - |y| \Rightarrow x \geq 0$ and $|x| \geq |y|$ or $x \leq 0$ and $y \leq 0$ and $|x| \geq |y|$

(e) $|x \pm y| \leq |x| + |y|$

(f) $|x \pm y| \geq ||x| - |y||$

(v) Greatest Integer or Integral Part of a Real Number : For every $x \in R$, $[x]$ is the greatest integer $\leq x$, *i.e.*, $[x] = x$ if x is and integer and [x] equal to integer immediately to the left of x, if x is not an integer.

Greatest Integer Function The function f defined by $f(x) = [x]$ for all $x \in R$ is called the greatest integer function.

Properties of Greatest Integer Function

(a) If $f(x) = [x + n]$, where $n \in I$ and [.] denotes the greatest integer function, then $f(x) = n + [x]$

(b) $x = [x] + \{n\}$, [.] and {x} denote the integral and fractional part of x respectively, then $x - 1 < [x] \leq x$

$$[-x] = -[x],\ \text{if } x \in I$$
$$[-x] = -[x] - 1,\ \text{if } x \notin I$$
$$[x] - [-x] = 2n,\ \text{if } x < n, n \in I$$
$$[x] - [x] = 2n + 1,\ \text{if } x = n + \{x\}, x \in I$$
$$[x] \geq n \Rightarrow x \geq n, n \in I$$
$$[x] \leq n \Rightarrow x < n + 1, n \in I$$
$$[x] > n \Rightarrow x \geq n + 1, n \in I$$
$$[x] < n \Rightarrow x < n, n \in I$$

(c) $D_f = R$, $R_f = I$

(vi) **Signum function** The function f defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

is called the signum function

$$D_f = R;\ R_f = \{-1, 0, 1\}.$$

(vii) **Logarithmic function** The function f defined by $f(x) = \log_a x$, where $x > 0, a > 0, x, a \in R$ is called the logarithmic function

$$D_f = (0, \infty);\ R_f = R$$

(viii) **Exponential function** The function f defined $f(x) = a^x$, when $a > 0$ and $x \in R$ is called exponential function

$$D_f = R;\ R_f = (0, \infty)$$

(ix) **Square root function** The function f defined by $f(x) = \sqrt{x}$ is called the square root function

$$D_f = [0, \infty);\ R_f = [0, \infty)$$

(x) **Polynomial function** The function f defined by $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^x$, $a_n \neq 0$, where $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $n \in N$ is called a polynomial function of degree n.

$$D_f = R;\ R_f = R$$

(xi) **Rational function** The function f defined by $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial functions and $h(x) \neq 0$ is called a rational function.

The domain of f is the set of all real numbers except those values of x for which $h(x) = 0$

(xii) **Odd and even functions**

(a) A function is an odd function if $f(-x) = -f(x)$ for all x.

(b) A function is an even function if $f(-x) = f(x)$ for all x.

(xiii) **Explicit and implicit functions** A function which is expressed directly in terms of independent variable is called an explicit function.

If a function is not expressed directly in terms of independent variable, it is called an implicit function.

(xiv) **Single and many valued functions** A function $y = f(x)$ is said to be single valued function, if there is one and only one value of y corresponding to each value of x.

A function $y = f(x)$ is said to be many valued, if there are more than one value of y corresponding to each value of x.

(xv) **Periodic functions** A function $f(x)$ is said to be a periodic function, if there exists a positive real function T such that $f(x+T) = f(x)$ for all $x \in R$. If T is the smallest positive real number such that $f(x+T) = f(x)$ for all $x \in R$, then T is called the period of $f(x)$.

If $f(x)$ is a periodic function with period T and $a, b \in R$ such that $a > 0$, then $f(ax+b)$ is periodic with period $\frac{T}{a}$.

Example 1 *If $2f(x) + 3f\left(\frac{1}{x}\right) = x^2 - 1$, then prove that $f(x)$ is an even function.*

Solution Given,
$$2f(x) + 3f\left(\frac{1}{x}\right) = x^2 - 1 \qquad \text{...(i)}$$
Replacing x by $\frac{1}{x}$, we get
$$2f\left(\frac{1}{x}\right) + 3f(x) = \frac{1}{x^2} - 1 \qquad \text{...(ii)}$$
Solving Eqs. (i) and (ii), we have
$$f(x) = \frac{3 - 2x^4 - x^2}{5x^2}$$
Clearly, $f(-x) = f(x)$

Example 2 Prove that,
$$f(x) = \log(x + \sqrt{x^2 + 1})$$
is an odd function.

Solution $f(x) = \log(x + \sqrt{x^2+1})$
$$\Rightarrow \quad f(-x) = \log(-x + \sqrt{x^2+1})$$
$$\Rightarrow \quad f(x) + f(-x) = \log(x + \sqrt{x^2+1}) + \log(-x + \sqrt{x^2+1}) = \log 1 = 0$$
$$\Rightarrow \quad f(-x) = -f(x)$$
Hence, $f(x)$ is an odd function.

Example 3 *If $f : R \to R$ is a function satisfying $f(2x+3) + f(2x+7) = 2 \ \forall \ x \in R$, then prove that $f(x)$ is a periodic function.*

Solution Given,
$$f(2x+3) + f(2x+7) = 2 \qquad \text{...(i)}$$
Replacing x by $(x+1)$
$$f(2x+5) + f(2x+9) = 2 \qquad \text{...(ii)}$$
Now replace x by $(x+2)$
$$f(2x+7) + f(2x+11) = 2 \qquad \text{...(iii)}$$

Subtracting Eq. (iii) from Eq. (i), we get

$$f(2x+3) - f(2x+11) = 0$$

$$\Rightarrow \quad f(2x+3) = f(2x+11)$$

$\Rightarrow f(x)$ is a periodic function with period 4.

(xvi) **Composite function** Let $f : A \to R$ and $g : B \to R$, then the composite map of f and g denoted by gof is the map from A to R defined by $(gof)(x) = g(f(x))$.

Example $f : R \to R$ defined by $f(x) = 2x$

$g : R \to R$ defined by $g(x) = x^2$

Then, $gof : R \to R$ defined by

$$(gof)(x) = g(f(x)) = g(2x) = (2x)^2 = 4x^2$$

Note *gof* may not be equal to *fog* in the above example

$$(fog)(x) = f(g(x)) = f(x^2) = 2x^2$$

Example 1 *If* $f(x) = x^2 + 1, g(x) = \dfrac{1}{x-1}$, *then find* $(fog)(x)$ *and* $(gof)(x)$.

Solution Given,

$$f(x) = x^2 + 1 \quad \text{...(i)}$$

$$g(x) = \frac{1}{x-1} \quad \text{...(ii)}$$

Now, $$(fog)(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = f(z),$$

where $$z = \frac{1}{x-1} = z^2 + 1 \quad [\because f(x) = x^2 + 1]$$

$$= \left(\frac{1}{x-1}\right)^2 + 1 = \frac{1}{(x-1)^2} + 1$$

$$(gof)(x) = g(f(x)) = g(x^2+1) = g(u), \quad \text{where } u = x^2 + 1$$

$$= \frac{1}{u-1} = \frac{1}{x^2+1-1} = \frac{1}{x^2}$$

Example 2 *If* $f(x) = (2 - x^n)^{1/n}, n > 0$, *then show that for* $x > 0$

$$f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right) \ge 2$$

Solution Given, $f(x) = (2 - x^n)^{1/n}, x > 0$

Now, $$f(f(x)) = f(y), \quad \text{where } y = f(x)$$

$$= (2 - y^n)^{1/n}, \quad \text{where } y = (2 - x^n)^{1/n}$$

$$[2 - (2 - x^n)]^{1/n} = (x^n)^{1/n} = x, x > 0$$

$$f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right) = x + \frac{1}{x}$$

$$= \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2 \ge 2$$

Functional Equations

Equation which involves an unknown function is called a functional equation.

Example 1 *If $f(x)$ satisfies the relation $f(x+y)=f(x)+f(y)$ for all $x, y \in R$ and $f(1)=5$, then find $\sum_{n=1}^{m} f(n)$. Also prove that $f(x)$ is an odd function.*

Solution
$$f(x+y)=f(x)+f(y)$$

Now,
$$\begin{aligned} f(r) &= f(r-1+1)=f(r-1)+f(1) \\ &= f(r-1)+5=\{f(r-2)+5\}+5 \\ &= f(r-2)+2\cdot 5=\{f(r-3)+5\}+2\cdot 5 \\ &= f(r-3)+3\cdot 5 \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ &= f[r-(r-1)]+(r-1)\cdot 5 \\ &= f(1)+5(r-1)=5+5r-5=5r \end{aligned}$$

$$\therefore \quad \sum_{n=1}^{m} f(n)=\sum_{n=1}^{m} 5x=5\,\frac{m(m+1)}{2}$$

Now, putting $x=y=0$ in given relation we, get
$$f(0+0)=f(0)+f(0) \Rightarrow f(0)=0$$

By putting $y=-x$, we get
$$f(0)=f(x)+f(-x)$$
$$\Rightarrow \quad f(x)+f(-x)=0$$
$$\Rightarrow \quad f(-x)=-f(x)$$

$\therefore f(x)$ is an odd function.

Example 2 $f(x)=x^4+ax^3+bx^2+cx+d$, where $a, b, c, d \in R$. If $f(1)=10$, $f(2)=20$ and $f(3)=30$, find the value of $f(12)+f(-8)$.

Solution Given, $$f(x)=x^4+ax^3+bx^2+cx+d \quad \ldots(i)$$

Given, $f(1)=10$
$$\Rightarrow \quad 1+a+b+c+d=10$$
$$\Rightarrow \quad a+b+c+d=9 \quad \ldots(ii)$$
$$f(2)=20$$
$$\Rightarrow \quad 8a+4b+2c+d=4 \quad \ldots(iii)$$
$$f(3)=30$$
$$\Rightarrow \quad 27a+9b+3c+d=-51 \quad \ldots(iv)$$

Let $y=f(12)+f(-8)$

Then,
$$\begin{aligned} y &= 12^4+8^4+a(12^3-8^3)+b(12^2+8^2)+c(12-8)+2d \\ &= 4^4(81+16)+4\cdot 304a+208b+4c+2d \\ &= 256\cdot 97+1216a+208b+4c+2d \quad \ldots(v) \end{aligned}$$

On adding Eqs. (ii) and (iv), we get

$$28a + 10b + 4c + 2d = -42 \quad \text{...(vi)}$$

On multiplying by 2 in Eq. (iii) and subtracting from Eq. (vi)

$$12a + 2b = -50$$

$$\Rightarrow \quad 6a + b = -25 \quad \text{...(vii)}$$

From Eq. (v), $y = 256 \cdot 97 + 1200a + 200b + 16a + 8b + 4c + 2d$

$$= 256 \cdot 97 + 200(6a + b) + 2(8a + 4b + 2c + d)$$

$$= 256 \cdot 97 + 200(-25) + 2 \cdot 4 \quad \text{[from Eq. (iii)]}$$

$$= 24832 - 5000 + 8 = 19840$$

$$\therefore \quad \frac{f(12) + f(-8)}{10} = 1984$$

Example 3 *If $f(x)$ is a polynomial function such that*

$$f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right), \text{ show that } f(x) = 1 \pm x^n.$$

Solution Let $f(x) = a_0 + a_1x + a_2x^2 + \dots a_nx^n, a_n \neq 0$ ...(i)

Then, $$f\left(\frac{1}{x}\right) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_n}{x^n}$$

$$= \frac{1}{x^n}(a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n)$$

Given, $$f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad a_0x + a_1x + a_2x^2 + \dots a_{n-1}x^{n-1} + a_nx^n + \frac{1}{x^n}(a_0x^n + a_1x^{n-1} + \dots + a_n)$$

$$= (a_0 + a_1 x + a_2 x^2 + a_n x^n)\frac{1}{x^n}(a_0x^n + a_1x^{n-1} + \dots + a_n)$$

$$\Rightarrow \quad a_0x^n + a_1x^{n+1} + a_2x^{n+2} + \dots + a_{n-1}x^{2n-1} + a_nx^{2n} + a_0x^n + a_1x^{n-1} + \dots + a_n$$

$$= (a_0 + a_1x + a_2x^2 + \dots a_nx^n)(a_0x^n + a_1x^{n-1} + \dots + a_n)$$

Equating the coefficient of x^{2n}, we get

$$a_n = a_0a_n \Rightarrow a_0 = 1 \quad [\because a_n \neq 0]$$

Equating the coefficient of x^{2n-1}, we get

$$a_{n-1} = a_0 \, a_{n-1} + a_n \, a_1$$

$$\Rightarrow \quad a_na_1 = 0 \quad [\because a_0 = 1]$$

$$\Rightarrow \quad a_1 = 0 \quad [\because a_n \neq 0]$$

Similarly, equating the coefficients of other powers of x, we get

$$a_2 = a_3 = \dots = a_{n-1} = 0$$

Equating the coefficient of x^n, we get

$$2a_0 = a_0^2 + a_n^2$$

$$\Rightarrow \quad a_n^2 = 1 \quad [\because a_0 = 1]$$

$$\Rightarrow \quad a_n = \pm 1$$

Thus, $$f(x) = a_0 + a_1x + a_2x^2 + \dots a_nx^n = 1 \pm x^n$$

Example 4 *If* $f(x) = \dfrac{4^x}{4^x + 2}$, *find* $\sum_{r=1}^{2001} f\left(\dfrac{r}{2002}\right)$

Solution Given,
$$f(x) = \frac{4^x}{4^x + 2} \qquad \text{...(i)}$$

$\therefore$
$$f(1) - x) = \frac{4^{1-x}}{4^{1-x} + 2} = \frac{4}{4 + 2 \cdot 4^x} = \frac{2}{2 + 4^x} \qquad \text{...(ii)}$$

Clearly,
$$f(x) + f(1 - x) = 1$$

Now,
$$\sum_{r=1}^{2001} f\left(\frac{r}{2002}\right) = f\left(\frac{1}{2002}\right) + f\left(\frac{2}{2002}\right) + f\left(\frac{3}{2002}\right) + \ldots + f\left(\frac{2001}{2002}\right)$$
$$= \left\{f\left(\frac{1}{2002}\right) + f\left(\frac{2001}{2002}\right)\right\} + \left\{f\left(\frac{2}{2002}\right) + f\left(\frac{2000}{2002}\right)\right\}$$
$$+ \ldots + \left\{f\left(\frac{1000}{2002}\right) + f\left(\frac{1002}{2002}\right) + f\left(\frac{1001}{2002}\right)\right\}$$
$$= 1 + 1 + 1 + \ldots \text{ to 1000 terms} + f\left(\frac{1}{2}\right)$$
$$= 1000 + \frac{\sqrt{4}}{\sqrt{4} + 2} = 1000 + \frac{1}{2} = 1000.5$$

Example 5 *If* $2f(xy) = \{f(x)\}^y + \{f(y)\}^x$ *for all* $x, y, \in R$, *and* $f(1) = 2$, *then find the value of* $\sum_{r=1}^{100} f(r)$.

Solution Given,
$$2f(xy) = \{f(x)\}^y + \{f(y)\}^x \text{ for all } x, y, \in R \qquad \text{...(i)}$$

Putting $y = 1$, we get $2f(x) = f(x) + \{f(1)\}^x$

$\therefore$
$$f(x) = 2^x \therefore f(r) = 2^r$$

Now,
$$\sum_{r=1}^{100} f(r) = \sum_{r=1}^{100} 2^r = 2 + 2^2 + 2^3 + \ldots + 2^{100} = \frac{2(2^{100} - 1)}{2 - 1} = 2^{201} - 2$$

Example 6 *If a function satisfies* $2f(x - 1) - f\left(\dfrac{1 - x}{x}\right) = x; (x \neq 0)$, *then determine* $f(x)$.

Solution We have,
$$2f(x - 1) - f\left(\frac{1 - x}{x}\right) = x, x \neq 0$$

$\Rightarrow$
$$2f(x - 1) - f\left(\frac{1}{x} - 1\right) = x \qquad \text{...(i)}$$

Replacing x by $\dfrac{1}{x}$, we get $2f\left(\dfrac{1}{x} - 1\right) - f(x - 1) = \dfrac{1}{x}$ ...(ii)

Now, the operation $2x$(i) + (ii) gives $4f(x - 1) - f(x - 1) = 2x + \dfrac{1}{x}$

$\Rightarrow$
$$3f(x - 1) = \frac{2x^2 + 1}{x}$$

$\Rightarrow$
$$3f(x - 1) = \frac{2(x - 1 + 1)^2 + 1}{x - 1 + 1}$$

$\Rightarrow$
$$f(x) = \frac{2(x + 1)^2 + 1}{3(x + 1)} = \frac{2x^2 + 4x + 3}{3(x + 1)}$$

Additional Solved Examples

Additional Solved Examples

Example 1. *Let f be a function satisfying, $f(x+y)=f(x)f(y)\ \forall\ x,y\in R$. If $f(1)=3$, find, the value of $\sum_{r=1}^{n} f(r)$*

Solution $\because$ $$f(x+y)=f(x)f(y)$$

$\therefore$ $f(x)=a^{\lambda x}$, where λ is constant and $f(1)=a^{\lambda}=3$

$$\sum_{r=1}^{n}(r)=\sum_{r=1}^{n}a^{\lambda r}=a^{\lambda}+a^{2\lambda}+a^{3\lambda}+\ldots+a^{n\lambda}$$

$$=\frac{a^{\lambda}(a^{n\lambda}-1)}{a^{\lambda}-1}=\frac{3(3^n-1)}{3-1}=\frac{3}{2}(3^n-1)$$

Example 2. *Determine all functions $f:R\to R$ such that*

$$f(x-f(y))=f(f(y))+xf(y)+f(x)-1,\ \forall\ x,y\in R,$$

Solution $$f(x-f(y))=f(f(y))+xf(y)+f(x)-1 \quad \ldots\text{(i)}$$

Put $x=f(y)=0$

Then, $$f(0)=f(0)+0+f(0)-1$$

$\therefore$ $$f(0)=1 \quad \ldots\text{(ii)}$$

Again put $x=f(y)=\lambda$ in Eq. (i), then $f(0)=f(\lambda)+\lambda^2+f(\lambda)-1$

$\Rightarrow$ $$1=2f(\lambda)+\lambda^2-1$$

$\therefore$ $$f(\lambda)=\frac{2-\lambda^2}{2}=1-\frac{\lambda^2}{2}$$

Hence, $f(x)=1-\dfrac{x^2}{2}$ is the unique function.

Example 3. *Let n be a positive integer and define $f(n)=1!+2!+3!+\ldots+n!$ where $n!=n(n-1)(n-2)\ldots3\cdot2\cdot1$ find polynomials $P(x)$, $Q(x)$ such that*

$$f(n+2)=P(n)f(n+1)+Q(n)f(n)\ \forall\ n\geq 1.$$

Solution $\because$ $$f(n)=1!+2!+3!+\ldots+n! \quad \ldots\text{(i)}$$

$\therefore$ $$f(n+1)=1!+2!+\ldots+n!+(n+1)! \quad \ldots\text{(ii)}$$

Subtract Eq. (i) from Eq. (ii), then

$$f(n+1)-f(n)=(n+1)! \quad \ldots\text{(iii)}$$

Now, $$f(n+2)=1!+2!+\ldots+(n+1)!+(n+2)!$$
$$=f(n+1)+(n+2)(n+1)!$$
$$=f(n+1)+(n+2)\{f(n+1)-f(n)\}$$

$\therefore$ $$f(n+2)=(n+3)f(n+1)+(-n-2)f(n) \quad \ldots\text{(iv)}$$

But given

$$f(n+2)=P(n)f(n+1)+Q(n)f(n) \quad \ldots\text{(v)}$$

Comparing Eqs. (iv) and (v), we get

$$P(n)=n+3,\ Q(n)=-n-2$$

$\therefore$ $$P(x)=x+3,\ Q(x)=-x-2$$

Example 4. *A function f defined $\forall x, y \in R$ is such that $f(1) = 2; f(2) = 8$ and $f(x + y) - kxy = f(x) + 2y^2$ where k is constant. Find $f(x)$ and show that $f(x + y) f\left(\frac{1}{x + y}\right) = k$ for $x + y \neq 0$.*

Solution $$f(x + y) - kxy = f(x) + 2y^2$$

Replace y by $-x$, we get

$$f(0) + kx^2 = f(x) + 2x^2$$

$\Rightarrow$ $$f(x) = f(0) + kx^2 - 2x^2 \quad \text{...(i)}$$

$\therefore$ $$f(1) = f(0) + k - 2 = 2 \quad \text{...(ii)}$$

and $$f(2) = f(0) + 4k - 8 = 8 \quad \text{...(iii)}$$

Solve Eqs. (ii) and (iii), we get

$$k = 4 \text{ and } f(0) = 0 \quad \text{...(iv)}$$

$\therefore$ from Eq. (i), we get $f(x) = 2x^2$

Also $$f(x + y) \cdot f\left(\frac{1}{x + y}\right) = 2(x + y)^2 \cdot 2\left(\frac{1}{x + y}\right)^2$$

$$= 4 = k \quad \text{[from Eq. (iv)]}$$

Example 5. *Consider a real valued function $f(x)$ satisfying*

$$2f(xy) = (f(x))^y + (f(y))^x \ \forall x, y \in R$$

and $f(1) = a$, where $a \neq 1$. Show that

$$(a - 1) \sum_{i = 1^n} f(i) = a^{n+1} - a$$

Solution $2f(xy) = (f(x))^y + (f(y))^x$

Replace y by 1, we get

$$2f(x) = f(x) + (f(1))^x$$

$\Rightarrow$ $$f(x) = (f(1))^x = a^x \quad [\because f(1) = a]$$

$\therefore$ $$\text{LHS} = (a - 1) \sum_{i=1}^{n} f(i) = (a - 1) \sum_{i=1}^{n} a^i$$

$$= (a - 1)(a + a^2 + a^3 + \ldots + a^n)$$

$$= (a - 1)\left\{\frac{a(a^n - 1)}{(a - 1)}\right\} \quad \text{(If } a > 1\text{)}$$

$$= a(a^n - 1) = (a^{n+1} - a) = \text{RHS}$$

Example 6. *If p and q are +ve integers, f is a function defined for +ve numbers and attains only positive values such that $f(xf(y)) = x^p y^q$.*

Prove that $$q = p^2.$$

Solution $f(xf(y)) = x^p y^q$

or $$[f(xf(y))]^{1/p} = xy^{q/p}$$

or $$x = \frac{[f(xf(y))]^{1/p}}{y^{q/p}} \quad \text{...(i)}$$

Let $$xf(y)=1 \Rightarrow x=\frac{1}{f(y)} \quad \text{...(ii)}$$

from Eq. (i) $$f(y)=\frac{y^{q/p}}{\{f(1)\}^{1/p}}$$

$\therefore$ $$f(1)=\frac{1}{\{f(1)\}^{1/p}} \Rightarrow f(1)=1$$

Then, $$f(y)=y^{q/p} \quad \text{...(iii)}$$

Hence, $$f(xy^{q/p})=x^p y^q$$

put $$y^{q/p}=z$$

Then, $$f(xz)=(xz)^p$$

$\Rightarrow$ $$f(\lambda)=\lambda^p \quad \text{...(iv)}$$

from Eqs. (iii) and (iv), we get

$$y^p=y^{q/p} \Rightarrow p=q/p$$

Hence, $$p^2=q$$

Example 7. *If $f(x+y)=f(x).f(y) \forall$ real x, y and $f(0)\neq 0$, then prove that the function $F(x)=\dfrac{f(x)}{1+[f(x)]^2}$ is an even function.*

Solution $$f(x+y)=f(x)\cdot f(y) \quad \text{...(i)}$$

Put $x=y=0$, then $(f(0))^2=f(0)$

$\Rightarrow$ $$f(0)\{f(0)-1\}=0 \quad \therefore \quad f(0)=1$$

Putting $y=-x$ in Eq. (i), then

$$f(0)=f(x)\cdot f(-x)$$

$\Rightarrow$ $$1=f(x)\cdot f(-x)$$

$\therefore$ $$f(-x)=\frac{1}{f(x)} \quad \text{...(ii)}$$

Also, $$F(x)=\frac{f(x)}{1+[f(x)]^2}$$

$\therefore$ $$F(-x)=\frac{f(-x)}{1+(f(-x))^2}=\frac{\frac{1}{f(x)}}{1+\frac{1}{(f(x))^2}}=\frac{f(x)}{1+(f(x))^2}=F(x)$$

Hence, $F(x)$ is an even function.

Example 8. *If the function f satisfies the relation $f(x+y)+f(x-y)=2f(x)\cdot f(y) \forall x, y \in R$ and $f(0)\neq 0$. Prove that $f(x)$ is an even function.*

Solution $$f(x+y)+f(x-y)=2f(x)\cdot f(y) \quad \text{...(i)}$$

Replace x by y and y by x in Eq. (i)

Then, $$f(y+x)+f(y-x)=2f(y)f(x) \quad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii), we get

$$f(y-x)=f(x-y)$$

Put $y = 2x$, then $f(x) = f(-x)$

Hence, $f(x)$ is an even function.

Example 9. *If $f(x)$ satisfied the relation $f(x+y) = f(x) + f(y)\ \forall x, y \in R$ and $f(1) = 7$, find value of $\sum_{i=1}^{n} f(i)$.*

Solution
$$f(x+y) = f(x) + f(y)$$

Put $x = y = 1$, then $f(2) = 2f(1)$

Put $x = 1, y = 2$, then
$$f(3) = f(1) + f(2) = 3f(1)$$

Similarly, $f(i) = i\, f(1) = 7i$ $\quad [\because f(1) = 7]$

$\therefore$
$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} 7i = 7\sum_{i=1}^{n} i = \frac{7n(n+1)}{2}$$

Also $f(x+y) = f(x) + f(y)$

Put $y = -x$

then $f(0) = f(x) + f(-x)$ $\quad$ [for $x = y = 0$ $f(0) = 0$]

$\therefore$ $0 = f(x) + f(-x)$

$\therefore$ $f(-x) = -f(x)$

Hence, $f(x)$ is an odd function.

Example 10. *If $f(a-x) = f(a+x)$ and $f(b-x) = f(b+x)\ \forall$ real x where $a, b\ (a > b)$ are constants, then prove that $f(x)$ is a periodic function.*

Solution
$$f(b+x) = f(b-x) \quad \text{(given)}$$

Replace x by $x + b$

Then, $f(x+2b) = f(b-(x+b)) = f(-x)$

Replace x by $-x$, then
$$f(x) = f(2b - x) = f(2b + x)$$

$[\because f(a-x) = f(a+x)$ and $a > b$, Here, $a = 2b]$

$\Rightarrow$ $f(2b + x) = f(x)$

Hence, $f(x)$ is periodic with period $2b$.

Example 11. *If f be a function defined on the set of non-negative integers and taking values in the same set. Given that,*

(i) $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$ *$\forall$ non-negative integers.*

(ii) $1900 < f(1990) < 2000$

Find the possible values of $f(1990)$ can take.

Solution $\because$
$$1900 < f(1990) < 2000$$

$\therefore$
$$\frac{1900}{90} < \frac{f(1990)}{90} < \frac{2000}{90}$$

$\Rightarrow$
$$\left[\frac{1900}{90}\right] < \left[\frac{f(1990)}{90}\right] < \left[\frac{2000}{90}\right]$$

$\Rightarrow$
$$21 < \left[\frac{f(1990)}{90}\right] < 22$$

$\{\because [21.111] = 21$ and $[22.222] = 22\}$

Case I Suppose $\left[\frac{f(1990)}{90}\right] = 21$, then substitute $x = 1990$ in Eq. (i)

$$1990 - f(1990) = 19\left[\frac{1990}{19}\right] - 90\left[\frac{f(1990)}{90}\right]$$

$$\Rightarrow \quad 1990 - f(1990) = 19.104 - 90.21$$

$$\Rightarrow \quad f(1990) = 1904$$

Case II Suppose $\left[\frac{f(1990)}{90}\right] = 22$

then substitute $x = 1990$ in Eq. (i)

$$1990 - f(1990) = 19\left[\frac{1990}{19}\right] - 90\left[\frac{f(1990)}{90}\right]$$

$$\Rightarrow \quad 1990 - f(1990) = 19.104 - 90.22$$

$$\Rightarrow \quad f(1990) = 1994$$

Both the value satisfying given conditions.

$\therefore f(1990)$ can take values 1904 and 1994.

Example 12. *Let f be a real valued function with domain R. Now if for some +ve constant a, the equation. $f(x+a) = 1 + \{2 - 3f(x) + 3(f(x))^2 - (f(x))^3\}^{1/3}$ holds good for $x \in R$. Prove that $f(x)$ is a periodic function*

Solution

$$f(x+a) = 1 + [2 - 3f(x) + 3(f(x))^2 - (f(x))^3]^{1/3}$$

$$= 1 + \{1 + (1 - f(x))^3\}^{1/3} \quad \text{...(i)}$$

Replace x by $(x + a)$, then

$$f(x+2a) = 1 + \{1 + (1 - f(x+a))^3\}^{1/3}$$

$$= 1 + \{1 - (f(x+a) - 1)^3\}^{1/3}$$

$$= 1 + \{1 - (1 + (1 - f(x))^3)\}^{1/3}$$

$$= 1 - \{(1 - f(x))^3\}^{1/3}$$

$$= 1 - (1 - f(x)) = f(x)$$

Hence, $f(x)$ is periodic with period $2a$.

Example 13. *Determine all functions f satisfying the functional relation*

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$$

where x is a real number $x \neq 0, x \neq 1$.

Solution

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)} = \frac{2}{x} - \frac{2}{1-x}$$

Let $\quad y = \frac{1}{1-x} \Rightarrow x = 1 - \frac{1}{y}$ ...(i)

from Eq. (i), $\quad f(x) + f(y) = \frac{2}{x} - 2y$ ...(ii)

from Eq. (i), $\quad f(y) + f\left(\frac{1}{1-y}\right) = \frac{2}{y} - \frac{2}{1-y}$ ...(iii)

Let $\quad z = \frac{1}{1-y} \Rightarrow y = 1 - \frac{1}{z}$

from Eq. (iii) we have $$f(y)+f(z)=\frac{2}{y}-2z \quad \text{...(iv)}$$

whenever $y, z \neq 0$ or 1.

$$y=\frac{1}{1-x} \quad \text{and} \quad z=\frac{1}{1-y}$$

Now, $$x=\frac{1}{1-z}$$

$$\Rightarrow \quad z=1-\frac{1}{x}$$

Similarly, $$f(x)+f(z)=\frac{2}{z}-2x \quad \text{...(v)}$$

Adding corresponding sides of Eqs. (ii), (iv) and (v) and divide throughout by 2, we get

$$f(x)+f(y)+f(z)=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z)$$

$$\Rightarrow \quad f(x)+\frac{2}{y}-2z=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z)$$

$$\Rightarrow \quad f(x)=\left(\frac{1}{x}-\frac{1}{y}+\frac{1}{z}\right)-(x+y-z)$$

$$=\left(\frac{1}{x}-x\right)-\left(y+\frac{1}{y}\right)+\left(z+\frac{1}{z}\right)$$

$$=\left(\frac{1}{x}-x\right)-\left(\frac{1}{1-x}+1-x\right)+\left(1-\frac{1}{x}+\frac{x}{x+1}\right)$$

$$=-\frac{1}{1-x}+\frac{x}{x-1}$$

$$=\left(\frac{x+1}{x-1}\right) \text{ for real values of } x \text{ except } x=0 \text{ or } 1.$$

Aliter

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}=\frac{2}{x}-\frac{2}{1-x} \quad \text{...(i)}$$

Replace x by $\frac{1}{1-x}$, we get

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{1}{1-\frac{1}{1-x}}\right)=2(1-x)-\frac{2}{1-\frac{1}{1-x}}$$

$$\Rightarrow \quad f\left(\frac{1}{1-x}\right)+f\left(1-\frac{1}{x}\right)=2(1-x)-2\left(1-\frac{1}{x}\right)$$

$$=-2x+\frac{2}{x} \quad \text{...(ii)}$$

Replace x by $1-\frac{1}{x}$ in Eq. (i), we get

$$f\left(1-\frac{1}{x}\right)+f\left(\frac{1}{1-\left(1-\frac{1}{x}\right)}\right)=\frac{2}{\left(1-\frac{1}{x}\right)}-\frac{2}{1-\left(1-\frac{1}{x}\right)}$$

$\Rightarrow$
$$f\left(1-\frac{1}{x}\right)+f(x)=\frac{2x}{x-1}-2x \quad \text{...(iii)}$$

Subtract Eq. (ii) from Eq. (i), we get

$$f(x)-f\left(1-\frac{1}{x}\right)=2x-\frac{2}{1-x} \quad \text{...(iv)}$$

Adding Eqs. (iii) and (iv), we get

$$2f(x)=\frac{2x}{x-1}-\frac{2}{1-x}$$

$$f(x)=\frac{x+1}{x-1}$$

Example 14. *If $f: R \to R$ is a function satisfying the properties (i) $f(-x)=-f(x)$ (ii) $f(x+1)=f(x)+1$ (iii) $f\left(\frac{1}{x}\right)=\frac{f(x)}{x^2}, x \neq 0$. Prove that $f(x)=x \ \forall x \in R$.*

Solution Let us observe that if $f(x)$ is known $\forall x > 0$, then $f(x)$ can be found $\forall x < 0$ by using (i) ...(i)

By putting $x = 0$ in (1), we find that $f(-0)=-f(0)$ *i.e.*, $f(0)=0$ ...(ii)

from Eqs. (i) and (ii) we find that it enough to find $f(x) \forall x > 0$.

We shall take $x > 0$ and compute $f\left(\frac{1}{x+1}\right)$ in terms of $f(x)$ in two different ways. By equating the two expressions for $f\left(\frac{1}{x+1}\right)$ thus obtained, we shall find $f(x)$ ...(iii)

Now,
$$f\left(\frac{1}{x+1}\right)=\left[\frac{f(x+1)}{(x+1)^2}\right] \quad \text{[from Eq. (iii)]}$$

$$=\left[\frac{f(x)+1}{(x+1)^2}\right] \quad \text{[by Eq. (ii)] ...(A)}$$

Again,
$$f\left(\frac{1}{x+1}\right)=f\left(1-\frac{x}{x+1}\right)=f\left(\frac{-x}{x+1}\right)+1 \quad \text{[by Eq. (i)]}$$

$$=-f\left(\frac{x}{x+1}\right)+1 \quad \text{[by Eq. (i)]}$$

$$=-\left[\frac{f\left(\frac{x+1}{x}\right)}{\left(\frac{x+1}{x}\right)^2}\right]+1 \quad \text{[by Eq. (iii)]}$$

$$=\frac{-x^2}{(x+1)^2}f\left(1+\frac{1}{x}\right)+1=\frac{-x^2}{(x+1)^2}\left[\frac{f(x)}{x^2}+1\right]+1 \quad \text{[by Eq. (iii)] ...(B)}$$

from Eqs. (A) and (B), we have

$$\frac{f(x)+1}{(x+1)^2} = -\frac{x^2}{(x+1)^2}\left[\frac{f(x)}{x^2}+1\right]+1$$

$$\frac{2f(x)}{(x+1)^2} = \frac{-x^2}{(x+1)^2}+1-\frac{1}{(x+1)^2}$$

$$\Rightarrow \qquad f(x) = x$$

Thus we find that $f(x) = x \ \forall x > 0$

Let $\qquad x < 0 \qquad$ ...(iv)

Put $x = -y$ so that $y > 0$

We have
$$f(x) = f(-y) = -f(y) \qquad \text{[by Eq. (i)]}$$
$$= -y \qquad \text{[by Eq. (iii)]}$$
$$= x$$

$\because$ We have already seen that $f(x) = x$ when $x = 0$, it follows that $f(x) = x \ \forall x \in R$.

Example 15. *Let $f(x, y)$ be a periodic function satisfying $f(x, y) = f(2x + 2y, 2y - 2x) \ \forall \ x, y$. Define $g(x) = f(2^x, 0)$. Show that $g(x)$ is a periodic function with period* 12.

Solution
$$f(x, y) = f(2x + 2y, 2y - 2x) \qquad \text{...(i)}$$

or
$$f(I, II) = f(2I + 2II, 2II - 2I)$$

$\therefore$
$$f(2x + 2y, 2y - 2x)$$
$$= f(2(2x + 2y) + 2(2y - 2x), 2(2y - 2x) - (2x + 2y))$$
$$= f(8y, -8x) \qquad \text{...(ii)}$$

from Eqs. (i) and (ii), we get
$$f(x, y) = f(8y, -8x) \qquad \text{...(iii)}$$

or
$$f(I, II) = f(8II, -8I)$$

$\therefore$
$$f(8y, -8x) = f(8(-8x), -8(8y))$$
$$= f(-64x, -64y) \qquad \text{...(iv)}$$

from Eqs. (iii) and (iv), we get,
$$f(x, y) = f(-64x, -64y) \qquad \text{...(v)}$$

or
$$f(I, II) = f(-64I, -64II)$$

$\therefore$
$$f(-64x, -64y) = f(-64(-64x), -64(-64y))$$
$$= f(2^{12}x, 2^{12}y) \qquad \text{...(vi)}$$
$$f(x, y) = f(2^{12}x, 2^{12}y) \qquad \text{[from Eqs. (v) and (vi)]}$$
$$\Rightarrow \qquad f(x, 0) = f(2^{12}x, 0)$$

Replace x by 2^x, then $f(2^x, 0) = f(2^{x+12}, 0)$
$$\Rightarrow \qquad g(x) = g(x + 12) \qquad [\because g(x) = f(2^x, 0)]$$

Hence, $g(x)$ is periodic with period 12.

Example 16. *If for all real values of u and v, $2f(u)\cos v = f(u+v) + f(u-v)$*

Prove that ∀ real values of x.

(i) $f(x) + f(-x) = 2a\cos x$ *(ii) $f(\pi - x) + f(-x)$*

(iii) $f(\pi - x) + f(x) = 2b\sin x$.

Deduce that $f(x) = a\cos x + b\sin x$ where, a, b are arbitrary constant.

Solution $$2f(u)\cos v = f(u+v) + f(u-v) \quad \text{...(i)}$$

put $u = 0$ and $v = x$ in Eq. (i)

we get, $$f(x) + f(-x) = 2f(0)\cos x = 2a\cos x \quad \text{...(ii)}$$

a is arbitrary constant.

Put $u = \frac{\pi}{2} - x$ and $v = \frac{\pi}{2}$ in Eq. (i), we get

$$f(\pi - x) + f(-x) = 0 \quad \text{...(iii)}$$

Again put $u = \pi/2$ and $v = (\pi/2) - x$ in Eq. (i), we get

$$f(\pi - x) + f(x) = 2f(\pi/2)\sin x = 2b\sin x \quad \text{...(iv)}$$

b is arbitrary constant.

Adding Eqs. (ii) and (iv), then

$$2f(x) + f(\pi - x) + f(-x) = 2a\cos x + 2b\sin x$$

$$2f(x) + 0 = 2a\cos x + 2b\sin x$$

$$\therefore \quad f(x) = a\cos x + b\sin x$$

Example 17. *Let f and g be real valued functions such that $f(x+y) + f(x-y) = 2f(x)\cdot g(y) \; \forall x, y \in R$. Prove that if f is not identically zero and $|f(x)| \le 1 \; \forall x \in R$, then $|g(y)| \le 1 \; \forall y \in R$.*

Solution $$0 < |f(x)| \le 1 \quad \text{(given)}$$

Let $$\max|f(x)| = M \; \forall \; 0 < M \le 1 \quad \text{...(i)}$$

$$\therefore \quad |f(x+y)| \le M \text{ and } |f(x-y)| \le M$$

Then, $$|f(x+y)| + |f(x-y)| \le 2M \quad \text{...(ii)}$$

$$2f(x)g(y) = f(x+y) + f(x-y)$$

$$\Rightarrow \quad |2f(x)g(y)| = |f(x+y) + f(x-y)|$$

$$\le |f(x+y)| + |f(x-y)| \le 2M \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad 2|f(x)||g(y)| \le 2M$$

$$\Rightarrow \quad |f(x)||g(y)| \le M \le 1 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad |f(x)||g(y)| \le 1$$

$$|f(x)| \le 1$$

$$\therefore \quad |g(y)| \le 1 \; \forall y \in R$$

Example 18. *Let f be a real valued function defined ∀ real numbers, such that for some $a > 0$ it satisfies*

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$$

Prove that f is periodic i.e., there exists a +ve real b such that $f(x+b) = f(x)$ holds for every x. Also gives an example of such a non constant f for $a = 1$.

Solution
$$f(x+a)=\frac{1}{2}+\sqrt{f(x)-(f(x))^2} \quad \text{...(i)}$$

Note that
$$\frac{1}{2}\le f(x)\le 1 \quad \text{...(ii)}$$

holds for all $x \because f(x)$ is the sum of $\frac{1}{2}$ and a non negative real and the expression under the square root is non negative only, if $f(x)(1-f(x))\ge 0$

which according to $f(x)\ge\frac{1}{2}\Rightarrow f(x)\le 1$

First Solution

$\because$ Eq. (i) provides a relation between $f(x+a)$ and $f(x)$ substitute $x\to x+a$ in Eq. (i)

$$\begin{aligned}
f(x+2a)&=\frac{1}{2}+\sqrt{f(x+a)-(f(x+a))^2}\\
&=\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{f(x)-(f(x))^2}-\left(\frac{1}{2}-\sqrt{f(x)-(f(x))^2}\right)^2}\\
&=\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{f(x)-(f(x))^2}-\frac{1}{4}-f(x)+(f(x))^2-\sqrt{f(x)-(f(x))^2}}\\
&=\frac{1}{2}+\sqrt{\frac{1}{4}-f(x)+(f(x))^2}\\
&=\frac{1}{2}+\sqrt{\left(f(x)-\frac{1}{2}\right)^2}=\frac{1}{2}+f(x)-\frac{1}{2}=f(x)
\end{aligned}$$

It shows that $b=2a$ is a period of f.

Second Solution

Consider Eq. (i) as a quadratic equation with $f(x)$ as unknown.

Taking its square root, we get

$$f(x+a)-\frac{1}{2}=\sqrt{f(x)-(f(x))^2}=f(x)^2-f(x)+\left(f(x+a)-\frac{1}{2}\right)^2=0$$

Now use quadratic formula

$$\frac{1}{2}+\sqrt{f(x+a)-(f(x+a))^2},$$
$$\frac{1}{2}-\sqrt{f(x+a)-(f(x+a))^2}$$

Second root is less than $\frac{1}{2}$ hence the unique solution for $f(x)$ is

$$f(x)=\frac{1}{2}+\sqrt{f(x+a)-(f(x+a))^2}$$

Substitute $x\to x-a$ and we get

$$f(x-a)=\frac{1}{2}+\sqrt{f(x)-(f(x))^2}$$

According to Eq. (i) this shows that

$$f(x-a)=f(x+a)$$

Hence, finally substitute $x\to x+a$

$$\Rightarrow \qquad f(x)=f(x+2a)$$

Consequently $b=2a$ is a period of f.

Example 19. *The function $f(x, y)$ satisfies*

$$f(0, y) = y + 1, f(x + 1, 0) = f(x, 1)$$
$$f(x + 1, y + 1) = f(x, f(x + 1, y))$$

for every integer x and y find $f(4, 1981)$.

Solution Let y be an arbitrary +ve integer. We want to find the values $f(1, y), f(2, y), f(3, y)$.

$$f(1, 0) = f(0, 1) = 2$$
$$f(1, y + 1) = f(0, f(1, y)) = f(1, y) + 1$$
$$= f(0, f(1, y - 1)) + 1 = f(1, y - 1) + 2$$

Thus, $$f(1, y) = f(1, y - 1) + 1$$

Again,

$$f(1, y) = f(1, 0) + y = f(0, 1) + y = y + 2$$

Now, we determine the value of $f(2, y)$

$$f(2, y) = f(1, f(2, y - 1)) = f(2, y - 1) + 2$$
$$f(2, y) = f(2, 0) + 2y = f(1, 1) + 2y = 2y + 3$$
$$f(3, y) = f(2, f(3, y - 1)) = 2f(3, y - 1) + 3$$
$$f(3, y) = 2(2f(3, y - 2) + 3) + 3$$
$$= 2^2 f(3, y - 2) + 3 + 2 . 3$$
$$= 3 + 2.3 + 2^2 . 3 + \dots + 2^{y-1} . 3 + 2^y f(3, 0)$$
$$= 3(2^y - 1) + 2^y f(2, 1)$$
$$= 3(2^y - 1) + 2^y 5$$
$$f(3, y) = 2^{y+3} - 3$$

finally we determine value of $f(4, y) \cdot f(4, y) = f(3, f(4, y - 1)) = 2^{f(4, y-1)+3} - 3$

$$= 2^{(2f(4, y-2)+3)} - 3$$
$$= 2^{2f(4, y-2)+3)} - 3$$
$$= f(4, y) = 2^{2^{2^{\dots 2f(4, 0)+3}}} - 3$$

and there are y many

$$f(4, 0) = f(3, 1)$$
$$= 2^4 - 3 = 2^{2^2} - 3$$
$$f(4, y) = 2^{2^{2 \dots 2}} - 3$$

Example 20. *The function $f(x)$ is defined on the +ve integers and takes on non - ve integers values $\forall\, n, m$*

$$f(m + n) - f(m) - f(n) = 0 \text{ or } 1$$
$$f(2) = 0, f(3) > 0$$
$$f(9999) = 3333$$

find $f(1982)$.

Solution

$$f(m + n) - f(m) - f(n) = 0 \text{ or } 1 \quad \text{...(i)}$$
$$f(2) = 0, f(3) > 0 \quad \text{...(ii)}$$
$$f(9999) = 3333 \quad \text{...(iii)}$$

Let $m = n = 1$, then from Eq. (i) it follows that $f(2) - 2f(1) = 0$ or Eq. (ii) implies that $2f(1) = 0$ or -1

Hence, $f(1) = 0.$

Similarly, if $m = 2, n = 1$

$$f(3) - f(2) - f(1) = f(3) = 0 \text{ or } 1$$

and by Eq. (ii), we have $f(3) = 1$

Using induction we show that for every +ve integer k

$$f(3k) \geq k \quad \text{...(iv)}$$

Assume that Eq. (iv) holds until some k. Let $m = 3k, n = 3$ then Eq. (i) implies $f(3(k+1))$

$$= f(3k+3) \geq f(3k) + f(3) \geq k + 1$$

Hence, Eq. (iv) holds for every k. If in the last step $f(3k) > k$, then $f(3(k+1)) > k+1$ holds *i.e.*, if for some k_0 we have $f(3k_0) > k_0$ in Eq. (iv), then for every $k > k_0, f(3k) > k$ holds as well.

This observation implies that in case $k < 3333, f(3k) > k$ cannot hold as, then $f(3.3333) > 3333$ follows contradicting Eq. (iii)

$$\therefore \quad f(3.1982) = 1982.$$

Now Eq. (i) implies.

$$1982 = f(3.1982) = f(2.1982 + 1982)$$

$$\geq f(2.1982) + f(1982)$$

$$f(1982 + 1982) \geq 2f(1982)$$

Combine last two results, we get $1982 \geq 3f(1982)$,

$$f(1982) \leq \frac{1982}{3} < 661 \quad \text{...(v)}$$

Using Eq. (i) again $f(1982) = f(1980 + 2)$

$$\geq f(1980) + f(2) = f(3.660) + 0 = 660$$

from Eqs. (v), $660 \leq f(1982) < 661$

so $f(1982) = 660$

Example 21. *Prove that there is no function f from the set of non - ve integers into itself such that* $f(f(n)) = n + 1987 \ \forall n.$

Solution

$$f(f(n)) = n + 1987 \quad \text{...(i)}$$

Substitute $f(n)$ for n in Eq. (i), we get

$$f(f(f(n))) = f(n) + 1987 \quad \text{...(ii)}$$

Again $f(f(f(n))) = f(n + 1987)$ ...(iii)

from Eqs. (ii) and (iii), we get

$$f(n + 1987) = f(n) + 1987 \quad \text{...(iv)}$$

Let t be any arbitrary +ve integer let us now use mathematical induction.

$$f(n + 1987t) = f(n) + 1987t \quad \text{...(v)}$$

for $t = 1$ $f(n + 1987(t-1)) = f(n) + 1987(t-1)$

Substitute here $n + 1987$ for n

Then from Eq. (iv)

$$f(n + 1987t) = f(n + 1987) + 1987(t-1) = f(n) + 1987 + 1987(t-1) = f(n) + 1987t$$

which is equal to Eq. (v)

Let s be an arbitrary non - ve integer less than 1987 and consider the remainder r when $f(s)$ is divided by 1987.

$$f(s) = 1987k + r \ (0 \leq k \text{ and } 0 \leq r \leq 1986)$$

By condition $f(s)$ is not - ve ...(vi)

$\Rightarrow$ $f(f(s)) = s + 1987$ [By Eq. (i)]

from Eqs. (v) and (vi)

$$f(f(s)) = f(1987k + r) = f(r) + 1987k$$

$\Rightarrow$ $$s + 1987 = f(r) + 1987k \quad \text{...(vii)}$$

$\because$ $$s < 1987$$

$$f(r) + 1987k < 2.1987$$

$$f(r) < 1987(2 - k)$$

Now, $$f(r) \geq 0$$

either $$k = 1 \text{ or } k = 0$$

from Eqs. (vi) and (vii) $$f(s) = 1987 + r \quad \text{...(viii)}$$

$$f(r) = s \quad \text{...(ix)}$$

$\Rightarrow 1987 = 0$ which is a contradiction.

In other case $k = 0$

Again from Eqs. (vi) and (vii) implies

$$f(s) = r \quad \text{...(x)}$$

$$f(r) = 1987 + s \quad \text{...(xi)}$$

So, $r = s$ leads to contradiction.

Eqs. (ix) and (x) together imply when acting on the numbers 0, 1, ...,1986.

When f is arranging them into pairs (a, b) in such a way that either

$$f(a) = b \text{ and } f(b) = a + 1987$$

$$f(b) = a \text{ and } f(a) = b + 1987$$

We observe that numbers in each pair are different. Which is a contradiction. Since, the number of elements of the set 0, 1, ..., 1986 is odd.

Example 22. *Find all function f defined on the non -ve reals and taking non -ve real values such that,*

$$f(x \cdot f(y)) \cdot f(y) = f(x + y)$$

for every non - ve x and y $f(2) = 0$,

$$f(x) \neq 0, \text{ if } 0 \leq x < 2$$

Solution Let $$x \geq 2 \text{ and } y = 2 \quad \text{...(A)}$$

from $f(x \cdot f(y)) \cdot f(y) = f(x + y)$, we have $f((x - 2) f(2)) \cdot f(2) = f(x - 2 + 2) = f(x)$

from $$f(2) = 0 \quad \text{...(B)}$$

for every $x \geq 2$ $$f(x) = 0 \quad \text{...(i)}$$

and this holds only in case $x \geq 2$

Now, let $$0 \leq y < 2$$

LHS of Eq. (A) is 0 if $xf(y) \geq 2$ *i.e.*, $$x \geq \frac{2}{f(y)} \quad \text{...(ii)}$$

RHS equals to 0 if $x + y \geq 2$

Thus $$x \geq 2 - y \quad \text{...(iii)}$$

$\Rightarrow$ Under the given conditions the equality

$$\frac{2}{f(y)} = 2 - y \quad \text{...(iv)}$$

If Eq. (iv) does not hold for some y, then there is a smaller number on the LHS of Eq. (iv) than on RHS.

Hence there were an x such that $$\frac{2}{f(y)} < x < 2 - y \quad \text{...(v)}$$

But now from Eq. (ii) it implies there is 0 on LHS of Eq. (A)

It is given $$f(x) \neq 0 \text{ if } 0 \leq x < 2 \quad \text{...(C)}$$

$\Rightarrow$ RHS $\neq 0$ which is a contradiction

$\therefore$ for every $0 \le y < 2$ Eq. (iv) holds

$$f(y) = \frac{2}{2-y} \quad (0 \le y < 2) \qquad \text{...(vi)}$$

So only function which satisfies A,B,C, is

$$f(x) = \begin{cases} \frac{2}{2-x} & \text{if} \quad 0 \le x < 2 \\ 0 & \text{if} \quad x \ge 2 \end{cases}$$

if $y \ge 2$, Eqs. (B) and (C) are true

$\therefore$ Both sides of Eq. (A) is 0

Considering the case $\quad 0 \le y < 2$

Now, assume that $\quad x + y \ge 2$

Hence $\quad$ RHS $= 0$

then $\quad x \ge 2 - y$

and $\quad x \cdot f(y) = \frac{2x}{2-y} \ge \frac{2(2-y)}{2-y} = 2$

So, $$f(x \cdot f(y)) \cdot f(y) = \frac{2}{2 - \frac{2x}{2-y}} \cdot \frac{2}{2-y} = \frac{2(2-y) \cdot 2}{(4 - 2y - 2x)(2-y)} = \frac{2}{2-(x+y)} = f(x+y)$$

So, $f(x)$ satisfies Eq. (A).

Example 23. *Find all functions of defined on the set of all real numbers with real values, such that,*

$$f(x^2 + f(y)) = y + (f(x))^2 \forall x, y$$

Solution

$$f(x^2 + f(y)) = y + (f(x))^2 \qquad \text{...(i)}$$

Assume that for some real number y

$$f(y) < y \text{ i.e., } y - f(y) > 0 \qquad \text{...(ii)}$$

Let x be such that $x^2 = y - f(y)$

i.e., $$y = x^2 + f(y)$$

from Eq. (i)

$$f(y) = f(x^2 + f(y)) = y + f^2(x) \ge y$$

which contradicts Eq. (ii)

Hence for every real number y

$$f(y) \ge y \qquad \text{...(iii)}$$

Set now $y_0 - t$ to be smaller than $-f^2(0)$ denote $f(y_0)$ by a

from Eqs. (i) and (iii), we get

$$a \le f(a) = f(0^2 + f(y_0)) = y_0 + f^2(0) < 0$$

i.e., $$a \le f(a) < 0$$

$\therefore$ $$a^2 \ge f^2(a) \qquad \text{...(iv)}$$

Let x be now arbitrary

Using Eqs. (i), (iii) and (iv), we get

$$x + a^2 \le a^2 + f(x) \le f(a^2 + f(x))$$
$$= x + f^2(a) \le x + a^2 \qquad \text{...(v)}$$

$\because$ The lowest and highest terms are equal there is equality every where in Eq. (v)

Thus, $$f(x) = x$$

for every real number x.

Let us Practice

Let us Practice

Level 1

1. If $f(x)=\dfrac{1+x}{1-x}$, prove that
$$\frac{f(x)\cdot f(x^2)}{1+[f(x)]^2}=\frac{1}{2}$$

2. If $f(x+y, x-y)=xy$, then find the arithmetic mean of
$$f(x, y) \text{ and } f(y, x).$$

3. If $f(x)=64x^3+\dfrac{1}{x^3}$ and a, b are roots of $4x+\dfrac{1}{x}=3$, then prove that
$$f(a)=f(b).$$

4. If $f(x+y)=f(x)+f(y)-1$ for all $x, y \in R$ and $f(1)=1$, then find the number of solutions of $f(n)=n, n \in N$.

5. Find the value of natural number a for which $\sum\limits_{k=1}^{n} f(a+k)=16(2^n-1)$ where the function f satisfies the relation $f(x+y)=f(x)f(y)$ for all natural numbers x, y and further $f(1)=2$.

6. A real valued function $f(x)$ satisfies the functional equation
$$f(x-y)=f(x)f(y)-f(a-x)f(a+y)$$
where a is given constant and $f(0)=1$, then prove that
$$f(2a-x)=-f(x).$$

7. If $f(x)=\dfrac{\alpha x}{x+1}, x \neq -1$, then for what value of α is $f(f(x))=x$.

8. If $g(x)=1+\sqrt{x}$ and $f(g(x))=3+2\sqrt{x}+x$, then find $f(x)$.

9. If f is an even function defined on the interval $[-5, 5]$, then find real values of x satisfying
$$f(x)=f\left(\frac{x+1}{x+2}\right)$$

10. Determine the function f satisfying
$$f(x)+f\left(\frac{x-1}{x}\right)=1+x, \forall x \in R-\{0, 1\}$$

11. Let X be the set of all positive integers greater than or equal to 8 and let $f: X \to X$ be a function such that $f(x+y)=f(xy)$ for all $x \geq 4, y \geq 4$. If $f(8)=9$, determine $f(9)$.
(RMO 2006)

Level 2

1. If $f(x)=\dfrac{a^x}{a^x+\sqrt{a}}, a>0$, then prove that
$$f(x)+f(1-x)=1$$
and $$\sum_{k=1}^{2n-1} 2f\left(\frac{k}{2n}\right)=2n-1$$
and $$\sum_{k=1}^{2n} 2f\left(\frac{k}{2n+1}\right)=2n$$

2. Determine a function $f: R \to R$ such that $f(x-f(y))=f(f(y))+xf(y)+f(x)-1$ for all $x, y \in R$.

3. Let $f(x)$ be a polynomial function of degree n satisfying the condition $f(x)f\left(\dfrac{1}{x}\right)=f(x)+f\left(\dfrac{1}{x}\right)$. Find $f(x)$.

4. Let f be a function from the set of natural numbers to the set of real numbers *i.e.*, $f: N \to R$ such that
(i) $f(1)=1$
(ii) $f(1)+2f(2)+3f(3)+\ldots+nf(n)=n(n+1)f(n)$ for all $n \geq 2$, then find $f(2008)$.

5. Let f be polynomial function such that $f(x)f(y)+2=f(x)+f(y)+f(xy); \quad \forall x, y \in R.$

Suppose f is one-one. If $f(3) = 82$ and $f(9) \neq 2$, then show that

$$f(x) = x^4 + 1, \forall x \in R.$$

6. Determine all functions f satisfying the functional relation

$$2f(x) + 3f\left(\frac{2x + 29}{x - 2}\right) = 100x + 80,$$

where x is a real number different from 2.

7. For any natural number $n, (n \geq 3)$, let $f(n)$ denote the number of non-congruent integer-sided triangles with perimeter n (*e.g.*, $f(3) = 1, f(4) = 0, f(7) = 2$). Show that

(a) $f(1999) > f(1996)$;

(b) $f(2000) = f(1997)$. (INMO 2000)

8. Let R denote the set of all real numbers. Find all functions $f : R \to R$ satisfying the condition

$$f(x + y) = f(x)f(y)f(xy) \text{ for all } x, y \text{ in } R.$$

(INMO 2001)

9. Find all functions $f : R \to R$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y)$$

for all x, y, z in R. (Here R denotes the set of all real numbers). (INMO 2005)

10. Let X denote the set of all triples (a, b, c) of integers. Define a function $f : X \to X$ by

$$f(a, b, c) = (a + b + c, ab + bc + ca, abc).$$

Find all triples (a, b, c) in X such that

$$f(f(a, b, c)) = (a, b, c)$$ (INMO 2006)

11. Prove that for every positive integer n there exists a unique ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a + b - 1)(a + b - 2) + a$$

(INMO 2006)

Solutions

Solutions

Level 1

1. $\because \quad f(x) = \frac{1+x}{1-x}$

$\therefore \quad f(x^2) = \frac{1+x^2}{1-x^2}$

Now, $\quad \text{LHS} = \frac{f(x)\cdot f(x^2)}{1+[f(x)]^2} = \frac{\left(\frac{1+x}{1-x}\right)\left(\frac{1+x^2}{1-x^2}\right)}{1+\left(\frac{1+x}{1-x}\right)^2}$

$$= \frac{\frac{1+x^2}{(1-x)^2}}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}}$$

$$= \frac{1+x^2}{2(1+x^2)} = \frac{1}{2}$$

2. Let $X = x+y, Y = x-y$

$\Rightarrow \quad x = \frac{X+Y}{2}, y = \frac{X-Y}{2}$

Given functional relation becomes

$$f(X,Y) = \left(\frac{X+Y}{2}\right)\left(\frac{X-Y}{2}\right) = \frac{X^2-Y^2}{4}$$

$\therefore \quad f(x,y) = \frac{x^2-y^2}{4}$ and $f(y,x) = \frac{y^2-x^2}{4}$

$\therefore$ Arithmetic mean $= \frac{f(x,y)+f(y,x)}{2}$

$$= \frac{\frac{x^2-y^2}{4}+\frac{y^2-x^2}{4}}{2} = 0$$

3. $64x^3 + \frac{1}{x^3} = \left(4x+\frac{1}{x}\right)^3 - 3\cdot 4x\cdot\frac{1}{x}\left(4x+\frac{1}{x}\right)$

$$= 27 - 12\times 3 = 27 - 36 = -9$$

$\therefore \quad f(x) = -9 \Rightarrow f(a) = f(b) = -9$

4. Since, $f(x+y) = f(x) + f(y) - xy - 1 \ \forall x, y \in R$

Putting $y = 1$

$$f(x+1) = f(x) + f(1) - x - 1$$

$\Rightarrow \quad f(x+1) = f(x) - x$

$\therefore \quad f(n+1) = f(n) - n < f(n)$

So, $f(n) < f(n-1) < f(n-2) < \dots$

$$< f(3) < f(2) < f(1) = 1$$

$\therefore f(n) = n$ holds only for $n = 1$.

5. $\because f(x+y) = f(x)f(y)$

Putting $x = y = 1, f(2) = f(1)\cdot f(1) = 2^2$

Putting $x = 2, y = 1, f(3) = f(2)f(1) = 2^3$

....

$$f(k) = 2^k$$

Now, from given $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$

$\Rightarrow \quad \sum_{k=1}^{n} f(a)f(k) = 16(2^n - 1)$

$\Rightarrow \quad f(a)\sum_{k=1}^{n} 2^k = 16(2^n - 1)$

$\Rightarrow \quad f(a)\cdot\frac{2(2^n-1)}{2-1} = 16(2^n - 1)$

$\Rightarrow \quad f(a) = 8 = 2^a \Rightarrow a = 3$

6. Given, $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$

Putting $x = y = 0$

$\Rightarrow \quad f(0) = (f(0))^2 - (f(a))^2$

$\Rightarrow \quad 1 = 1 - (f(a))^2$

$\Rightarrow \quad f(a) = 0$

$\therefore \quad f(2a-x) = f(a-(x-a))$

$$= f(a)f(x-a) - f(a-a)f(x)$$

$$= 0 - f(x)\cdot 1 = -f(x)$$

7. $f(f(x)) = \frac{\alpha f(x)}{f(x)+1} = \frac{\alpha\cdot\frac{\alpha x}{x+1}}{\frac{\alpha x}{x+1}+1}$

$\Rightarrow \quad x = \frac{\alpha^2 x}{(\alpha+1)x+1}$

which holds only when $\alpha = -1$

8. $f(1+\sqrt{x}) = 3 + 2\sqrt{x} + x = (1+\sqrt{x})^2 + 2$

$\Rightarrow \quad f(x) = x^2 + 2$

9. $f(x) = f\left(\frac{x+1}{x+2}\right)$

$\Rightarrow \quad x = \frac{x+1}{x+2}$

$\Rightarrow \quad x(x+2) - (x+1) = 0$

$\Rightarrow \quad x^2 + x - 1 = 0$

$\therefore \quad x = \dfrac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} = \dfrac{-1 \pm \sqrt{5}}{2} \quad \ldots(i)$

Replacing x by $-x$

$$f(-x) = f\left(\frac{-x+1}{-x+2}\right)$$

$\Rightarrow \quad f(x) = f\left(\dfrac{-x+1}{-x+2}\right) \quad [\because f(x) \text{ is even}]$

$\Rightarrow \quad x = \dfrac{-x+1}{-x+2}$

$\Rightarrow \quad -x^2 + 2x + x - 1 = 0$

$\Rightarrow \quad x^2 - 3x + 1 = 0$

$\therefore \quad x = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2} \quad \ldots(ii)$

$\therefore$ All real values of x satisfying given relation is given by Eqs. (i) and (ii).

10. Let f be a given real valued function satisfying the condition,

$f(x) + f\left(\dfrac{x-1}{x}\right) = 1 + x, \forall x \in R - \{0, 1\} \quad \ldots(i)$

Let us put $\dfrac{x-1}{x}$ in place of x so that

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}}\right) = 1 + \frac{x-1}{x}$$

$\Rightarrow \quad f\left(\dfrac{x-1}{x}\right) + f\left(\dfrac{-1}{x-1}\right) = 1 + \dfrac{x-1}{x} \quad \ldots(ii)$

Subtracting Eq. (ii) from Eq. (i), we get

$f(x) - f\left(-\dfrac{1}{x-1}\right) = x - \dfrac{x-1}{x} \quad \ldots(iii)$

Replacing x by $-\dfrac{1}{x-1}$ in Eq. (i), we get

$$f\left(-\frac{1}{x-1}\right) + f\left(\frac{-\frac{1}{x-1} - 1}{\frac{-1}{x-1}}\right) = 1 + \frac{-1}{x-1}$$

$\Rightarrow \quad f\left(-\dfrac{1}{x-1}\right) + f(x) = 1 - \dfrac{1}{x-1} \quad \ldots(iv)$

Adding Eqs. (iii) and (iv), we get

$$2f(x) = x - \frac{x-1}{x} + 1 - \frac{1}{x-1}$$

$$= \frac{x^3 - x^2 - x^2 + 2x - 1 + x^2 - x - x}{x(x-1)}$$

$$= \frac{x^3 - x^2 - 1}{x(x-1)}$$

$\Rightarrow \quad f(x) = \dfrac{x^3 - x^2 - 1}{2x(x-1)}$

which is the required function.

11. We observe that,

$$f(9) = f(4+5) = f(4 \cdot 5) = f(20)$$
$$= f(16+4) = f(16 \cdot 4) = f(64)$$
$$= f(8 \cdot 8) = f(8+8) = f(16)$$
$$= f(4 \cdot 4) = f(4+4) = f(8)$$

Hence, if $f(8) = 9$, then $f(9) = 9$. (This is one string. There may be other different ways of approaching $f(8)$ from $f(9)$. The important thing to be observed is the fact that the rule $f(x+y) = f(xy)$ applies only when x and y are at least 4. One may get strings using numbers x and y which are smaller then 4 but that is not valid. For example $f(9) = f(3 \cdot 3) = f(3+3) = f(6) = f(4+2) = f(4 \cdot 2) = f(8)$, is not a valid string.)

Level 2

1. We have $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$

$\Rightarrow f(1-x) = \dfrac{a^{1-x}}{a^{1-x} + \sqrt{a}} = \dfrac{a}{a + a^x\sqrt{a}} = \dfrac{\sqrt{a}}{a^x + \sqrt{a}}$

$\therefore \quad f(x) + f(1-x) = \dfrac{a^x}{a^x + \sqrt{a}} + \dfrac{\sqrt{a}}{a^x + \sqrt{a}}$

$$= \frac{a^x + \sqrt{a}}{a^x + \sqrt{a}} = 1$$

Now, $\displaystyle\sum_{k=1}^{2n-1} 2f\left(\frac{k}{2n}\right) = 2\left[f\left(\frac{1}{2n}\right) + f\left(\frac{2}{2n}\right) + f\left(\frac{3}{2n}\right) + f\left(\frac{4}{2n}\right) + \ldots + f\left(\frac{2n-2}{2n}\right) + f\left(\frac{2n-1}{2n}\right)\right]$

Putting the terms equidistant from ends together, we get

$$= 2\left[\left(f\left(\frac{1}{2n}\right) + f\left(1 - \frac{1}{2n}\right)\right) + \left(f\left(\frac{2}{2n}\right) + f\left(1 - \frac{2}{2n}\right)\right) + \left(f\left(\frac{3}{2n}\right) + f\left(1 - \frac{3}{2n}\right)\right) + \ldots + \left(f\left(\frac{n-1}{2n}\right) + f\left(1 - \frac{n-1}{2n}\right)\right) + f\left(\frac{n}{2n}\right)\right]$$

$$= 2\left[(n-1) + f\left(\frac{1}{2}\right)\right]$$

$$= 2n - 2 + 2\left(\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}} + \sqrt{a}}\right) = 2n - 2 + 2 \cdot \frac{1}{2} = 2n - 1$$

Also, $\sum_{k=1}^{2n} 2f\left(\frac{k}{2n+1}\right)$

$$= 2\left[f\left(\frac{1}{2n+1}\right) + f\left(\frac{2}{2n+1}\right) + f\left(\frac{3}{2n+1}\right) + f\left(\frac{4}{2n+1}\right) + \ldots + f\left(\frac{2n-1}{2n+1}\right) + f\left(\frac{2n}{2n+1}\right)\right]$$

Putting terms equidistant from the ends together, we get

$$= 2\left[\left(f\left(\frac{1}{2n+1}\right) + f\left(1 - \frac{1}{2n+1}\right)\right) + \left(f\left(\frac{2}{2n+1}\right) + f\left(1 - \frac{2}{2n+1}\right)\right) + \left(f\left(\frac{3}{2n+1}\right) + f\left(1 - \frac{3}{2n+1}\right)\right) + \ldots + \left(f\left(\frac{n}{2n+1}\right) + f\left(1 - \frac{n}{2n+1}\right)\right)\right]$$

$$= 2\left[\left\{f\left(\frac{1}{2n+1}\right) + f\left(1 - \frac{1}{2n+1}\right)\right\} + \left\{f\left(\frac{2}{2n+1}\right) + f\left(1 - \frac{2}{2n+1}\right)\right\} + \left\{f\left(\frac{3}{2n+1}\right) + f\left(1 - \frac{3}{2n+1}\right)\right\} + \ldots + \left\{f\left(\frac{n}{2n+1}\right) + f\left(1 - \frac{n}{2n+1}\right)\right\}\right]$$

$= 2[1 + 1 + 1 + \ldots$ upto n term$] = 2n$

2. First Method

Let $f : R \to R$ such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1,\ \forall x, y \in R$

Let $a, b \in R$ such that $f(b) = a$.

Let us put $x = a$, $y = b$ in the given equation, we get

$$f(a - f(b)) = f(f(b)) + af(b) + f(a) - 1$$

$$\Rightarrow \quad f(a - a) = f(a) + a^2 + f(a) - 1$$

$$\Rightarrow \quad f(0) = 2f(a) + a^2 - 1$$

$$\Rightarrow \quad f(a) = \frac{f(0) + 1}{2} - \frac{a^2}{2} \qquad \ldots\text{(i)}$$

Again let $y_1 \in R$ such that $f(y_1) = c$

Let us put $x = a$, $y = y_1$ in the given equation, we get

$$f(a - f(y_1)) = f(f(y_1)) + af(y_1) + f(a) - 1$$

$$\Rightarrow \quad f(a - c) = f(c) + ac + f(a) - 1$$

$$\Rightarrow \quad f(a - c) = \frac{f(0) + 1}{2} - \frac{c^2}{2} + ac + \frac{f(0) + 1}{2} - \frac{a^2}{2} - 1$$

$$\Rightarrow f(a - c) = f(0) - \frac{a^2 + c^2 - 2ac}{2} = f(0) - \frac{(a - c)^2}{2}$$

As f is defined from R to R and $a, c \in R$ so $\exists x \in R$ such that $x = a - c$ and hence

$$f(x) = f(0) - \frac{x^2}{2}$$

$$\Rightarrow \quad f(a) = f(0) - \frac{a^2}{2} = \frac{f(0) + 1}{2} - \frac{a^2}{2}$$

[from Eq. (i)]

$$\Rightarrow \quad 2f(0) = f(0) + 1$$

$$\Rightarrow \quad f(0) = 1$$

Hence, $f(x) = 1 - \dfrac{x^2}{2}$

It can be easily verified that the function $f(x) = 1 - \dfrac{x^2}{2}$ satisfies the given functional relation.

Second Method

According to the given condition

$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1,\ \forall x, y \in R$

Let us put $x = f(y) = 0$ so that

$$f(0) = f(0) + 0f(0) + f(0) - 1 \Rightarrow f(0) = 1$$

Again putting $x = f(y) = k$ so that

$$f(0) = f(k) + k^2 + f(k) - 1 = 1$$

$$\Rightarrow \quad 2f(k) = 2 - k^2 \Rightarrow f(k) = 1 - \frac{k^2}{2}$$

$\Rightarrow f(x) = 1 - \dfrac{x^2}{2}$ is the required function satisfying the given condition.

3. We have $f(x)f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$

$$\Rightarrow 1 - f(x) - f\left(\frac{1}{x}\right) + f(x) f\left(\frac{1}{x}\right) = 1$$

$$\Rightarrow \quad (1 - f(x)) - f\left(\frac{1}{x}\right)(1 - f(x)) = 1$$

$$\Rightarrow \quad (1 - f(x))\left(1 - f\left(\frac{1}{x}\right)\right) = 1$$

Since, $f(x)$ is a polynomial function of degree n and $(1 - f(x))\left(1 - f\left(\dfrac{1}{x}\right)\right) = 1$, constant, such that $1 - f(x)$ and $1 - f\left(\dfrac{1}{x}\right)$ are reciprocal in magnitude which will be possible when

$$1-f(x)=x^n \text{ or } 1-f(x)=-x^n$$
$$1-f\left(\frac{1}{x}\right)=\frac{1}{x^n} \text{ or } 1-f\left(\frac{1}{x}\right)=-\frac{1}{x^n}$$

Whence $f(x)=\pm x^n+1$

4. Let $f: N \to R$ be such that $f(1)=1$ and

$$f(1)+2f(2)+3f(3)+\ldots+nf(n)$$
$$=n(n+1)f(n),\ \forall n \geq 2 \quad \ldots(i)$$
$$\Rightarrow f(f)+2f(2)+3f(3)+\ldots+nf(n)$$
$$+(n+1)f(n+1)=(n+1)(n+2)f(n+1)\ldots(ii)$$

On subtraction Eq. (i) from Eq. (ii)

$$(n+1)f(n+1)=(n+1)(n+2)f(n+1)$$
$$-n(n+1)f(n)$$
$$\Rightarrow (n+1)(n+2-1)f(n+1)=n(n+1)f(n)$$
$$\Rightarrow \quad nf(n)=(n+1)f(n+1)$$
$$\Rightarrow \quad 2f(2)=3f(3)=4f(4)=5f(5)=\ldots=nf(n)$$

Thus, the given result reduces to

$$f(1)+(n-1)nf(n)=n(n+1)(n)$$
$$\Rightarrow f(1)=n(n+1-n+1)f(n)=2nf(n)$$
$$\Rightarrow f(n)=\frac{f(1)}{2n}=\frac{1}{2n} \quad [\because f(1)=1]$$
$$\Rightarrow f(2008)=\frac{1}{2\times 2008}=\frac{1}{4016}$$

5. We have, $f(x)f(y)+2=f(x)+f(y)+f(xy)$

Putting $x=y=0$, we get

$$f(0)f(0)+2=f(0)+f(0)+f(0)$$
$$\Rightarrow (f(0))^2-3f(0)+2=0$$
$$\Rightarrow (f(0)-2)(f(0)-1)=0$$
$$\Rightarrow f(0)=2 \text{ or } f(0)=1$$

But $f(0)\neq 2$ (given) hence $f(0)=1$

Again putting $x=y=1$, we get

$$(f(1))^2+2=f(1)+f(1)+f(1)$$
$$\Rightarrow (f(1)-2)(f(1)-1)=0$$
$$\Rightarrow f(1)=1 \text{ or } f(1)=2$$

But f is given to be injective (one-one) and $f(0)=1$, hence $f(1)\neq 1$

$$\therefore \quad f(1)=2$$

Now, putting $y=\frac{1}{x}$ in the given relation, we get

$$f(x)f\left(\frac{1}{x}\right)+2=f(x)+f\left(\frac{1}{x}\right)+f(1)$$
$$\Rightarrow f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right); \quad [\because f(1)=2]$$
$$\Rightarrow f(x)=\pm x^n+1$$

From the given condition we have

$$f(3)=82=\pm 3^n+1$$
$$\Rightarrow \quad 3^n=81=3^4 \text{ (neglecting -ve sign)}$$
$$\Rightarrow \quad n=4$$

Hence, $f(x)=x^4+1$

6. We have, $2f(x)+3f\left(\frac{2x+29}{x-2}\right)=100x+80$

Let us put $\frac{2x+29}{x-2}$ in place of x, we get

$$2f\left(\frac{2x+29}{x-2}\right)+3f\left(\frac{2\cdot\frac{2x+29}{x-2}+29}{\frac{2x+29}{x-2}-2}\right)$$
$$=100\cdot\frac{2x+29}{x-2}+80$$
$$\Rightarrow 2f\left(\frac{2x+29}{x-2}\right)+3f\left(\frac{4x+58+29x-58}{2x+29-2x+4}\right)$$
$$=100\left(\frac{2x+29}{x-2}\right)+80$$
$$\Rightarrow 2f\left(\frac{2x+29}{x-2}\right)+3f\left(\frac{33x}{33}\right)$$
$$=100\left(\frac{2x+29}{x-2}\right)+80$$
$$\Rightarrow f\left(\frac{2x+29}{x-2}\right)=\frac{-3}{2}f(x)+50\left(\frac{2x+29}{x-2}\right)+40$$

With this result, the given functional relation reduces to

$$2f(x)-\frac{9}{2}f(x)+150\left(\frac{2x+29}{x-2}\right)+120=100x+80$$
$$\Rightarrow f(x)=60\left(\frac{2x+29}{x-2}\right)+16-40x$$
$$=\frac{120x+1740+16x-40x^2-32+80x}{x-2}$$
$$=\frac{1708+216x-40x^2}{x-2}$$

7. (a) Let a, b, c be the sides of a triangle with $a+b+c=1996$, and each being a positive integer. Then, $a+1, b+1, c+1$ are also sides of a triangle with perimeter 1999 because

$$a<b+c \Rightarrow a+1<(b+1)+(c+1)$$

and so on. Moreover (999, 999, 1) form the sides of a triangle with perimeter 1999, which is not obtainable in the form $(a+1, b+1, c+1)$ where a, b, c are the integers and the sides of a triangle with $a+b+c=1996$. We conclude that $f(1999)>f(1996)$.

(b) As in the case (a) we conclude that $f(2000) \geq f(1997)$. On the other hand, if x, y, z are the integer sides of a triangle with $x + y + z = 2000$, and say $x \geq y \geq z \geq 1$, then we cannot have $z = 1$; for otherwise we would get $x + y = 1999$ forcing x, y to have opposite parity so that $x - y \geq 1 = z$ violating triangle inequality for x, y, z. Hence, $x \geq y \geq z \geq 1$. This implies that $x - 1 \geq y - 1 \geq z - 1 > 0$. We already have $x < y + z$. If $x \geq y + z - 1$, then we see that $y + z - 1 \leq x < y + z$, showing that $y + z - 1 = x$. Hence, we obtain $2000 = x + y + z = 2x + 1$ which is impossible. We conclude that $x < y + z - 1$. This shows that $x - 1 < (y - 1) + (z - 1)$ and hence $x - 1, y - 1, z - 1$ are the sides of a triangle with perimeter 1997. This gives $f(2000) \leq f(1997)$. Thus, we obtain the desired result.

8. Putting $x = 0, y = 0$, we get $f(0) = f(0)^3$ so that $f(0) = 0, 1$ or -1.

If $f(0) = 0$, then taking $y = 0$ in the given equation, we obtain $f(x) = f(x)f(0)^2 = 0$ for all x.

Suppose $f(0) = 1$. Taking $y = -x$, we obtain

$$1 = f(0) = f(x - x) = f(x)f(-x)f(-x^2)$$

This shows that $f(x) \neq 0$ for any $x \in R$. Taking $x = 1, y = x - 1$, we obtain

$$f(x) = f(1)f(x-1)^2 = f(1)[f(x)f(-x)f(-x)]^2$$

Using $f(x) \neq 0$, we conclude that $1 = kf(x)(f(-x))^2$ where $k = f(1)(f(-1))^2$. Changing x to $-x$ here, we also infer that $1 = kf(-x)(f(x))^2$. Comparing these expression we see that $f(-x) = f(x)$. It follows that $1 = kf(x)^3$. Thus, $f(x)$ is constant for all x. Since, $f(0) = 1$. We conclude that $f(x) = 1$ for all real x.

If $f(0) = -1$, a similar analysis shows that $f(x) = -1$ for all $x \in R$. We can verify that each of these functions satisfies the given functional equation. Thus, there are three solutions, all of them being constant functions.

9. $$f(x^2 + yf(z)) = xf(x) + zf(y) \quad \text{...(i)}$$

Taking $x = y = 0$ in Eq. (i), we get $zf(0) = f(0)$ for all $z \in R$. Hence, we obtain $f(0) = 0$. Taking $y = 0$ in Eq. (i), we get

$$f(x^2) = xf(x) \quad \text{...(ii)}$$

Similarly, $x = 0$ in Eq. (i) gives

$$f(yf(z)) = zf(y) \quad \text{...(iii)}$$

Putting $y = 1$ in Eq. (iii), we get

$$f(f(z)) = zf(1), \forall z \in R \quad \text{...(iv)}$$

Now, using Eqs. (ii) and (iv), we obtain

$$f(xf(x)) = f(f(x^2)) = x^2 f(1) \quad \text{...(v)}$$

Put $y = z = x$ in Eq. (iii) also given

$$f(xf(x)) = xf(x) \quad \text{...(vi)}$$

Comparing Eqs. (v) and (vi), it follows that $x^2 f(1) = xf(x)$. If $x \neq 0$, then $f(x) = cx$, for some constant c. Since, $f(0) = 0$, we have $f(x) = cx$ for $x = 0$ as well. Substituting this in Eq. (i), we see that

$$c(x^2 + cyz) = cx^2 + cyz$$

or $$c^2 yz = cyz, \forall y, z \in R$$

This implies that $c^2 = c$. Hence, $c = 0$ or 1. We obtain $f(x) = 0$ for all x or $f(x) = x$ for all x. It is easy to verify that these two are solutions of the given equation.

10. We show that the solution set consists of $\{(t, 0, 0); t \in Z\} \cup \{(-1, -1, 1)\}$. Let us put $a + b + c = d, ab + bc + ca = e$ and $abc = f$. The given condition $f(f(a, b, c)) = (a, b, c)$ implies that

$$d + e + f = a, de + ef + fd = b, def = c$$

Thus, $abcdef = fc$ and hence either $cf = 0$ or $abde = 1$.

Case I Suppose $cf = 0$. Then, either $c = 0$ or $f = 0$. However $c = 0$ implies $f = 0$ and *vice-versa*. Thus, we obtain $a + b = d, d + e = a, ab = e$ and $de = b$. The first two relations give $b = -e$. Thus, $e = ab = -ae$ and $de = b = -e$. We get either $e = 0$ or $a = d = -1$.

If $e = 0$, then $b = 0$ and $a = d = t$, say. We get the triple $(a, b, c) = (t, 0, 0)$, where $t \in Z$. If $e \neq 0$, then $a = d = -1$. But then $d + e + f = a$ implies that $-1 + e + 0 = -1$ forcing $e = 0$. Thus, we get the solution family $(a, b, c) = (t, 0, 0)$ where $t \in Z$.

Case II Suppose $cf \neq 0$. In this case $abde = 1$. Hence, either all are equal to 1; or two equal to 1 and the other two equal to -1; or all equal to -1.

Suppose $a = b = d = e = 1$. Then, $a + b + c = d$ shows that $c = -1$. Similarly, $f = -1$. Hence, $e = ab + bc + ca = 1 - 1 - 1 = -1$ contradicting $e = 1$.

Suppose $a = b = 1$ and $d = e = -1$. Then, $a + b + c = d$ gives $c = -3$ and $d + e + f = a$ gives $f = 3$. But, then $f = abc = 1 \cdot 1 \cdot (-3) = -3$, a contradiction. Similarly $a = b = -1$ and $d = e = 1$ is not possible.

If $a = 1, b = -1, d = 1, e = -1$, then $a + b + c = d$ gives $c = 1$. Similarly $f = 1$. But then $f = abc = 1 \cdot 1 \cdot (-1) = -1$ a contradiction. If $a = 1, b = -1, d = -1, e = 1$, then $c = -1$ and $e = ab + bc + ca = -1 + 1 - 1 = -1$ and a contradiction to $e = 1$. The symmetry between (a, b, c) and (d, e, f) shows that $a = -1, b = 1, d = 1, e = -1$ is not possible. Finally, if $a = -1, b = 1, d = -1$ and $e = 1$, then $c = -1$ and $f = -1$. But then $f = abc$ is not satisfied.

The only case left is that of a, b, d, e being all equal to -1. Then, $c = 1$ and $f = 1$. It is easy to check that $(-1, -1, 1)$ is indeed a solution.

Aliter $cf \neq 0$ implies that $|c| \geq 1$ and $|f| \geq 1$. Observe that

$d^2 - 2e = a^2 + b^2 + c^2$, $a^2 - 2b = d^2 + e^2 + f^2$

Adding these two, we get $-2(b + e) = b^2 + c^2 + e^2 + f^2$. This may be written in the form

$$(b + 1)^2 + (e + 1)^2 + c^2 + f^2 - 2 = 0.$$

We conclude that $c^2 + f^2 \leq 2$. Using $|c| \geq 1$ and $|f| \geq 1$, we obtain $|c| = 1$ and $|f| = 1, b + 1 = 0$ and $e + 1 = 0$. Thus, $b = e = -1$. Now $a + d = d + e + f + a + b + c$ and this gives $b + c + e + f = 0$. It follows that $c = f = 1$ and finally $a = d = -1$.

11. We have to prove that $f : N \times N \to N$ defined by

$$f(a, b) = \frac{1}{2}(a + b - 1)(a + b - 2) + a, \forall a, b \in N,$$

is a bijection. (Note that the right side is a natural number). To this end define

$$T(n) = \frac{n(n + 1)}{2}, n \in N \cup \{0\}.$$

An idea of the proof can be obtained by looking at the following table of values of $f(a, b)$ for some small values of a, b.

a \ b	1	2	3	4	5	6
1	1	2	4	7	11	16
2	3	5	8	12	17	
3	6	9	13	18		
4	10	14	19			
5	15	20				
6	21					

We observe that the nth diagonal runs from $(1, n)$th position to $(n, 1)$th position and the entries are n consecutive integers; the first entry in the nth diagonal is one more than the last entry of the $(n - 1)$th diagonal. For example the first entry in 5th diagonal is 11 which is one more than the last entry of 4th diagonal which is 10. Observe that 5th diagonal starts from 11 and ends with 15 which accounts for 5 consecutive natural numbers. Thus, we see that $f(n - 1, 1) + 1 = f(1, n)$. We also observe that the first n diagonals exhaust all the natural numbers from 1 to $T(n)$. (This a kind of visual bijection is already there. We formally prove the property.)

We first observe that

$$f(a, b) - T(a + b - 2) = a > 0,$$

and $$T(a + b - 1) - f(a, b) = \frac{(a + b - 1)(a + b)}{2} - \frac{(a + b - 1)(a + b - 2)}{2} - a = b - 1 \geq 0$$

Thus, we have

$$T(a + b - 2) < f(a, b) = \frac{(a + b - 1)(a + b - 2)}{2} + a \leq T(a + b - 1)$$

Suppose $f(a_1, b_1) = f(a_2, b_2)$. Then, the previous observation shows that

$$T(a_1 + b_1 - 2) < f(a_1, b_1) \leq T(a_1 + b_1 - 1),$$
$$T(a_2 + b_2 - 2) < f(a_2, b_2) \leq T(a_2 + b_2 - 1)$$

Since, the sequence $\langle T(n) \rangle_{n=0}^{\infty}$ is strictly increasing, it follows that $a_1 + b_1 = a_2 + b_2$. But then the relation $f(a_1, b_1) = f(a_2, b_2)$ implies that $a_1 = a_2$ and $b_1 = b_2$. Hence, f is one-one.

Let n be any natural number. Since, the sequence $\langle T(n) \rangle_{n=0}^{\infty}$ is strictly increasing we can find a natural number k such that

$$T(k - 1) < n \leq T(k)$$

Equivalently,

$$\frac{(k - 1)k}{2} < n \leq \frac{k(k + 1)}{2} \quad \ldots(i)$$

Now, set $a = n - \frac{k(k - 1)}{2}$ and $b = k - a + 1$.

Observe that $a > 0$. Now Eq. (i) shows that

$$a = n - \frac{k(k - 1)}{2} \leq \frac{k(k + 1)}{2} - \frac{k(k - 1)}{2} = k$$

Hence, $b = k - a + 1 \geq 1$. Thus, a and b are both positive integers and

$$f(a, b) = \frac{1}{2}(a + b - 1)(a + b - 2) + a$$
$$= \frac{k(k - 1)}{2} + a = n$$

This shows that every natural number is in the range of f. Thus, f is also onto. We conclude that f is a bijection.

Solved Paper 2016 RMO

Solved Paper 2016

RMO

Regional Mathematics Olympiad

Conducted by: Homi Bhabha Centre for Science Education, India

Exam Held on 09-10-2016

Regional Mathematics Olympiad is the first stage of 5 Stages of Mathematics Olympiad Program. On the basis of the performance in RMO, a certain number of students from each region is selected for Stage 2 (Indian National Mathematics Olympiad). Atmost 6 Class XII students from each region will be selected to appear for Stage 2 (INMO).

1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let I be the incentre of ΔABC. Draw a line perpendicular to AI at I. Let it intersect the line CB at D. Prove that CI is perpendicular to AD and prove that $ID = \sqrt{b(b-a)}$, where $BC = a$ and $CA = b$.

2. Let a, b and c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1$$

Prove that $abc \leq \frac{1}{8}$.

3. For any natural number n, expressed in base 10, let $S(n)$ denotes the sum of all digits of n. Find all natural numbers n such that $n^3 = 8(S(n))^3 + 6n\,S(n) + 1$.

4. How many 6-digit natural numbers containing only the digits 1, 2, 3 are there in which 3 occurs exactly twice and the number is divisible by 9?

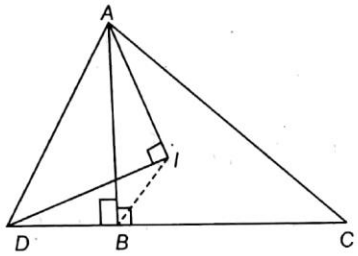
5. Let ABC be a right-angled triangle with $\angle B = 90°$. Let AD be the bisector of $\angle A$ with D on BC. Let the circumcircle of ΔACD intersect AB again in E and let the circumcircle of ΔABD intersect AC again in F. Let K be reflection of E in the line BC. Prove that $FK = BC$.

6. Show that the infinite arithmetic progression $\langle 1, 4, 7, 10, \ldots\ldots \rangle$ has infinitely many 3-term subsequences in harmonic progression such that for any two such triples $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ in harmonic progression, one has

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2} \neq \frac{a_3}{b_3}$$

Detailed Solutions

1. Let us draw the following diagram according to the question.



Since, $\angle AID = \angle ABD = 90°$

$\therefore ADBI$ is a cyclic quadrilateral.

$\therefore \quad \angle ADI = \angle ABI = 45°$

$\Rightarrow \quad \angle DAI = 45°$

We also have,

$$\angle ADB = \angle ADI + \angle IDB = 45° + \angle IAB$$
$$= \angle DAI + \angle IAC = \angle DAC$$

$\therefore \Delta CDA$ is an isosceles triangle.

$\therefore \quad CD = CA$

Since, CI bisects $\angle C$.

$\Rightarrow \quad CI \perp AD$ **Hence proved.**

This shows that $DB = CA - CB = b - a$

$$\therefore \quad AD^2 = c^2 + (b-a)^2$$
$$= c^2 + b^2 + a^2 - 2ba$$
$$= c^2 + a^2 + b^2 - 2ba$$
$$= b^2 + b^2 - 2ba$$
$$= 2b^2 - 2ba = 2b(b-a)$$

Again, $\quad 2ID^2 = AD^2 \quad \left[\because \dfrac{ID}{AD} = \cos 45°\right]$

$\Rightarrow \quad 2ID^2 = 2b(b-a)$

$\Rightarrow \quad ID^2 = b(b-a)$

$\Rightarrow \quad ID = \sqrt{b(b-a)}$ **Hence proved.**

2. We have, $\dfrac{a}{1+a} + \dfrac{b}{1+b} + \dfrac{c}{1+c} = 1$

$$\Rightarrow a(1+b)(1+c) + b(1+a)(1+c) + c(1+a)(1+b) = (1+a)(1+b)(1+c)$$

$$\Rightarrow a(1+b+c+bc) + b(1+a+c+ac) + c(1+a+b+ab) = (1+a)(1+b+c+bc)$$

$$\Rightarrow a + ab + ca + abc + b + ab + bc + abc + c + ca + bc + abc = 1 + b + c + bc + a + ab + ac + abc$$

$\Rightarrow \quad ab + bc + ca + 2abc = 1$

Now, we know that

AM$\geq$ GM

$$\Rightarrow \frac{ab + bc + ca + 2abc}{4} \geq (ab \cdot bc \cdot ca \cdot 2abc)^{\frac{1}{4}}$$

$$\Rightarrow \quad 1 \geq 4(2a^3b^3c^3)^{\frac{1}{4}}$$

$$\Rightarrow \quad \left(\frac{1}{4}\right)^4 \geq 2a^3b^3c^3$$

$$\Rightarrow \quad a^3b^3c^3 \leq \frac{1}{512}$$

$$\Rightarrow \quad abc \leq \frac{1}{8}$$ **Hence proved.**

3. We have,

$$n^3 = 8(S(n))^3 + 6n\,S(n) + 1$$

$\Rightarrow \quad n^3 - 8(S(n))^3 - 1 = 6n\,S(n)$

$\Rightarrow (n)^3 + (-2\,S(n))^3 + (-1)^3 = 3 \times n \times (-2\,S(n))(-1)$

$\Rightarrow n + (-2\,S(n)) + (-1) = 0$

[If $x^3 + y^3 + z^3 = 3xyz \Rightarrow x + y + z = 0$]

$\Rightarrow \quad n - 2\,S(n) - 1 = 0$

$\Rightarrow \quad n - S(n) = S(n) + 1 \quad \ldots$(i)

Again, we know that for every number $n - S(n)$ is always divisible by 9.

$\therefore S(n) + 1$ is also divisible by 9. [from Eq. (i)]

Now, for a three digit number maximum value of $S(n)$ can be 27.

$\therefore \quad 2\,S(n) + 1 = 2 \times 27 + 1 = 55$

$\therefore \quad 2\,S(n) + 1 \leq 55$

Hence, n is either a 1-digit number or a two digit number.

Hence, $S(n) \leq 18$. Since, $S(n) + 1$ must be divisible by 9.

$\therefore \quad S(n) = 8$ or $S(n) = 17$

$\therefore \quad n = 17$ or 35 [from Eq. (i)]

Among these $n = 17$ works but not 35,

as $S(35) = 8$

and $\quad 2S(n) + 1 = 17 \neq 35$

Hence, the only solution is $n = 17$.

4. Let the number be n and sum of its digit be $S(n)$.

Since, n is divisible by 9.

$\therefore \quad n \equiv S(n) \pmod 9$

We have, n contains 1, 2, 3 and 3 can occurs exactly twice.

$\therefore \quad 10 \leq S(n) \leq 14$

Since, there is no value of $S(n)$ that is a multiple of 9.

Thus, no such n exists.

5. Consider the ΔEBD and ΔCFD.

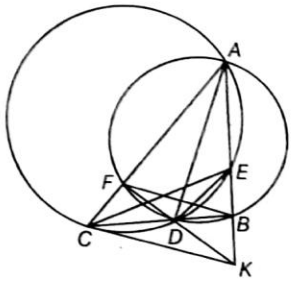
$\angle CFD = 90°$

$[\because \angle AFD = 90°, \angle CFD + \angle AFD = 180°]$

$\therefore \quad \angle CFD = \angle EBD$

Since, $ACDE$ is a cyclic quadrilateral.

$\therefore \quad \angle CDE = 180° - \angle A$



Similarly, $AFDB$ is a cyclic quadrilateral and therefore, $\angle FDB = 180° - \angle A$

$\therefore \quad \angle CDE = \angle FDB$

$\Rightarrow \quad \angle FDC = \angle BDE$

$\therefore \quad \Delta EBD \sim \Delta CFD$

Now, AD is bisector of $\angle A$.

$\therefore \quad DB = DF$

$\therefore \quad \Delta EBD \cong \Delta CFD$

Hence, $\quad FC = EB = BK$

Also, $\quad AF = AB$ as $\Delta ABD \cong \Delta AFD$

$\Rightarrow \quad FB \parallel CK$

Since, $FC = BK$, we can say that $CKDF$ is an isosceles trapezium.

$\therefore \quad FK = BC$ **Hence proved.**

6. Consider the triplet $< 4, 7, 28 >$

Then, $\dfrac{1}{4} + \dfrac{1}{28} = \dfrac{7+1}{28}$

$= \dfrac{8}{28}$

$= \dfrac{2}{7}$

$\therefore$ 4, 7, 28 are in HP.

Thus, we can say that a, b, ab are in HP and required condition is

$$\frac{1}{a} + \frac{1}{ab} = \frac{2}{b}$$

$$\Rightarrow \quad \frac{b+1}{ab} = \frac{2}{b}$$

$$\Rightarrow \quad b + 1 = 2a$$

or $\quad 2a = b + 1$

Given, AP is 1, 4, 7, 10, ...

$\therefore \quad a_k = 3k + 1$

If we take $a = 3k + 1$, then

$b = 2a - 1 = 2(3k + 1) - 1$

$= 6k + 1$

We observe that b is also a term of the given AP.

Again, $\quad ab = (3k + 1)(6k + 1)$

$= 18k^2 + 9k + 1$

$= 3(6k^2 + 3k) + 1$

which is again a term of the given AP.

Thus, $3k + 1$, $6k + 1$, $(3k + 1)(6k + 1)$ are in HP.

Also, we have,

$$\frac{3m+1}{3n+1} \neq \frac{6m+1}{6n+1} \neq \frac{(3m+1)(6m+1)}{(3n+1)(6n+1)}$$

Solved Paper 2017 INMO

Solved Paper 2017

INMO

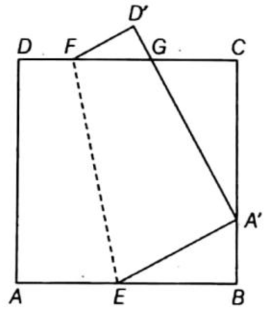
Indian National Mathematics Olympiad

Conducted by: Homi Bhabha Centre for Science Education, India

Exam Held on 15-01-2017

Indian National Mathematics Olympiad is organised by HBCSE (Homi Bhabha Centre for Science Education). On the basis of performance in INMO, top 35 students are selected from which final 5 students are selected to participate in International Mathematics Olympiad.

1. In the given figure, $ABCD$ is a square paper. It is folded along EF such that A goes to a point $A' \neq C, B$ on the side BC and D goes to D'. The line $A'D'$ cuts CD in G. Show that the inradius of the $\Delta GCA'$ is the sum of the inradii of the $\Delta GD'F$ and $\Delta A'BE$.



2. Suppose $n \geq 0$ is an integer and all the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of α.

3. Find the number of triples (x, a, b) where x is a real number and a, b belong to the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that

$$x^2 - a\{x\} + b = 0,$$

where $\{x\}$ denotes the fractional part of the real number x. (For example $\{1.1\} = 0.1 = \{-0.9\}$.)

4. Let $ABCDE$ be a convex pentagon in which $\angle A = \angle B = \angle C = \angle D = 120°$ and whose side lengths are 5 consecutive integers in some order. Find all possible values of $AB + BC + CD$.

5. Let ABC be a triangle with $\angle A = 90°$ and $AB < AC$. Let AD be the altitude from A on BC. Let P, Q and I denote respectively the incentres of Δ's ABD, ACD and ABC. Prove that AI is perpendicular to PQ and $AI = PQ$.

6. Let $n \geq 1$ be an integer and consider the sum

$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \ldots.$$

Show that $2x-1, 2x, 2x+1$ form the sides of a triangle whose area and inradius are also integers.

Detailed Solutions

1. From the figure drawn below,

$\angle 1 + \angle 2 = 90°$

and $\angle 2 + \angle 3 = 90°$

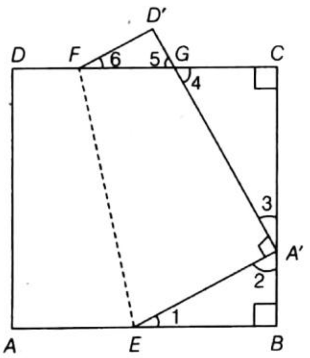
$\Rightarrow \angle 1 = \angle 3$

Also, $\angle 3 + \angle 4 = 90°$

and $\angle 2 + \angle 3 = 90°$

$\Rightarrow \angle 2 = \angle 4$

Again, $\angle 4 = \angle 6$ and $\angle 3 = \angle 6$



$\therefore \Delta GCA'$ and $\Delta A'BE$ are similar to the $\Delta GD'F$.
If $GF = u$, $GD' = v$ and $D'F = w$,

then we have

$A'G = pu, CG = pv, A'C = pw$

and $A'E = qu, BE = qw, A'B = qv$

Let r be the inradius of $\Delta GD'F$, then pr and qr will be the inradius of $\Delta GCA'$ and $\Delta A'BE$ respectively.

Now, $AE = EA'$ and $DF = FD'$

Again, $AB = BC = CD = AD$

$\therefore \quad pw + qv = qw + qu = w + u + pv = v + pu$

$\therefore \quad pw + qv = qw + qu$

$\Rightarrow \quad (p-q)w = q(u-v)$

$\Rightarrow \quad \dfrac{p-q}{q} = \dfrac{u-v}{w}$ ...(i)

$\because \quad w + u + pv = v + pu$

$\Rightarrow \quad w = v - u + pu - pv$

$\Rightarrow \quad w = p(u-v) - (u-v)$

$= (u-v)(p-1)$

$\Rightarrow \quad \dfrac{u-v}{w} = \dfrac{1}{p-1}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get

$\dfrac{p-q}{q} = \dfrac{u-v}{w} = \dfrac{1}{p-1}$

Now, $\dfrac{p-q}{q} = \dfrac{1}{p-1}$

$\Rightarrow (p-q)(p-1) = q$

$\Rightarrow p(p-q-1) = 0$

But $p \neq 0$

$\therefore \quad p - q - 1 = 0$

$\Rightarrow \quad p = q + 1$

$\Rightarrow \quad pr = qr + r$

2. Let u, v, w be the roots of

$$x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$$

$\therefore$ Sum of the zero's $= u + v + w = 0$

Sum of the product of zero's taken two at a line

$= uv + vw + wu = \alpha$

and product of the zero's

$= uvw = -(4 - 2 \times 2016^n)$

$= 2 \times 2016^n - 4$

Now, $u + v + w = 0$

$\Rightarrow \quad w = -(u+v)$

$\therefore \quad uvw = -uv(u+v) = 2 \times 2016^n - 4$

$\Rightarrow \quad uv(u+v) = 4 - 2 \times 2016^n$

Let $n \geq 1$

Then, $uv(u+v) \equiv 4 \pmod{2016^n}$

$\therefore$ $uv(u+v) \equiv 1 \pmod 3$ or $1 \pmod 9$

$\Rightarrow$ $u = 2 \pmod 3$ and $v = 2 \pmod 3$

$\therefore$ (u, v) can have values

(2, 2), (2, 5), (2, 8), (5, 2), (5, 5), (5, 8), (8, 2), (8, 5), (8, 8)

But in each case $uv(u+v) \not\equiv 4 \pmod 9$

$\therefore$ n must be equal to 0.

$\therefore$ $uv(u+v) = 4 - (2 \times 2016^0) = 4 - 2 \times 1 = 2$

Hence, $(u, v) = (1, 1), (1, -2), (-2, 1)$

$$\begin{aligned}\therefore \quad \alpha &= uv + vw + wu = uv + w(u+v)\\ &= uv - (u+v)^2\\ &= -3 \text{ for } (1, 1), (1, -2), (-2, 1)\end{aligned}$$

3. Let us write $x = n + f$, where $n = [x]$ and $f = \{x\}$.

Then, $f^2 + (2n - a)f + n^2 + b = 0$ ...(i)

The product of the roots of above equation is $n^2 + b$ which is greater than or equal to 1.

$\therefore$ $n^2 + b \geq 1$

For solution of Eq. (i), $0 \leq f < 1$, the larger root must be greater than 1. The Eq. (i) has a real root less than 1 only if

$1 + 2n - a + n^2 + 2b < 0$

$\Rightarrow$ $(n+1)^2 + b < a$

- If $n \geq 2$, then $(n+1)^2 + b \geq 10 > a$. Hence, $n \leq 1$. If $n \leq -4$, then again $(n+1)^2 + b \geq 10 > a$. Thus, we have the range for n: $-3, -2, -1, 0, 1$.
- If $n = -3$ or $n = 1$, we have $(n+1)^2 = 4$. Thus, we must have $4 + b < a$. If $a = 9$, we must have $b = 4, 3, 2, 1$ giving 4 values. For $a = 8$, we must have $b = 3, 2, 1$ giving 3 values.

Similarly, for $a = 7$ we get 2 values of b and $a = 6$ leads to 1 value of b. In each case we get a real value of $f < 1$ and this leads to a solution for x. Thus, we get totally $2(4 + 3 + 2 + 1) = 20$ values of the triple (x, a, b).

For $n = -2$, $n = 0$, we have $(n+1)^2 = 1$. Hence, we require $1 + b < a$. We again count pairs (a, b) such that $a - b > 1$. For $a = 9$, we get 7 values of b; for $a = 8$ we get 6 values of b and so on.

Thus, we get

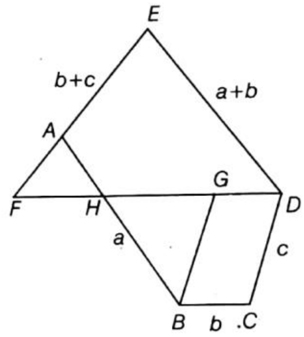
$2(7 + 6 + 5 + 4 + 3 + 2 + 1) = 56$ values for the triple (x, a, b).

Suppose $n = -1$ so that $(n+1)^2 = 0$. In this case we require $b < a$.

We get $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ values for the triple (x, a, b).

Thus, total triples $= 20 + 56 + 36 = 112$

4. Let $AB = a$, $BC = b$ and $CD = c$. By symmetry, we may assume that $c < a$. We have, $DE = a + b$ and $EA = b + c$



Draw a line parallel to BC through D. Extend EA to meet this line at F. Draw a line parallel to CD through B and let it intersect DF at G. Let AB intersect DF at H. Now, we have $\angle FDE = 60°$ and $\angle E = 60°$. Hence, EFD is an equilateral triangle. Similarly, AFH and BGH are also equilateral triangles. Hence, $HG = GB = c$. Moreover, $DG = b$, therefore, $HD = b + c$. But $HD = AE$, since $FH = FA$ and $FD = FE$.

Also, $AH = a - BH = a - c$

Hence, $ED = EF = EA + AF = b + c + AH$

$= (b + c) + (a - c) = b + a$

Now, we have five possibilities

(1) $b < c < a < b + c < a + b$;

(2) $c < b < a < b + c < a + b$;

(3) $c < a < b < b + c < a + b$;

(4) $b < c < b + c < a < a + b$;

(5) $c < b < b + c < a < a + b$;

In case (1) If $b = 2$, then $c < a < b + c$ are three consecutive integers.

$\therefore$ $c = 3$ and $a = 4$

$\therefore$ $b + c = 5$, $a + b = 6$

Hence, we have five consecutive integers 2, 3, 4, 5, 6 as side lengths.

In case (2) If $c = 2$, then $b < a < b + c$ form three consecutive integers.

$\therefore$ $b = 3$, $a = 4$

But $b + c = 5$ and $a + b = 7$

Thus, sides will be 2, 3, 4, 6, 7 which are not consecutive integers.

In case (3) We have $b < b + c$ are two consecutive integers so that $c = 1$.

$\therefore \quad a = 2$ and $b = 3$

Also, $b + c = 4$ and $a + b = 5$

Thus, sides will be 1, 2, 3, 4, 5.

In case (4) We have $c < b + c$ are two consecutive integers and $b = 1$.

$\therefore\ c = 2, b + c = 3, a = 4$ and $a + b = 5$

Thus, sides will be 1, 2, 3, 4, 5

In case (5) We have $b < b + c$ are two consecutive integers, so that $c = 1$

$\therefore \quad b = 2, b + c = 3, a = 4, a + b = 6$

Thus, sides will be 1, 2, 3, 4, 6 which are not consecutive integers.

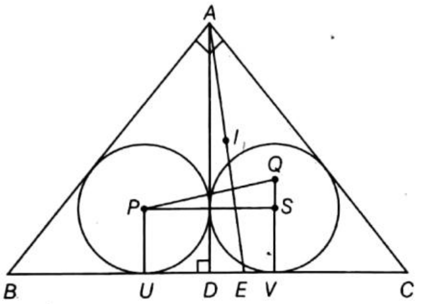
Therefore, the possible values of (a,b,c) are (4, 2, 3), (2, 3, 1), (4, 1, 2).

$\therefore \quad a + b + c = 9, 6$ or 7

Thus, possible sum $AB + BC + CA = 6, 7$ or 9

5. Draw $PS \| BC$ and $QS \| AD$

$\therefore$ The ΔPSQ is right-angled triangle with $\angle PSQ = 90°$.



From figure, we have

$PS = r_1 + r_2$ and $SQ = r_2 - r_1$, where r_1 and r_2 are the inradii of ΔABD and ΔACD respectively.

By AA similarity, $\Delta DAB \sim \Delta ACB$ and $\Delta DCA \sim \Delta ACB$

Hence, $\quad \frac{r_1}{r} = \frac{c}{a}$ and $\frac{r_2}{r} = \frac{b}{a}$

where, r is the inradius of ΔABC. Thus, we get

$$\frac{PS}{SQ} = \frac{r_2 + r_1}{r_2 - r_1} = \frac{b + c}{b - c}$$

Also, $\quad AD = h = \frac{bc}{a}$ and $BE = \frac{ca}{b + c}$

$$\text{Now, } BD^2 = c^2 - h^2 = c^2 - \frac{b^2c^2}{a^2} = c^2\left[\frac{a^2 - b^2}{a^2}\right]$$

$$= c^2 \times \frac{c^2}{a^2} = \frac{c^4}{a^2}$$

[$\because \Delta ABC$ is right-angled triangle]

$$\therefore \quad BD = \frac{c^2}{a}$$

$$\text{Now, } DE = BE - BD = \frac{ca}{b + c} - \frac{c^2}{a}$$

$$= \frac{cb(b - c)}{a(b + c)}$$

Thus, we get

$$\frac{AD}{DE} = \frac{b + c}{b - c} = \frac{PS}{SQ}$$

Since, $\angle ADE = 90° = \angle PSQ$, we can have

$$\Delta ADE \sim \Delta PSQ$$

Again, $\quad AD \perp PS$

$\Rightarrow \quad AE \perp PQ$

We also observe that

$$PQ^2 = PS^2 + SQ^2 = (r_2 + r_1)^2 + (r_2 - r_1)^2$$

$$= 2(r_1^2 + r_2^2)$$

But $\ r_1^2 + r_2^2 = \frac{c^2 + b^2}{a^2} \cdot r^2 = r^2 \quad [\because a^2 = b^2 + c^2]$

$\therefore \quad PQ = \sqrt{2}\, r$

Also, we have,

$AI = r \operatorname{cosec}(A/2) = r \operatorname{cosec} 45° = \sqrt{2}\, r$

$\therefore \quad PQ = AI \quad$ **Hence proved.**

6. Consider the binomial expansion of $(2 + \sqrt{3})^n$

Let $\quad (2 + \sqrt{3})^n = x + y\sqrt{3} \quad \ldots(i)$

and $\quad (2 - \sqrt{3})^n = x - y\sqrt{3} \quad \ldots(ii)$

On multiplying Eqs. (i) and (ii), we get

$\Rightarrow \quad (4 - 3)^n = x^2 - 3y^2$

$\Rightarrow \quad x^2 - 3y^2 = 1$

Since, in the expansion of $(2 + \sqrt{3})^n$, all the terms will be positive.

$$\therefore \quad 2x = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n$$

$$= 2\left(2^n + \binom{n}{2}2^{n-2} \cdot 3 + \cdots\right) \geq 4$$

Thus, $\quad x \geq 2$

$\therefore \quad 2x + 1 < 2x + (2x - 1)$

$\therefore (2x - 1), 2x, (2x + 1)$ will form a triangle.

By Heron's formula, we have

$$\Delta^2 = 3x(x + 1)(x)(x - 1) = 3x^2(x^2 - 1)$$

$$= 9x^2y^2 \qquad [\because x^2 - 3y^2 = 1]$$

$\Rightarrow \quad \Delta = 3xy$, which is an integer.

Again, inradius $= \frac{\text{area}}{\text{perimeter}} = \frac{3xy}{3x}$

$= y$, which is also an integer.

Solved Paper 2017 RMO

Solved Paper 2017

RMO

Regional Mathematical Olympiad

Conducted by: Homi Bhabha Centre for Science Education, India

(Exam Held on 08-10-2017)

Regional Mathematics Olympiad is the first stage of 5 Stages of Mathematics Olympiad Program. On the basis of the performance in RMO, a certain number of students from each region is selected for Stage 2 (Indian National Mathematics Olympiad). Atmost 6 Class XII students from each region will be selected to appear for Stage 2 (INMO).

1. Let AOB be a given angle less than $180°$ and let P be an interior point of the angular region determined by $\angle AOB$. Show, with proof, has to construct, using only ruler and compasses, a line segment CD passing through P such that C lies on the ray OA and D lies on the ray OB and $CP : PD = 1 : 2$.

2. Show that the equation

$a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3 + (a+4)^3 + (a+5)^3 + (a+6)^3 = b^4 + (b+1)^4$ has no solutions in integers a, b.

3. Let $P(x) = x^2 + \frac{1}{2}x + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)\,Q(x) = Q\,(P(x))$ for all real x. Find all the real roots of $P(Q(x)) = 0$.

4. Consider n^2 unit squares in the xy-plane centred at point (i, j) with integer coordinates, $1 \le i \le n$, $1 \le j \le n$. It is required to colour each unit square in such a way that when ever $1 \le i < j \le n$ and $1 \le k < l \le n$, the three squares with centres at (i, k), (j, k), (j, l) have distinct colours. What is the least possible number of colours needed?

5. Let Ω be a circle with chord AB which is not a diameter. Let T_1 be a circle on one side of AB such that it is tangent to AB at C internally tangent to Ω at D. Likewise. Let T_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the lines DC intersects Ω at $X \ne D$ and the line FE intersects Ω at $Y \ne F$. Prove that XY is a diameter of Ω.

6. Let x, y, z be real numbers, each greater than 1. Prove that

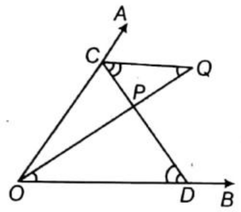
$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \le \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}$$

Detailed Solutions

1. We have,

$\angle AOB$ is less than 180° and P be an any interior point.

Therefore



Join OP and extend OP to Q such that

$$\frac{OP}{PQ} = \frac{2}{1} \quad \ldots\text{(i)}$$

Draw a line through Q parallel to OB which meets OA at C.

Join CP and extend to meet OB at D. CD is required line.

In ΔCPQ and ΔDPO

$\angle CQP = \angle POD$

[$\because$ alternate interior angles as $CQ \| OD$]

$\angle QCP = \angle PDO$

[$\because$ alternate interior angles as $CQ \| OD$]

$\angle DPO = \angle CPQ$ [vertically opposite angles]

$\therefore \quad \Delta CPQ \sim \Delta DPO$ [by AAA similarity criterion]

$\therefore \quad \dfrac{CP}{DP} = \dfrac{PQ}{OP} = \dfrac{1}{2}$ [from eq. (i)]

$\therefore \quad PC : PD = 1:2$

2. We have,

$$a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3 + (a+4)^3 + (a+5)^3 + (a+6)^3 = b^4 + (b+1)^4$$

Since, 7 consecutive number appears on left side, therefore apply modulo of

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$$

$$= 1 + 8 + 27 + 64 + 125 + 216 + 343$$

When divide by 7, we get

$$1 \pmod 7 + 1 \pmod 7 + (-1) \pmod 7 + 1 \pmod 7 - 1 \pmod 7 - 1 \pmod 7 + 0 \pmod 7$$

$$= (1 + 1 - 1 + 1 - 1 - 1 + 0) \pmod 7$$

$$= 0 \pmod 7$$

So, LHS is always divisible by 7.

Or

$$a^3 + (a+1)^3 + \ldots (a+6)^3 = \sum_{r=1}^{7} r^3 \pmod 7$$

$$= \left(\frac{7(7+1)}{2}\right)^2 \pmod 7$$

$$= (28)^2 \pmod 7$$

$$= 0 \pmod 7$$

Now, RHS $= b^4 + (b+1)^4$ for any integral b.

$b = r \pmod 7$, where $0 \le r < 7$

$b^4 + (b+1)^4 = r^4 + (r+1)^4 \not\equiv 0 \pmod 7$

for any $r = 0, 1, 2, 3, 4, 5, 6$

$\therefore b^4 + (b+1)^4$ is not divisible by 7

Hence, no solution for any integral value of a and b.

3. We have, $P(x) = x^2 + \dfrac{1}{2}x + b \quad \ldots$(i)

and $\quad Q(x) = x^2 + cx + d \quad \ldots$(ii)

Let $P(x) = 0$

$\therefore \quad P(x)\,Q(x) = Q(P(x))$

$0 \cdot Q(x) = Q(0)$

$\Rightarrow \quad Q(0) = 0$

$\Rightarrow \quad d = 0$

$\therefore \quad Q(x) = x^2 + cx$ [from eq. (ii) and $d = 0$]

Now, $P(x)\,.\,Q(x) = Q(P(x)) = (P(x))^2 + cP(x)$

[$\because d = 0$]

$\Rightarrow \quad Q(x) = P(x) + c$ [dividing both sides by $P(x)$]

$\Rightarrow \quad x^2 + cx = x^2 + \dfrac{1}{2}x + b + c$

$\Rightarrow \quad cx = \dfrac{1}{2}x + b + c$

On comparing the coefficient of x and constant terms both sides, we get

$\Rightarrow \quad c = \dfrac{1}{2}, b + c = 0,$

$\therefore \quad b = -\dfrac{1}{2}$

Hence, $\quad P(x) = x^2 + \dfrac{1}{2}x - \dfrac{1}{2}$ [from eq (ii)]

and $\quad Q(x) = x^2 + \dfrac{1}{2}x \quad \ldots$(iii)

Now, since $P(Q(x)) = 0$, $Q(x)$ is root of $P(x) = 0$

i.e. $$x^2 + \frac{x}{2} - \frac{1}{2} = 0$$

$$\Rightarrow \quad (x+1)\left(x - \frac{1}{2}\right) = 0$$

$$x = -1, \frac{1}{2}$$

$\therefore Q(x)$ has to be either -1 or $\frac{1}{2}$.

Case 1. $Q(x) = -1$

$\Rightarrow x^2 + \frac{1}{2}x + 1 = 0$ [from eq. (iii)]

$\Rightarrow 2x^2 + x + 2 = 0$ [multiplying both sides by 2]

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{15}\,i}{4}$$

[by quadrotic formula]

Case 2. $Q(x) = \frac{1}{2}$

$\Rightarrow x^2 + \frac{1}{2}x - \frac{1}{2} = 0$ [from eq (iii)]

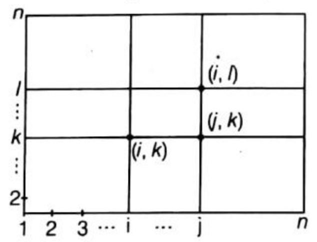
$$\Rightarrow \quad (x+1)\left(x - \frac{1}{2}\right) = 0$$

$$\Rightarrow \quad x = \frac{1}{2}, -1$$

So total 4 roots, 2 real and 2 imaginary.

Real roots are $\frac{1}{2}, -1$.

4. We have n^2 unit squares in the xy- plane.



Here, $1 \le i < j \le n$

$1 \le k < l \le n$

According to problems

(i, k), (j, k), (j, l) are of distinct colours in the k^{th} row no two square will be same colour as (i, k) and (j, k) are of distinct colours similarly in i^{th} column no two square of same colour as (j, l) and (j, k) are of distinct colour.

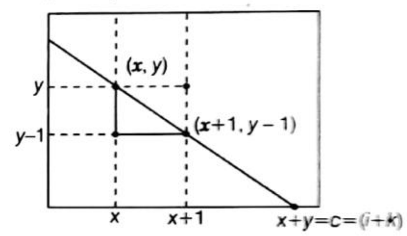
Therefore, all square of type in Ist column i.e. $(x, 1)$ for $x = 1, 2, 3, \ldots, n$ are of distinct colour

$\Rightarrow$ total n distinct colours.

Similarly all square of type (x, y) for $y = 1, 2, 3, \ldots, n$ are of distinct colours, also no square of $(x, 1)$ and (x, y) with same colour otherwise take square (x, x) with $(x, 1)$ and (x, y), we will get contradiction to given condition.

There will be atleast $(2n - 1)$ colours.

Now, let us prove $2n - 1$ colours are sufficient.



We can see that (x, y) and $(x+1, y-1)$ can have same colours. So, point all squares with same colour for which $x + y$ is same.

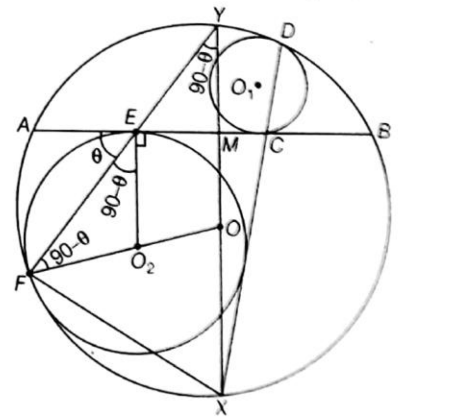
Here, minimum $x + y$ will be 2(for $x = 1$, $y = 1$) and maximum $x + y = 2n$

As from 2 to $2n$ three $2n - 1$ values.

So, $2n - 1$ colours will be sufficient.

5. Let O_1, O and O_2 be the centres of circles T_1, Ω and T_2 respectively.

Join OY it to meet AB at M. Join OO_2 it will passes through F (as both circles are tangent)



Now let $\angle AEF = \theta$

$\therefore \quad \angle O_2EF = 90° - \angle AEF$

$\Rightarrow \quad \angle O_2EF = 90° - \theta$

Since, O_2E and O_2F are radius of circle T_2

$\therefore \quad \angle O_2EF = \angle O_2FE = 90° - \theta$

[$\because$ angles opposite to equal sides are equal]

Also $\quad \angle OFY = \angle OYF$

[$OY = OF$ radii of same circle O]

$\angle FOX = 2\angle OYF$

[$\because$ angle subtended by an arc at the centre of the circle is thrice the angle subtended by the same arc at any point on the remaining part of the circle]

$\therefore \quad \angle FOX = 2(90° - \theta)$

$= 180° - 2\theta$

In ΔOFX

$\angle OFX + \angle OXF + \angle FOX = 180°$

[by angle sum property of triangle]

$\Rightarrow \quad \angle OFX + \angle OFX = 180° - \angle FOX$

[$\because \angle OFX = \angle OXF$ as $OF = OX$, radii of same circle]

$\Rightarrow \quad 2\angle OFX = 180° - 180° + 2\theta$

$\Rightarrow \quad \angle OFX = \theta$

$\therefore \quad \angle XFY = \angle OFX + \angle OFY$

$= \theta + 90° - \theta = 90°$

[$\because \angle OFY = \angle O_2FE = 90° - \theta$]

$\angle XFY = 90°$

$\therefore XY$ is a diameter of circle O.

6. We have x, y, z are real numbers and each greater than 1

Let $x \geq y \geq z > 1$

$\Rightarrow \quad x^2 - 1 \geq y^2 - 1 \geq z^2 - 1 > 0$

On taking reciprocal of each term except zero, we get

$$\Rightarrow \quad \frac{1}{z^2 - 1} \geq \frac{1}{y^2 - 1} \geq \frac{1}{x^2 - 1} > 0$$

[$\because$ sign of inequalities change due to reciprocal]

Also, $x - y \geq 0$, $y - z \geq 0$

$$\therefore \quad \frac{x - y}{y^2 - 1} \geq \frac{x - y}{x^2 - 1}$$

$$\text{and} \quad \frac{y - z}{z^2 - 1} \geq \frac{y - z}{x^2 - 1}$$

$$\Rightarrow \frac{x - y}{y^2 - 1} + \frac{y - z}{z^2 - 1} \geq \frac{x - y}{x^2 - 1} + \frac{y - z}{x^2 - 1}$$

$$\Rightarrow \frac{x - y}{y^2 - 1} + \frac{y - z}{z^2 - 1} \geq \frac{x - z}{x^2 - 1}$$

$$\Rightarrow \frac{x - y}{y^2 - 1} + \frac{y - z}{z^2 - 1} + \frac{z - x}{x^2 - 1} \geq 0$$

$$\Rightarrow \frac{1}{2}\left[\frac{x - 1}{y - 1} - \frac{x + 1}{y + 1}\right] + \frac{1}{2}\left[\frac{y - 1}{z - 1} - \frac{y + 1}{z + 1}\right] + \frac{1}{2}\left[\frac{z - 1}{x - 1} - \frac{z + 1}{x + 1}\right] \geq 0$$

$$\Rightarrow \frac{x - 1}{y - 1} - \frac{x + 1}{y + 1} + \frac{y - 1}{z - 1} - \frac{y + 1}{z + 1} + \frac{z - 1}{x - 1} - \frac{z + 1}{x + 1} \geq 0$$

$$\Rightarrow \frac{x - 1}{y - 1} + \frac{y - 1}{z - 1} + \frac{z - 1}{x - 1} \geq \frac{x + 1}{y + 1} + \frac{y + 1}{z + 1} + \frac{z + 1}{x + 1}$$

Solved Paper 2018 INMO

Solved Paper 2018

INMO

Indian National Mathematical Olympiad

Conducted by: Homi Bhabha Centre for Science Education, India

(Exam Held on 21-01-2018)

Indian National Mathematics Olympiad is organised by HBCSE (Homi Bhabha Centre for Science Education). On the basis of performance in INMO, top 35 students are selected from which final 5 students are selected to participate in International Mathematics Olympiad.

1. Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points of BC and CA; let G be the centroid of triangle ABC. Suppose D, C, E, G are concyclic. Find the least possible perimeter of triangle ABC.

2. For any natural number n, consider a $1 \times n$ rectangular board made up of n unit squares. This is covered by three types of tiles; 1×1 red tile, 1×1 green tile and 1×2 blue domino. Let t_n denote the number of ways of covering $1 \times n$ rectangular board by these three types of tiles. Prove that t_n divides t_{2n+1}.

3. Let Γ_1 and Γ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ in an obtuse angle. Let the circumcircle of triangle O_1AO_2 intersect Γ_1 and Γ_2 respectively in points C and D. Let the line CB intersect Γ_2 in E; let the line DB intersect Γ_1 in F. Prove that the points C, D, E, F are concyclic.

4. Find all polynomials with real coefficients $P(x)$ such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

5. There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that this process stops after a finite number of steps. (Assume that the teacher has an abundant supply of apples.)

6. Let N denote the set of all natural numbers and let $f: N \to N$ be a function such that

(a) $f(mn) = f(m)\, f(n)$ for all m, n in N

(b) $(m + n)$ divides $f(m) + f(n)$; for all m, n in N

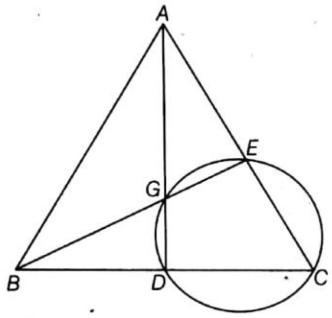
Prove that there exists an odd natural number k such that $f(n) = n^k$ for all n in N.

Detailed Solutions

1. We have,

ABC is non-equilateral triangle.

D and E are mid-point of BC and AC respectively.



$DCEG$ are concyclic

$\therefore \quad AG \cdot AD = AE \times AC$

$\Rightarrow \quad \frac{2}{3}AD \cdot AD = \frac{1}{2}AC \cdot AC$

[$\because$ G is centroid of ΔABC, $AG = \frac{2}{3}AD$

Also E is mid-point of AC]

$\Rightarrow \quad 2AD^2 = \frac{3}{2}AC^2$

In ΔABC by Appolonius theorem, we get

$$AB^2 + AC^2 = 2AD^2 + \frac{BC^2}{2}$$

$$\Rightarrow \quad AB^2 + AC^2 = \frac{3}{2}AC^2 + \frac{BC^2}{2}$$

$$\Rightarrow \quad AC^2 + BC^2 = 2AB^2$$

$$\Rightarrow \left(\frac{AC - BC}{2}\right)^2 + \left(\frac{AC + BC}{2}\right)^2 = AB^2$$

Thus $\left(\frac{AC - BC}{2}, \frac{AC + BC}{2}, AB\right)$ are pythagorean triplet. Consider the first triplet (3, 4, 5) this gives $AC = 7$, $BC = 1$, $AB = 5$. But AC, BC and AB are not forming a triangle.

The next triplet is (5, 12, 13), we get $AC = 17$, $BC = 7$ and $AB = 13$. In this case we get a triangle and its perimeter is $17 + 7 + 13 = 37$.

Since $2AB < AB + BC + CA < 3AB$. It is sufficient to verify upto $AB = 19$.

2. Consider a $1 \times (2n + 1)$ board and imagine the board to be placed horizontally.

Let us label the squares of the board as

$C_{-n}, C_{-(n-1)}, \ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots C_{n-1}, C_n$

from left to right. The 1×1 tiles will be referred to as red and green tiles, and the blue 1×2 tile will be referred to as a domino.

Let us consider the different ways in which the centre square C_0 can be covered.

There are four different ways in which this can be done.

(i) Let blue domino covering the square C_{-1}, C_0. In this case there is $1 \times (n-1)$ board remaining on left side which can be covered in t_{n-1} ways and $1 \times n$ board remaining on right side which can be covered in t_n ways.

(ii) If blue domino covering the square C_0, C_1, then $1 \times (n-1)$ board remaining on right side which can be covered in t_{n-1} ways and $(1 \times n)$ board remaining on left side which can be covered in t_n ways.

(iii) If red tile covering the square C_0. In this case there is $1 \times n$ board remaining on both sides of this tile, each can be covered in t_n ways.

(iv) If green tile covering the square C_0. In this case, there is $1 \times n$ board remaining on both sides of this tile each can be covered in t_n ways.

Now, putting all the possibilities mentioned above together, we get

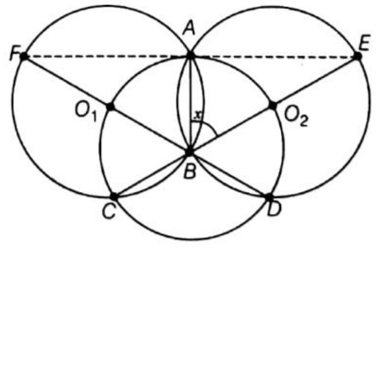
$$t_{2n+1} = t_{n-1} \cdot t_n + t_{n-1} \cdot t_n + t_n^2 + t_n^2$$

$$= 2(t_{n-1}t_n + t_n^2)$$

$$\Rightarrow \quad t_{2n+1} = 2t_n(t_{n-1} + t_n)$$

which implies that t_n divides t_{2n+1}

3. We will first prove that C, B, O_2, E are collinear and this line is bisector of $\angle ACD$



Let $\angle ABO_2 = x$

Then, $\angle AO_2B = 180° - 2x$

O_2O_1 is angle bisector of $\angle AO_2B$.

$\because \quad \angle AO_2O_1 = \frac{1}{2}\angle AO_2B = 90° - x$

Since A, O_1, C, O_2 are concylic.

$\therefore \quad \angle AO_2O_1 = \angle ACO_1 = 90° - x$

$\Rightarrow \quad \angle AO_1C = 180° - 2(90° - x) = 2x$

From this

$$\angle AFC = \frac{1}{2}\angle AO_1C = \frac{1}{2}(2x) = x$$

and $\measuredangle ABC = 180° - x$

Thus $\angle ABC$ and $\angle ABO_2$ are supplementary. Hence C, B, O_2, E are collinear. Now, $O_2A = O_2D$ implies that O_2 is the mid-point of arc AO_2D. Hence, CO_2 is the bisector of $\angle ACD$. Similarly we obtain that D, B, O_1, F are collinear. Thus, BE and BF are diameters of the respective circles.

$\Rightarrow \angle BAE = \angle BAF = 90°$;

$\Rightarrow$ FAE are collinear.

Finally, we get $\angle ECD = \angle BCD$

$= \angle ACB = \angle AFB = \angle EFD$

Therefore C, D, E, F are concylic.

4. If $P(x)$ is constant, say C, then $P(x^2 + x + 1) = C$ and $P(x^3 - 1) = C$, where $C \neq O$. Thus, in this case $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

If $P(x)$ is non-constant polynomial, then $P(x^3 - 1) = P(x^2 + x + 1) \cdot Q(x)$, where $Q(x)$ is some polynomial in x.

$\Rightarrow \quad P((x - 1)(x^2 + x + 1)) = P(x^2 + x + 1) \cdot Q(x)$

$\Rightarrow$ Whenever $x^2 + x + 1$ is a root of $P(x)$, then $(x - 1)(x^2 + x + 1)$ is also a root.

We prove that the only root of $p(x)=0$ is O

Suppose that there is a root 'α' with $|\alpha| > 0$

Now, take $x^2 + x + 1 = \alpha$

Let β and γ are root of $x^2 + x + 1 - \alpha = 0$

Then, $\beta + \gamma = -1$

Since, $P(x^2 + x + 1)$ divides $P(x^3 - 1)$

$\therefore \quad P(\beta^3 - 1) = 0, P(\gamma^3 - 1) = 0$

We also see that

$$\beta^3 - 1 + \gamma^3 - 1 = (\beta - 1)(\beta^2 + \beta + 1) + (\gamma - 1)(\gamma^2 + \gamma + 1)$$

$$\Rightarrow \beta^3 - 1 + \gamma^3 - 1 = (\beta - 1)(\alpha) + (\gamma - 1)\alpha = \alpha(\beta + \gamma - 2)$$

$[\because \beta^2 + \beta + 1 = \gamma^2 + \gamma + 1 = \alpha]$

Thus we have

$$|\beta^3 - 1| + |\gamma^3 - 1| \geq |\beta^3 - 1 + \gamma^3 - 1|$$

$$= |\alpha||\beta + \gamma - 2| = 3|\alpha|$$

$[\because \beta + \gamma = -1]$

This show that the absolute value of atleast one of $\beta^3 - 1$ and $\gamma^3 - 1$ is not less than $3|\alpha|/2$

It we take this as α_1, we have $|\alpha_1| > |\alpha|$

Now, α_1 is a root of $P(x) = 0$ and we repeat the argument with α_1 in place of α. We get infinite sequence of distinct roots of $P(x)=0$, which is not possible for any polynomial. This contradiction proves that all the roots of non-constant polynamial must be 'O'.

$\Rightarrow \quad P(x) = ax^n, a \in R, n \in N.$

or $\quad P(x) = C, c \in R - \{0\}$

5. We have $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Let i^{th} girl have a_i apples at any given time.

Consider two quantities with this distribution.

$s = a_1 + a_2 + a_3 + \ldots + a_n$ and

$t = a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2$

Using Cauchy - Schwartz inequality, we get

$$a_1^2 + a_2^2 + a_3^2 + \ldots a_n^2 \geq \frac{(a_1 + a_2 + a_3 + \ldots a_n)^2}{n}$$

$$\Rightarrow \qquad t \geq \frac{s^2}{n}$$

Therefore, $t \geq \frac{s^2}{n}$ at any stage of the above process. Whenever teacher makes a move, s increases by 1. Suppose the girl with a_m apples has more than the sum of her neighbours. Then the change in t equals to

$$(a_m - 1)^2 + (a_{m-1} + 1)^2 + (a_{m+1} + 1)^2 - a_m^2 - a_{m-1}^2 - a_{m+1}^2$$

$$= (a_m - 1)^2 - a_m^2 + (a_{m-1} + 1)^2 - a_{m-1}^2 + (a_{m+1} + 1)^2 - a^2{}_{m+1}$$

$$= (a_m - 1 + a_m)(a_m - 1 - a_m) + (a_{m-1} + 1 + a_{m-1})(a_{m-1} + 1 - a_{m-1}) + (a_{m+1} + 1 + a_{m+1})(a_{m+1} + 1 - a_{m+1})$$

$$= -2a_m + 1 + 2a_{m-1} + 1 + 2a_{m+1} + 1$$

$$= 2(a_{m+1} + a_{m-1} - a_m) + 3$$

$$= 3 + 2(a_{m+1} + a_{m-1} - a_m) \leq 3 + 2(-1) = 1$$

$[\because a_m > a_{m-1} + a_{m+1}]$

If s_1 and t_1 denote the corresponding sums after one move, we observe that

$$s_1 = s + 1 \text{ and } t_1 \leq t + 1$$

Thus after teacher performs k moves, if the corresponding sums and s_k and t_k, we get

$$s_k = s + k, t_k \leq t + k$$

$$t_k \geq \frac{s_k^2}{n} \quad \Rightarrow \quad t + k \geq \frac{(s+k)^2}{n}$$

$\Rightarrow \quad nt + nk \geq s^2 + k^2 + 2sk$

$\Rightarrow \quad k^2 + 2sk - nk + s^2 - nt \leq 0$

$\Rightarrow \quad k^2 + (2s - n)k + s^2 - nt \leq 0$

Since this cannot hold for large k, we see that the process must stop at some stage.

6. We have,

$$f(mn) = f(m)\, f(n),\ m, n \in N$$

Putting $\quad m = n = 1$, we get

$$f(1) = f(1)\, f(1)$$

$\Rightarrow \quad f(1)^2 - f(1) = 0$

$\Rightarrow \quad f(1)\,(f(1) - 1) = 0$

$\Rightarrow \quad f(1) = 1,\ f(1) \neq 0$

Now, put $m = 2$ and $n = n$, we get

$$f(2n) = f(2)\, f(n)$$

Also given $(m + n)$ divides $(f(m) + f(n))$

$\therefore \quad$ Putting $m = 2n$ and $n = 1$, we get

$$2n + 1 \text{ divides } f(2n) + f(1)$$

$\Rightarrow \quad (2n + 1)$ divides $f(2)\, f(n) + 1$

$[\because f(2n) = f(2)\, f(n) \text{ and } f(1) = 1]$

This shows that

HCF of $f(2)$ and $(2n + 1)$ is 1 for all n.

Since, $2n + 1$ is an odd number.

$\therefore f(2)$ must be even number i.e. $f(2) = 2^k$ for some natural number k.

Since $3 = 1 + 2$ divides $f(1) + f(2) = 1 + 2^k$, k is odd

Now take any arbitrary power of 2, say 2^m, and an arbitrary integer n.

$\therefore \quad 2^m + n$ divides $f(2^m) + f(n)$.

$\Rightarrow \quad 2^m + n$ divides $(f(2))^m + f(n)$

$[\because f(mn) = f(m)\,(n) f(2^m) = f(2) \cdot f(2) \ldots m \text{ times}]$

$[\because f(2) = 2^k]$

Thus $\quad 2^m + n$ divides $2^{km} + f(n)$

But $\quad 2^{km} + f(n) = 2^{km} + n^k + f(n) - n^k$

$$= M(2^m + n) + f(n) - n^k,\ k \text{ is odd}$$

This means $2^m + n$ divides $f(n) - n^k$.

By varying m over N, we conclude that

$$f(n) - n^k = 0$$

$\therefore \quad f(n) = n^k$ from all $n \in N$.

Solved Paper 2018 RMO

Solved Paper 2018

RMO

Regional Mathematical Olympiad

Conducted by: Homi Bhabha Centre for Science Education, India

Max Marks : 102 **(Exam Held on 07-10-2018)** **Time : 3 hours**

Regional Mathematics Olympiad is the first stage of 5 Stages of Mathematics Olympiad Program. On the basis of the performance in RMO, a certain number of students from each region is selected for Stage 2 (Indian National Mathematics Olympiad). Atmost 6 Class XII students from each region will be selected to appear for Stage 2 (INMO).

1. Let ABC be a triangle with integer sides in which $AB < AC$. Let the tangent to the circumcircle of triangle ABC at A intersect the line BC at D. Suppose, AD is also an integer. Prove that gcd $(AB, AC) > 1$

2. Let n be a natural number. Find all real numbers x satisfying the equation

$$\sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}$$

3. For a rational number r, its period is the length of the smallest repeating block in its decimal expansion. For example, the number $r = 0.123123123....$ has period 3. If S denotes the set of all rational number r of the form $r = 0 \cdot \overline{abcdefgh}$ having period 8, find the sum of all the elements of S.

4. Let E denote the set of 25 points (m,n) in the xy-plane, where m, n are natural numbers, $1 \le m \le 5$, $1 \le n \le 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form (a, b), $(a+k, b)$, $(a+k, b+k)$, $(a, b+k)$ for some positive integer k such that at least three of these four points have the same colour (i.e. there always exist four points in the set E which form the vertices of a square with sides parallel to axes and having at least three points of the same colour.)

5. Find all natural number n such that $1 + [\sqrt{2n}]$ divides $2n$. (For any real number x, $[x]$ denotes the largest integer not exceeding x.)

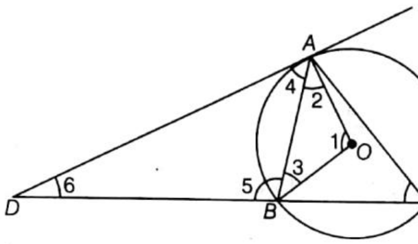
6. Let ABC be an acute-angled triangle with $AB < AC$. Let I be the incentre of triangle ABC, and let D, E and F be the points at which its incircle touches the sides BC, CA and AB respectively. Let BI, CI meet the line EF at Y, X, respectively. Further assume that both X and Y are outside the triangle ABC. Prove that

(i) B, C, Y, X are concyclic, and

(ii) I is also the incentre of triangle DYX.

Detailed Solutions

1.



We know that,

$$\angle 1 = 2\angle C$$

[Angle subtended at the centre by a chord is double the angle subtended by it at remaining part]

Now, $OA = OB$

$\Rightarrow \angle 2 = \angle 3$

$\therefore \angle 2 = \angle 3 = \frac{\pi}{2} - \angle C$ $\quad [\because OA \perp AD]$

Again, $\angle 4 = 90° - \angle 2$

$\Rightarrow \angle 4 = 90° - (90° - \angle C) = \angle C$

Also, $\angle 5 = 180° - \angle B$

In ΔABD,

$\angle 4 + \angle 5 + \angle 6 = 180°$

$\Rightarrow \angle 6 = 180° - [\angle C + 180° - \angle B]$

$\Rightarrow \angle 6 = \angle B - \angle C$

In ΔABD,

Using sine formula,

$$\frac{DB}{\sin C} = \frac{AB}{\sin D}$$

$\Rightarrow \frac{DB}{\sin C} = \frac{c}{\sin(B - C)}$ $[\because$ In ΔABC, $AB = c]$

$\Rightarrow DB = \frac{c\sin C}{\sin B\cos C - \cos B\sin C}$

$\Rightarrow DB = \frac{c \cdot Kc}{Kb \cdot \cos C - \cos B \cdot Kc}$

$\left[\text{In } \Delta ABC, \text{ by sine formula } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (let)}\right]$

$$= \frac{c^2}{b\cos C - c\cos B}$$

By cosine formula, in ΔABC

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{c^2}{b\left[\frac{a^2 + b^2 - c^2}{2ab}\right] - c\left[\frac{a^2 + c^2 - b^2}{2ac}\right]}$$

$$= \frac{c^2 a}{b^2 - c^2}$$

Also, we know that

$$DA^2 = DB \times DC$$

$$= \frac{c^2 a}{b^2 - c^2}\left(\frac{c^2 a}{b^2 - c^2} + a\right)$$

$[\because DC = DB + BC]$

$\therefore DA = \frac{abc}{b^2 - c^2} = \frac{abc}{(b + c)(b - c)}$

Since, DA is integer and let us assume b and c are coprime, then $\frac{abc}{(b + c)(b - c)} \in Z$, if $(b - c)$ and $(b + c)$ both are divisible by a, which is not possible as $b + c > a$ (triangle inequality).

$\therefore$ b and c can't be coprime.

Hence, $GCD(AB, AC) > 1$ [by contradiction]

2. We have

$$\sum_{k=1}^{n} \frac{kx^k}{1 + x^{2k}} = \frac{n(n + 1)}{4}$$

$\Rightarrow \frac{x}{1 + x^2} + \frac{2x^2}{1 + x^4} + \frac{3x^3}{1 + x^6} + \dots\dots\dots + \frac{nx^n}{1 + x^{2n}} = \frac{n(n + 1)}{4}$

$\Rightarrow \frac{1}{x^{-1} + x} + \frac{2}{x^{-2} + x^2} + \frac{3}{x^{-3} + x^3} + \dots\dots + \frac{n}{x^{-n} + x^n} = \frac{n(n + 1)}{4}$ $\quad \dots$(i)

Case I When $x > 0$

We know that, AM $\geq$ GM

$\therefore x^{-k} + x^k \geq 2\sqrt{x^{-k}x^k} \geq 2$ $\quad \dots$(ii)

From Eqs. (i) and (ii), we get

$$\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \dots + \frac{n}{2} \leq \sum_{k=1}^{n} \frac{k}{2} \leq \frac{n(n + 1)}{4}$$

$\therefore \sum_{k=1}^{n} \frac{kx^k}{1 + x^{2k}} = \frac{n(n + 1)}{4}$ is only possible when,

$x^k = 1, \forall k \in N$

$\therefore x = 1$

Case II When $x = 0$

$$\text{LHS} = \sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} = 0, \text{ but}$$

$$\text{RHS} = \frac{n(n+1)}{4} \neq 0 \text{ for any natural number } n.$$

So, $x = 0$ is not a solution.

Case III When $x < 0$

from the case I, it is clear that LHS is less than $\frac{n(n+1)}{4}$, if $x < 0$. Since, the odd indexed terms would becomes negative.

$\therefore$ No negative solution x exists.

Hence, $x = 1$ is only solution.

3. We have,

$$r = 0.\overline{abcdefgh}$$

$$\Rightarrow \quad 10^8 r = abcdefgh.\overline{abcdefgh}$$

$$\Rightarrow \quad (10^8 - 1)r = abcdefgh$$

$$\Rightarrow \quad r = \frac{abcdefgh}{10^8 - 1}$$

$$\text{Total sum} = \frac{1}{10^8 - 1} + \frac{2}{10^8 - 1} + \dots + \frac{999\dots(8\text{ times})}{10^8 - 1}$$

$$= \frac{1 + 2 + 3 + \dots + \underset{(8\text{ times})}{99\dots99}}{10^8 - 1}$$

$$= \frac{(10^8 - 1)10^8}{2(10^8 - 1)} = 5 \times 10^7$$

This sum contains all the numbers having period 1, 2, 4 and 8.

Now, period = 4

$\boxed{- \; - \; - \; -}$ $\boxed{- \; - \; - \; -}$

If unit digit is 1, then the next three places can be filled in 10^3 ways.

$\therefore$ Sum of all the digits at unit places

$$= \frac{(1 + 2 + 3 \dots + 9)10^3}{10^8 - 1} = \frac{45 \times 10^3}{10^8 - 1}$$

Hence, sum of all such numbers

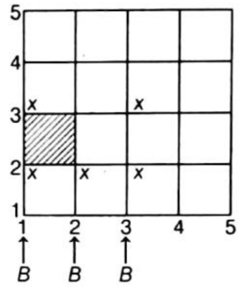
$$= \frac{(1 + 10 + \dots + 10^7)(45 \times 10^3)}{10^8 - 1} = 5 \times 10^3$$

It include numbers with period 1 and period 2.

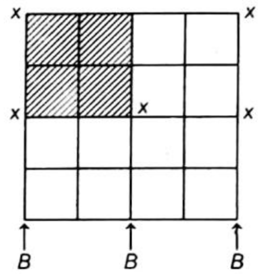
$\therefore$ Required sum $= 5 \times 10^7 - 5 \times 10^3$

$= 49995000$.

4. 5 vertices of each row has to be coloured using exactly two colours. We shall start with first row in which 3 vertices have one colour say blue and rest 2 vertices have other colour, red. If this initial condition ensures at least one square with at least 3 red vertices, then obviourly, the other two starting conditions for first row (i.e. 4 vertices coloured in blue or all 5 vertices coloured in blue) would automatically ensure at least one square with at least 3 red vertices.



(1)



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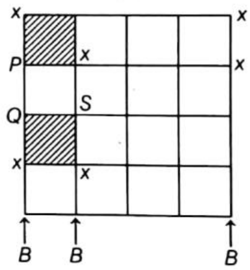

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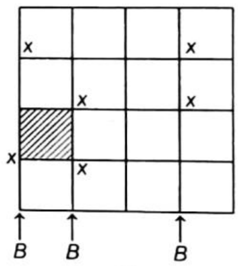

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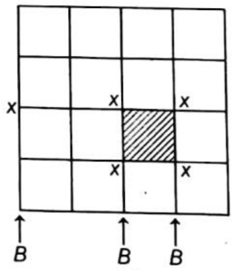


(5)

Then the vertices marked with (x) can not be coloured by blue, so they has to be coloured in red, marking at least one square with at least 3 red vertices (as shown in marked). In figure 3, position p can either be red or blue. If P is red, then we have upper left $|x|$ square with 3 red and if P is blue, then either Q or S is red, making another $|x|$ square with 3 red.

5. Let $\frac{2n}{1} = x$

$$\Rightarrow \quad 2n = x \quad \dots(\text{I})$$

We know that,

$$a-1<[a]\le a$$

$\therefore \quad \sqrt{2n}-1\le\sqrt{2n}$

$\Rightarrow x[\sqrt{2n}-1]<x\le x\sqrt{2n}$

$\Rightarrow x[\sqrt{2n}-1]<2n-x\le x\sqrt{2n}$ [from eq (i)]

Let $\sqrt{2n}=t$

$\therefore xt-x<t^2-x$ and $t^2-x\le xt$

$\Rightarrow \quad xt<t^2$ and $t^2-xt\le x$

$\Rightarrow \quad x<t$ and $\left(t-\frac{x}{2}\right)^2\le x+\frac{x^2}{4}$ $[\because t=\sqrt{2n}>0]$

$\Rightarrow \quad x<t$ and $\left(t-\frac{x}{2}\right)^2<\left(\frac{x}{2}+1\right)^2$

$\Rightarrow \quad x<t$ and $t-\frac{x}{2}<\frac{x}{2}+1$

$\Rightarrow \quad x<t$ and $t<x+1$

$\Rightarrow \quad t-1<x<t$ $[\because t=\sqrt{2n}]$

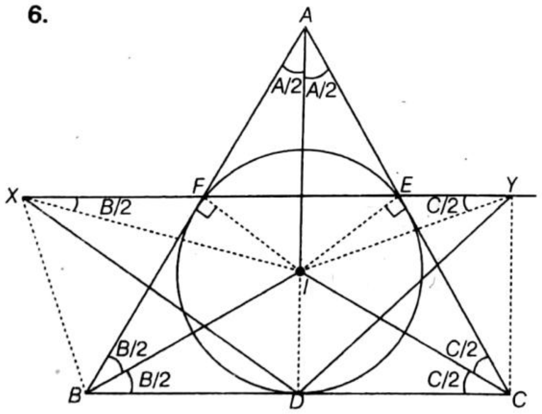
$\Rightarrow \quad =x$ ...(II)

from Eqs. (i) and (ii), we get

$$2n=x+x^2$$

$\Rightarrow \quad n=\frac{x(x+1)}{2},\ x\in N.$

6.



We have, $\Delta AIF\cong\Delta AIE$

$\therefore \quad \angle AIF=\angle AIE=90^\circ-\frac{A}{2}$

$\therefore \quad \angle EIF=180^\circ-A$

$\therefore \quad \angle IEF=\angle IFE=\frac{A}{2}$

$\therefore \quad \angle FEC=\angle FEI+\angle IEC$

$$=\frac{A}{2}+90^\circ$$

Now, in ΔXEC,

$$\angle EXC+\angle XEC+\angle ECX=180^\circ$$

$\Rightarrow \quad \angle EXC+\frac{A}{2}+90^\circ+\frac{C}{2}=180^\circ$

$\Rightarrow \quad \angle EXC=90^\circ-\frac{A}{2}-\frac{C}{2}$

$$=\frac{180^\circ-(A+C)}{2}$$

$$=\frac{180^\circ-(180^\circ-B)}{2}=\frac{B}{2}$$

$\therefore \quad \angle EXC=\angle EBC=\frac{B}{2}$

$\therefore$ $BCYX$ are concyclic .

$IDCE$ is a cyclic quadrilateral.

$\therefore$ $ICYE$ is also a cyclic quadrilateral.

Hence I, D, C, E, Y lie on same circle.

$\therefore \quad \angle ICD=\angle IYD$

$\Rightarrow \quad \angle IYD=\frac{C}{2}$

Similarly, $\angle IXO=\frac{B}{2}$

$\therefore$ IX, IY, ID are angle bisector.

$\therefore$ I is also incentre of ΔDYX.

Solved Paper 2019 INMO

Solved Paper 2019

INMO

Indian National Mathematical Olympiad

Conducted by: Homi Bhabha Centre for Science Education, India

Max Marks : 102 **(Exam Held on 20-1-2019)** **Time : 4 hours**

Indian National Mathematics Olympiad is organised by HBCSE (Homi Bhabha Centre for Science Education). On the basis of performance in INMO, top 35 students are selected from which final 5 students are selected to participate in International Mathematics Olympiad.

1. Let ABC be a triangle with $\angle BAC > 90°$, let D be a point on the segment BC and E be a point on the line AD such that AB is tangent to the circumcircle of triangle ACD at A and BE is perpendicular to AD. Given that $CA = CD$ and $AE = CE$, determine $\angle BCA$ in degrees.

2. Let $A_1B_1C_1D_1E_1$ be a regular pentagon. For $2 \le n \le 11$, let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoints of the sides of $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}$. All the 5 vertices of each of the 11 pentagons are arbitrarily coloured red or blue. Prove that four points among these 55 points have the same colour and form the vertices of a cyclic quadrilateral.

3. Let m, n be distinct positive integers. Prove that,

$gcd(m, n) + gcd(m+1, n+1) + gcd(m+2, n+2) \le 2|m-n| + 1.$

Further, determine when equality holds.

4. Let n and M be positive integers such that $M > n^{n-1}$. Prove that there are n distinct primes $p_1, p_2, p_3, \ldots, p_n$ such that p_j divides $M + j$ for $1 \le j \le n$.

5. Let AB be a diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB. Let K be a point of the segment CD such that AC is equal to the semiperimeter of the triangle ADK. Show that the excircle of triangle ADK opposite A is tangent to Γ.

6. Let f be a function defined from the set $\{(x, y) : x, y \text{ real}, xy \ne 0\}$ to the set of all positive real numbers such that

(i) $f(xy, z) = f(x, z)f(y, z)$, for all $x, y \ne 0$

(ii) $f(x, 1-x) = 1$, for all $x \ne 0, 1$.

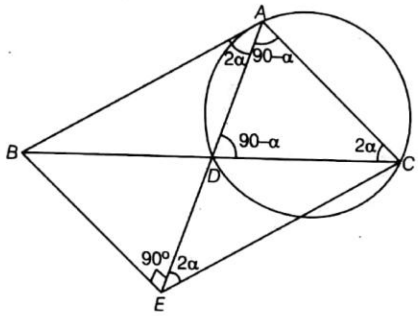
Prove that,

(a) $f(x, x) = f(x, -x) = 1$, for all $x \ne 0$;

(b) $f(x, y)f(y, x) = 1$, for all $x, y \ne 0$.

Detailed Solutions

1.



Let $\angle C = 2\alpha$. Then, $\angle CAD = \angle CDA = 90° - \alpha$. Moreover $\angle BAD = 2\alpha$ as BA is tangent to the circumcircle of ΔCAD. Since $AE = CE$. It gives $\angle AEC = 2\alpha$. Thus ΔAEC is similar to ΔACD. Hence,

$$\frac{AE}{AC} = \frac{AC}{AD}.$$

But the condition that $BE \perp AD$ gives $AE = AB\cos 2\alpha = c\cos 2\alpha$. It is easy to see that $\angle B = 90° - 3\alpha$. Using sine rule in triangle ADC, we get

$$\frac{AD}{\sin 2\alpha} = \frac{AC}{\sin(90-\alpha)}.$$

This gives $AD = 2b\sin\alpha$. Thus we get,

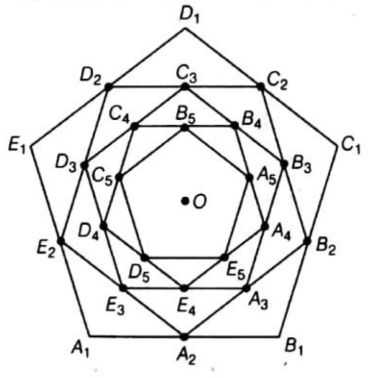
$$b^2 = AC^2 = AE \cdot AD = (c\cos 2\alpha)\cdot 2b\sin\alpha.$$

Using, $b = 2R\sin B$ and $c = 2R\sin C$, this leads to

$$\cos 3\alpha = 2\sin 2\alpha \cos 2\alpha \sin\alpha = \sin 4\alpha \sin\alpha$$

Writing $\cos 3\alpha = \cos(4\alpha - \alpha)$ and expanding, we get, $\cos 4\alpha \cos\alpha = 0$. Therefore, $\alpha = 90°$ or $4\alpha = 90°$. But $\alpha = 90°$ is not possible as $\angle C = 2\alpha$. Therefore $4\alpha = 90°$, which gives $\angle C = 2\alpha = 45°$.

2.



We first observe that all the eleven pentagons are regular. Moreover, there are 5 fixed directions and all the 55 sides are in one of these directions.

If we consider any two sides which are parallel, they are the parallel sides of an isosceles trapezium, which is cyclic.

If we consider any pentagon, its two adjacent vertices have the same colour. Consider all such 11 sides whose end points are of the same colour. These are in 5 fixed directions. By Pigeon-hole principle, there are 3 sides which are in the same directions and therefore parallel to each other. Among these three sides, two must have end points having one colour (again by P-H principle). Thus, there are two parallel sides among the 55 and the end points of these have one fixed colour. But these two sides are parallel sides of an isosceles trapezium. Hence the four end points are concyclic.

3. We know that, for $a > b$

$$gcd(a,b) = gcd(a, a-b).$$

$\therefore$ let $gcd(m,n) = gcd(m, m-n) = x$

$\Rightarrow \quad m - n = \alpha x \quad \ldots$(i)

$gcd(m+1, n+1) = gcd(m+1, m-n) = y$

$\Rightarrow \quad m - n = \beta y \quad \ldots$(ii)

and $gcd(m+2, n+2) = gcd(m+2, m-n) = z$

$\Rightarrow \quad m - n = \gamma z \quad \ldots$(iii)

$\therefore$ $gcd(x,y) = 1$, $gcd(y,z) = 1$ and $gcd(x,z) = 1$ or 2.

Eq. (i) Now, let gcd of $(x,y) = 1$, $gcd\,(y,z) = 1$

and $\quad gcd(x,z) = 1$

$\Rightarrow$ $m - n$ is divisible by xyz

$\Rightarrow \quad m - n \geq xyz$

$\Rightarrow \quad 2(m-n) + 1 \geq 2xyz + 1 \quad \ldots$(iv)

If $x = y = z = 1$, then

$2xyz + 1 = 3 = x + y + z$

Again, m and n are two consecutive integers.

$\therefore \quad 2(m-n) + 1 = 3$

$\therefore \quad x + y + z = 2(m-n) + 1$

If atleast one of $x, y, z \neq 1$, let $z > 1$

$\Rightarrow \quad z \geq 2$

$\therefore \quad 2xyz + 1 = 4xyz + 1 > x + y + z$

$\Rightarrow \quad x + y + z < 2(m-n) + 1$

Eq. (ii) Now, $gcd(x,y) = 1$, $gcd(y,z) = 1$

and $\quad gcd(z,x) = 2$

$\Rightarrow$ $z = 2p$, $x = 2q$ for relative prime p and q

$\therefore \quad m - n \geq 2pqy$

$\Rightarrow \; 2(m-n)+1 \geq 4pqy+1$

Again, equality satisfy only when $p=q=y=1$.

If $x=2=z$ and $y=1$ only when m and n are two consecutive even integers.

If $y>1 \Rightarrow y \geq 2$, then

$$4pqy+1 > 2p+2q+y = x+y+z$$

$\therefore \; 2(m-n)+1 > x+y+z$

$\Rightarrow \quad x+y+z < 2(m-n)+1.$

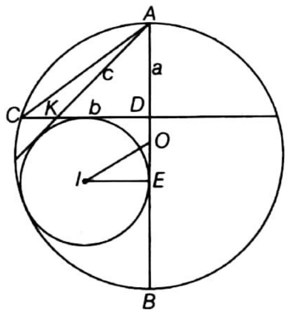
4. If some number $M+k, 1 \leq k \leq n$, has at least n distinct prime factors, then we can associate a prime factor of $M+k$ with the number $M+k$ which is not associated with any of the remaining $n-1$ numbers.

Suppose, $M+j$ has less than n distinct prime factors. Write

$$M+j = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}, r<n.$$

But $M+j > n^{n-1}$. Hence, there exist $t, 1 \leq t \leq r$ such that $p_t^{\alpha_t} > n$. Associate p_t with this $M+j$. Suppose p_t is associated with some $M+l$. Let $p_t^{\beta_t}$ be the largest power of p_t dividing $M+l$. Then, $p_t^{\beta_t} > n$. Let $T = gcd(p_t^{\alpha_t}, p_t^{\beta_t})$. Then $T>n$. Since $T \mid (M+j)$ and $T \mid (M+l)$, it follows that $T \mid (j-i) \mid$. But $|j-i| < n$ and $T>n$, and we get a contradiction. This shows that p_t cannot be associated with any other $M+l$. Thus each $M+j$ is associated with different primes.

5.



Let $AD=a$, $KD=b$, $AK=c$, $AC=x$, $EI=r$, $AB=2R$

Now, we have

$$AC = x = \frac{a+b+c}{2}$$

Again, Ex-radius of

$$\Delta ADK = r = \frac{KD+AK-AD}{2}$$

$$\therefore \quad r = \frac{b+c-a}{2} = \frac{b+c+a-2a}{2}$$

$$r = \frac{a+b+c}{2} - a = x-a$$

Again, $\quad AC^2 = AD \times AB$

$\Rightarrow \quad x^2 = a \times 2R$ ...(i)

Also, $\quad OE = |AD+DE-AO| = |a+r-R|$

$\quad = |x-R|$

Now, in ΔOIE

$OI^2 = OE^2 + EI^2$

$\Rightarrow \quad OI^2 = (x-R)^2 + r^2$

$\Rightarrow \quad OI^2 = x^2 + R^2 - 2xR + r^2$

$\Rightarrow \quad OI^2 = R^2 + r^2 + 2aR - 2xR$

$\Rightarrow \quad OI^2 = R^2 + r^2 - 2R(x-a)$ [From Eq. (i)]

$\Rightarrow \quad OI^2 = R^2 + r^2 - 2rR$

$\Rightarrow \quad OI^2 = (R-r)^2$

$\Rightarrow \quad OI = R-r.$

$\therefore$ The two circles touch each other internally.

6. We have,

$$f(xy, z) = f(x,z) \cdot f(y,z)$$

Put $x=y=1$, we get

$f(1,z) = f(1,z) \cdot f(1,z)$

$\Rightarrow \quad f(1,z) = (f(1,z))^2$

$\Rightarrow \quad f(1,z) = 1 \; \forall \; z \neq 0.$ ...(i)

Again, put $x=y=-1$, we get

$f(1,z) = f(-1,z) \cdot f(-1,z)$

$\Rightarrow \quad 1 = (f(-1,z))^2$

$\Rightarrow \quad f(-1,z) = 1 \; \forall \; z \neq 0$ ...(ii)

from Eq. (i)

$f(1,x) = 1 \; \forall \; x \neq 0$

$\Rightarrow \lim\limits_{n\to\infty} f(x^{1/2x}, x) = 1$

$\Rightarrow \lim\limits_{n\to\infty} f(x^{1/2^{n-1}}, x) = 1$

$\Rightarrow f(x^{1/2}, x) = 1$

$\Rightarrow f(x,x) = 1 \; \forall \; x \in (0, \infty)$

Similarly from Eq. (ii)

$f(1,-x) = 1 \; \forall \; x \neq 0$

$\Rightarrow \lim\limits_{n\to\infty} f(x^{1/2^n}, -x) = 1$

$\Rightarrow f(x,-x) = 1 \; \forall \; x \in (0, \infty)$

$\therefore \quad f(x,x) = f(x,-x) = 1 \; \forall \; x \neq 0$

Again,

$1 = f(xy, xy) = f(x, xy) f(y, xy)$

$= f(x,x) f(x,y) f(y,x) f(y,y)$

$\Rightarrow f(x,y) \cdot f(y,x) = 1 \; \forall \; x, y \neq 0.$